A NOTE ON SOME VARIANTS OF JENSEN’S INEQUALITY

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Abstract

In this paper, we present a refined Steffensen’s inequality for convex functions and further prove some variants of Jensen’s inequality using the new Steffensen’s inequality.

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1. Introduction

The inequality

\[ \int_{b-\lambda}^{b} g(x) \, dx \leq \int_{a}^{b} g(x) f(x) \, dx \leq \int_{a}^{a+\lambda} g(x) \, dx \]

was discovered in 1918 by Steffensen [10], where \( \lambda = \int_{a}^{b} f(x) \, dx \), \( f \) and \( g \) are integrable functions defined on \((a, b)\), \( g \) is decreasing and \( 0 \leq f(x) \leq 1 \) for each \( x \in (a, b) \). See also [6], [7], [8] and [9].

Let \( I \) be an interval in \( \mathbb{R} \). If \( \psi : I \to \mathbb{R} \) is convex, then for all \( x_1, x_2 \in I \) and all positive numbers \( a_1 \) and \( a_2 \) satisfying \( a_1 + a_2 = 1 \), we have

\[ \psi(a_1 x_1 + a_2 x_2) \leq a_1 \psi(x_1) + a_2 \psi(x_2). \]

Jensen [3] proved the inequality

\[ \psi \left( \sum_{i=1}^{n} a_i x_i \right) \leq \sum_{i=1}^{n} a_i \psi(x_i), \]

where \( \psi \) is convex on an interval containing the real variables \( x_1, x_2, ..., x_n \) and \( a_i \) (\( 1 \leq i \leq n \)) are positive weights such that \( \sum_{i=1}^{n} a_i = 1 \).

Mercer [5] proved the inequality

\[ \psi \left( x_1 + x_n - \sum_{j=1}^{n} \lambda_j x_j \right) \leq \psi(x_1) + \psi(x_n) - \sum_{j=1}^{n} \lambda_j \psi(x_j), \]

where \( \psi \) is convex on an interval containing the real variables \( x_1, x_2, ..., x_n \) with \( 0 < \lambda_j < 1 \) (\( j = 1, ..., n \)) such that \( \sum_{j=1}^{n} \lambda_j = 1 \).

The inequality (3) was first published by Mercer in 2003 as a variant of Jensen’s inequality. One year later, Witkowski simply recovered the inequality in [11]. Thereafter, the inequality went through various refinements and generalizations. See for instance Matković [4] and the references therein.
The aim of this short note is to first provide a further proof of the following refined Steffensen’s inequality (4) established in [2] and also recover the inequality (3) through the new inequality (4). Furthermore, another variant of the Jensen’s inequality will be provided.

2. Preliminary Results

The following auxiliary results are presented.

**Theorem 2.1** [2]. Let $0 \leq f(x) \leq 1$ and $\psi : [0, 1] \to \mathbb{R}$ be a convex and differentiable function with $\psi(0) = 0$. If $f : [0, 1] \to \mathbb{R}$ is continuous, then

$$
\psi\left(\int_0^1 f(x) \, dx\right) \leq \int_0^1 f(x) \psi'(x) \, dx
$$

(4)

for all $x \in [0, 1]$.

**Proof.** Let $\psi'(x)$ be the derivative of $\psi(x)$. Then $\psi'(x)$ is an increasing function on the interval $[0, 1]$ since $\psi(x)$ is an increasing function on $[0, 1]$. This implies that $-\psi'(x)$ is decreasing. Then by making the substitution $g(x) = -\psi'(x)$, $a = 0$ and $b = 1$ into (1) yields

$$
\int_0^\lambda \psi'(x) \, dx \leq \int_0^1 f(x) \psi'(x) \, dx \leq \int_1^{1-\lambda} \psi'(x) \, dx,
$$

which simplifies to

$$
\psi(\lambda) - \psi(0) \leq \int_0^1 f(x) \psi'(x) \, dx \leq \psi(1) - \psi(1-\lambda). \tag{5}
$$

Since $\lambda = \int_0^1 f(x) \, dx$ and $\psi(0) = 0$, the first part of inequality (5) yields the required result

$$
\psi\left(\int_0^1 f(x) \, dx\right) \leq \int_0^1 f(x) \psi'(x) \, dx.
$$

**Proposition 2.2** ([1], p. 62.). Let $f_n$ be a sequence of functions. If $f_n \to f$ in $L^1$, there is a subsequence $f_{n_j}$ such that $f_{n_j} \to f$ almost everywhere (a.e.).
3. Main Results

This section begins as follows:

**Theorem 3.1.** Let \( f \in L^1([0, 1]) \) with \( 0 \leq f(x) \leq 1 \) for all \( x \in [0, 1] \). If \( \psi : [0, 1] \to \mathbb{R} \) is a convex and differentiable function with \( \psi(0) = 0 \), then

\[
\psi\left( \int_0^1 f(x) \, dx \right) \leq \int_0^1 f(x) \psi'(x) \, dx. \tag{6}
\]

**Proof.** Let \( f_n \to f \) in \( L^1 \). Then by Proposition 2.2, there is subsequence \( f_{n_j} \to f \) a.e.. Let there exists \( h \in L^1 \) such that \( |f_{n_j}(x)| \leq h(x) \). Then by the dominated convergence theorem, \( f \in L^1 \) and

\[
\int_0^1 f(x) \, dx = \lim_{n \to \infty} \int_0^1 f_{n_j}(x) \, dx.
\]

By the boundedness of \( \psi'(x) \) on \([0, 1]\), we have

\[
f_{n_j}(x) \psi'(x) \leq h(x) \psi'(x) \in L^1
\]

which implies that

\[
\int_0^1 f(x) \psi'(x) \, dx = \lim_{n \to \infty} \int_0^1 f_{n_j}(x) \psi'(x) \, dx. \tag{7}
\]

Since \( f_n \) is continuous, then by (4), we have

\[
\psi\left( \int_0^1 \lim_{n \to \infty} f_{n_j}(x) \, dx \right) \leq \lim_{n \to \infty} \int_0^1 f_{n_j}(x) \psi'(x) \, dx,
\]

which yields the required result

\[
\psi\left( \int_0^1 f(x) \, dx \right) \leq \int_0^1 f(x) \psi'(x) \, dx.
\]

**Example 3.2.** Let \( \psi(x) = e^x \). Then \( \psi'(x) = e^x \). Thus

\[
\exp\left( \int_0^1 f(x) \, dx \right) \leq \int_0^1 f(x) \exp(x) \, dx.
\]
Example 3.3. Let

\[
f(x) = \begin{cases} 
  b_1 & \text{if } 0 \leq x < x_1 \\
  b_2 & \text{if } x_1 \leq x < x_2 \\
  \vdots \\
  b_n & \text{if } x_{n-1} \leq x \leq 1,
\end{cases}
\]

where \(0 < b_1 < b_2 < \cdots < b_n < 1\), \(x_0 = 0\) and \(x_n = 1\). Since

\[
\int_{x_{i-1}}^{x_i} \psi'(x) \, dx = \psi(x_i) - \psi(x_{i-1}),
\]

then by (6), we obtain

\[
\psi \left( \sum_{i=1}^{n} b_i (x_i - x_{i-1}) \right) \leq \sum_{i=1}^{n} b_i [\psi(x_i) - \psi(x_{i-1})]. \tag{8}
\]

Theorem 3.4. Let the function \(\psi\) be convex and differentiable on an interval containing an \(n\)-tuple \(x = (x_1, \ldots, x_n)\) such that \(0 < x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\) and \(a = (a_1, \ldots, a_n)\), a positive \(n\)-tuple with \(\sum_{j=1}^{n} a_j = 1\). If \(\psi(0) = 0\), then

\[
\psi \left( x_1 + x_n - \sum_{j=1}^{n} a_j x_j \right) \leq \psi(x_1) + \psi(x_n) - \sum_{j=1}^{n} a_j \psi(x_j).
\]

Proof. Consider \(0 < a_j < 1\) (\(1 \leq j \leq n\)) such that \(\sum_{j=1}^{n} a_j = 1\). Let \(b_n = 1 - a_n = \sum_{j=1}^{n-1} a_j\), such that \(b_2 = a_1\) and \(b_1 = 1\). Expansion of inequality (8) yields

\[
\psi \left( b_1 (x_1 - x_0) + b_2 (x_2 - x_1) + \cdots + b_n (x_n - x_{n-1}) \right) \leq b_1 [\psi(x_1) - \psi(x_0)] + b_2 [\psi(x_2) - \psi(x_1)] + \cdots + b_n [\psi(x_n) - \psi(x_{n-1})].
\]
Since \( x_0 = 0 \) and \( \psi(0) = 0 \),

\[
\psi\{ (b_1 - b_2) x_1 + (b_2 - b_3) x_2 + \cdots + (b_{n-1} - b_n) x_{n-1} + b_n x_n \} \\
\leq (b_1 - b_2) \psi(x_1) + (b_2 - b_3) \psi(x_2) + \cdots + (b_{n-1} - b_n) \psi(x_{n-1}) + b_n \psi(x_n).
\]

Substitute \( b_1 - b_2 = 1 - a_1, \ b_n = 1 - a_n \) and \( -a_j = (b_j - b_{j+1}) \) for \( j = 2, \ldots, n - 1 \), we get

\[
\psi \left( (1 - a_1) x_1 + (1 - a_n) x_n - \sum_{j=2}^{n-1} a_j x_j \right) \\
\leq (1 - a_1) \psi(x_1) + (1 - a_n) \psi(x_n) - \sum_{j=2}^{n-1} a_j \psi(x_j).
\]

Hence

\[
\psi \left( x_1 + x_n - \sum_{j=1}^{n} a_j x_j \right) \leq \psi(x_1) + \psi(x_n) - \sum_{j=1}^{n} a_j \psi(x_j).
\]

**Remark 3.5.** It is observed that

\[
a_{n-1} = b_n - b_{n-1} \\
a_{n-2} = b_{n-1} - b_{n-2} \\
\vdots \\
a_2 = b_3 - b_2 \\
a_1 - 1 = b_2 - b_1.
\]

Thus

\[
a_{n-1} + a_{n-2} + \cdots + a_1 - 1 = b_n - b_1 \\
= 1 - a_n - 1.
\]

Therefore,

\[
\sum_{j=1}^{n} a_j = 1.
\]
Next is a Lemma before another variant of the Jensen’s inequality.

**Lemma 3.6.** Let $\psi(x)$ be a convex and differentiable function on an interval $I$ of real numbers. If $\psi(0) = 0$, then

$$
\psi(x) \leq \psi'(x)x,
$$

for all $x \in I$.

**Proof.** Let $x, y \in I$. Since $\psi$ is differentiable. Then by the definition of convexity, we have

$$
\psi(y) - \psi(x) \geq \psi'(x)(y - x).
$$

Putting $y = 0$ and $\psi(0) = 0$, we obtain

$$
\psi(x) \leq \psi'(x)x,
$$

for all $x \in I$.

**Theorem 3.7.** Let $\psi$ be a convex and differentiable function on an interval containing an $n$-tuple $x = (x_1, x_2, ..., x_n)$ such that $0 < x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$. If $\psi(0) = 0$ and $\sum_{j=1}^{n} a_j = 1$ for $0 < a_j < 1$, then

$$
\psi\left(\sum_{j=1}^{n} a_j x_j\right) \leq \sum_{j=1}^{n} a_j x_j \psi'(x_j).
$$

**Proof.** Recall from (2) that

$$
\psi\left(\sum_{j=1}^{n} a_j x_j\right) \leq \sum_{j=1}^{n} a_j \psi(x_j).
$$

Then substitution of (9) into (11) yields the required result

$$
\psi\left(\sum_{j=1}^{n} a_j x_j\right) \leq \sum_{j=1}^{n} a_j x_j \psi'(x_j).
$$

**Remark 3.8.** The inequality (10) is reversed if $\psi$ is concave.
Illustrative Example. Consider the convex function $\psi(x) = x^p$, $p > 1$, $x > 0$, then (10) becomes

$$\frac{1}{p} \left( \sum_{j=1}^{n} a_j x_j \right)^p \leq \sum_{j=1}^{n} a_j x_j^p. \quad (12)$$

For $p = 2$, we have

$$\frac{1}{2} \left( \sum_{j=1}^{n} a_j x_j \right)^2 \leq \sum_{j=1}^{n} a_j x_j^2.$$

4. Conclusion

This paper proved a refined Steffensen’s inequality for convex functions. The Jensen-Mercer’s inequality was also proved in this paper through exemplification of the refined Steffensen’s inequality. A further variant of the Jensen’s inequality was provided.

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