Generalizations of Some Sharp Inequalities for the Ratio of Gamma Functions

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Abstract

In this paper, we present some generalizations of the inequalities presented by the authors in [4].

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1 Introduction

The classical Euler’s Gamma function, \(\Gamma(t)\) is commonly defined as

\[
\Gamma(t) = \int_{0}^{\infty} e^{-x} x^{t-1} dx, \quad t > 0.
\]

The digamma function \(\psi(t)\), also known as the psi function is defined as follows.

\[
\psi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}
\]

The \((p,q)\)-analogue of the Gamma function, \(\Gamma_{(p,q)}(t)\) is defined by Krasniqi and Merovci [2] as

\[
\Gamma_{(p,q)}(t) = \frac{[p]_{q}^{[p]}[p]_{q}^{1}}{[t]_{q}[t + 1]_{q} \ldots [t + p]_{q}}, \quad t > 0, \quad p \in N, \quad q \in (0, 1)
\]
where \([p]_q = \frac{1-q^p}{1-q}\).

The \((p, q)\)-analogue of the digamma function, \(\psi_{(p, q)}(t)\) is also defined as follows.

\[
\psi_{(p, q)}(t) = \frac{d}{dt} \ln(\Gamma_{(p, q)}(t)) = \frac{\Gamma'_{(p, q)}(t)}{\Gamma_{(p, q)}(t)}
\]

Also, the \((q, k)\)-analogue of the Gamma function, \(\Gamma_{(q, k)}(t)\) is defined as (see \([1],[3]\))

\[
\Gamma_{(q, k)}(t) = \frac{(1 - q^k)^{\frac{k-1}{q}}}{(1 - q)^{\frac{k-1}{q}}} = \frac{(1 - q^k)^{\infty}_{q,k} q^t(1 - q)\frac{k-1}{q}}{1 - q^k}, \quad t > 0, \quad q \in (0, 1), \quad k > 0.
\]

Similarly, the \((q, k)\)-analogue of the digamma function, \(\psi_{(q, k)}(t)\) is defined as

\[
\psi_{(q, k)}(t) = \frac{d}{dt} \ln(\Gamma_{(q, k)}(t)) = \frac{\Gamma'_{(q, k)}(t)}{\Gamma_{(q, k)}(t)}
\]

The functions \(\psi(t)\), \(\psi_{(p, q)}(t)\) and \(\psi_{(q, k)}(t)\) as defined above possess the following series representations.

\[
\psi(t) = -\gamma + (t - 1) \sum_{n=0}^{\infty} \frac{1}{(1 + n)(n + t)}
\]

\[
\psi_{(p, q)}(t) = \ln[p]_q + (\ln q) \sum_{n=1}^{p} \frac{q^{nt}}{1 - q^n}
\]

\[
\psi_{(q, k)}(t) = -\ln(1 - q) + (\ln q) \sum_{n=1}^{\infty} \frac{q^{nk}}{1 - q^{nk}}
\]

where \(\gamma\) is the Euler-Mascheroni’s constant.

Recently, Nantomah and Prempeh \([4]\) established the following results.

\[
\frac{e^{-\gamma} \Gamma_{(\alpha)}}{[p]_q^{\alpha+t} \Gamma_{(p, q)}(\alpha)} < \frac{\Gamma(\alpha + t)}{\Gamma_{(p, q)}(\alpha + t)} < \frac{e^{\gamma(1-t)} \Gamma_{(\alpha + 1)}}{[p]_q^{\alpha+t-1} \Gamma_{(p, q)}(\alpha + 1)}
\]

for \(t \in (0, 1)\), where \(p \in N\), \(q \in (0, 1)\) and \(\alpha\) is a positive real number such that \(\alpha + t > 1\).

\[
\frac{e^{-\gamma} \Gamma_{(\alpha)}}{(1 - q)^{\frac{k}{q}} \Gamma_{(q, k)}(\alpha)} < \frac{\Gamma(\alpha + t)}{\Gamma_{(q, k)}(\alpha + t)} < \frac{e^{\gamma(1-t)} \Gamma_{(\alpha + 1)}}{(1 - q)^{\frac{k}{q}(1-t)} \Gamma_{(q, k)}(\alpha + 1)}
\]
for $t \in (0, 1)$, where $q \in (0, 1)$, $k > 0$ and $\alpha$ is a positive real number such that $\alpha + t > 1$.

The objective of this paper is to present some generalizations of the above inequalities.

2 Preliminary Results

Lemma 2.1. Let $a > 0$, $b > 0$ and $t > 1$, Then,

$$a\gamma + b\ln[p]_q + a\psi(t) - b\psi_{(p,q)}(t) > 0.$$

Proof. Using the series representations in equations (1) and (2) we have,

$$a\gamma + b\ln[p]_q + a\psi(t) - b\psi_{(p,q)}(t)
= a(t - 1)\sum_{n=0}^{\infty} \frac{1}{(n + 1)(n + t)} - b\ln q \sum_{n=1}^{\infty} \frac{q^{nt}}{1 - q^n} > 0$$

Lemma 2.2. Let $a > 0$, $b > 0$ and $\alpha + \beta t > 1$. Then,

$$a\gamma + b\ln[p]_q + a\psi(\alpha + \beta t) - b\psi_{(p,q)}(\alpha + \beta t) > 0.$$

Proof. Follows directly from Lemma 2.1.

Lemma 2.3. Let $a > 0$, $b > 0$ and $t > 1$, Then,

$$a\gamma - b\ln(1 - q) + a\psi(t) - b\psi_{(q,k)}(t) > 0.$$

Proof. Using the series representations in equations (1) and (3) we have,

$$a\gamma - b\frac{\ln(1 - q)}{k} + a\psi(t) - b\psi_{(q,k)}(t)
= a(t - 1)\sum_{n=0}^{\infty} \frac{1}{(n + 1)(n + t)} - b\ln q \sum_{n=1}^{\infty} \frac{q^{nkt}}{1 - q^{nk}} > 0$$

Lemma 2.4. Let $a > 0$, $b > 0$ and $\alpha + \beta t > 1$. Then,

$$a\gamma - b\frac{\ln(1 - q)}{k} + a\psi(\alpha + \beta t) - b\psi_{(q,k)}(\alpha + \beta t) > 0.$$

Proof. Follows directly from Lemma 2.3.
3 Main Results

**Theorem 3.1.** Define a function $F$ by

$$F(t) = \frac{e^{a\beta t} \Gamma(\alpha + \beta t)^a}{[p]_q^{-b\beta t} \Gamma_{(p,q)}(\alpha + \beta t)^b}, \quad t \in (0, \infty), \ p \in N, \ q \in (0, 1)$$

where $a, b, \alpha, \beta$ are positive real numbers such that $\alpha + \beta t > 1$. Then $F$ is increasing on $t \in (0, \infty)$ and the inequalities

$$\frac{e^{-a\beta t} \Gamma(\alpha)^a}{[p]_q^{b\beta t} \Gamma_{(p,q)}(\alpha)^b} < \frac{\Gamma(\alpha + \beta t)^a}{\Gamma_{(p,q)}(\alpha + \beta t)^b} < \frac{e^{a\beta(1-t)} \Gamma(\alpha + \beta)^a}{[p]_q^{b\beta t} \Gamma(\alpha + \beta)^b}$$

are valid for every $t \in (0, 1)$.

**Proof.** Let $\mu(t) = \ln F(t)$ for every $t \in (0, \infty)$. Then,

$$\mu(t) = \ln \frac{e^{a\beta t} \Gamma(\alpha + \beta t)^a}{[p]_q^{-b\beta t} \Gamma_{(p,q)}(\alpha + \beta t)^b} = a\beta t + b\beta \ln [p]_q + a \ln \Gamma(\alpha + \beta t) - b \ln \Gamma_{(p,q)}(\alpha + \beta t).$$

Then,

$$\mu'(t) = a\beta + b\beta \ln [p]_q + a \beta \psi(\alpha + \beta t) - b \psi_{(p,q)}(\alpha + \beta t) = \beta \left[ a\gamma + b \ln [p]_q + a \psi(\alpha + \beta t) - b \psi_{(p,q)}(\alpha + \beta t) \right] > 0$$

by Lemma 2.2. That implies $\mu$ is increasing on $t \in (0, \infty)$. Hence $F$ is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have,

$$F(0) < F(t) < F(1)$$

concluding the proof.

**Corollary 3.2.** If $t \in (1, \infty)$, then the following inequality is valid.

$$\frac{\Gamma(\alpha + \beta t)^a}{\Gamma_{(p,q)}(\alpha + \beta t)^b} > \frac{e^{a\beta(1-t)} \Gamma(\alpha + \beta)^a}{[p]_q^{b\beta t} \Gamma_{(p,q)}(\alpha + \beta)^b}$$

**Proof.** If $t \in (1, \infty)$, then we have $F(t) > F(1)$ concluding the proof.
Theorem 3.3. Define a function $G$ by

$$G(t) = \frac{e^{a\beta t} \Gamma(\alpha + \beta t)^a}{(1-q)^{k\frac{b}{k}} \Gamma(q,k)(\alpha + \beta t)^b}, \quad t \in (0, \infty), \; q \in (0,1), \; k > 0$$

where $a, b, \alpha, \beta$ are positive real numbers such that $\alpha + \beta t > 1$. Then $G$ is increasing on $t \in (0, \infty)$ and the inequalities

$$\frac{e^{-a\beta t} \Gamma(\alpha)^a}{(1-q)^{-k\frac{b}{k}} \Gamma(q,k)(\alpha)^b} < \frac{\Gamma(\alpha + \beta t)^a}{\Gamma(q,k)(\alpha + \beta t)^b} < \frac{e^{a\beta t(1-t)} \Gamma(\alpha + \beta)^a}{(1-q)^{k\frac{b}{k}(1-t)} \Gamma(q,k)(\alpha + \beta)^b}$$

are valid for every $t \in (0,1)$.

Proof. Let $\lambda(t) = \ln G(t)$ for every $t \in (0, \infty)$. Then,

$$\lambda(t) = \ln \left( \frac{e^{a\beta t} \Gamma(\alpha + \beta t)^a}{(1-q)^{k\frac{b}{k}} \Gamma(q,k)(\alpha + \beta t)^b} \right) = a\beta t - \frac{b\beta}{k} \ln(1-q) + a \ln \Gamma(\alpha + \beta t) - b \ln \Gamma(q,k)(\alpha + \beta t).$$

Then,

$$\lambda'(t) = a\beta - \frac{b\beta}{k} \ln(1-q) + a\beta \psi(\alpha + \beta t) - b \beta \psi(q,k)(\alpha + \beta t) = \beta \left[ a\gamma - \frac{b}{k} \ln(1-q) + a\psi(\alpha + \beta t) - b \psi(q,k)(\alpha + \beta t) \right] > 0$$

by Lemma 2.4. That implies $\lambda$ is increasing on $t \in (0, \infty)$. Hence $G$ is increasing on $t \in (0, \infty)$ and for every $t \in (0,1)$ we have,

$$G(0) < G(t) < G(1)$$

yielding the result.

Corollary 3.4. If $t \in (1, \infty)$, then the following inequality is valid.

$$\frac{\Gamma(\alpha + \beta t)^a}{\Gamma(q,k)(\alpha + \beta t)^b} > \frac{e^{a\beta t(1-t)} \Gamma(\alpha + \beta)^a}{(1-q)^{k\frac{b}{k}(1-t)} \Gamma(q,k)(\alpha + \beta)^b}$$

Proof. If $t \in (1, \infty)$, then we have $G(t) > G(1)$ yielding the result.

4 Concluding Remarks

Remark 4.1. By setting $a = b = \beta = 1$, then the entire results of the paper [4] are obtained as a special case. We have thus generalized our previous results.
References


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