SOME MATHEMATICAL INVESTIGATIONS INTO BOUNDARY LAYER FLOW
OVER FLAT SURFACES

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SOME MATHEMATICAL INVESTIGATIONS INTO BOUNDARY LAYER FLOW OVER FLAT SURFACES

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SEPTEMBER, 2011
DECLARATION

Student

I hereby declare that this thesis is the result of my own original work and that no part of it has been presented for another degree in this university or elsewhere.

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Supervisors’ Declaration

I hereby declare that the preparation and presentation of the thesis was supervised in accordance with the guidelines on supervision of thesis laid down by the University for Development Studies.

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ABSTRACT

The thesis presents a theoretical investigation into the combined effects of transverse magnetic field and chemical reaction on boundary layer flow control. The conservation laws of nature have been employed to derive the basic fluid dynamic equations of continuity, momentum, energy and concentration. The equations modeling the flow problems were expressed in partial derivatives and transformed to ordinary differential equations using similarity variables. A numerical method based on the fourth order Runge-Kutta integration scheme and the modified version of the Newton-Raphson shooting techniques were used to solve the boundary layer problems. The first problem analyzed the effects of chemical reaction on the magnetohydrodynamic (MHD) boundary layer flow over a stretching sheet. The second analyzed the MHD boundary layer flow past a vertical porous plate with Arrhenius chemical reaction and Ohmic heating, whilst the third examined the chemically reacting MHD boundary layer flow across a low-heat-resistant sheet moving vertically downwards. The higher order ordinary differential equations were reduced to first order equations and a program written in maple computer software package. Results for skin friction coefficient, heat and mass transfer rates at the surface were obtained. Graphical results were obtained for the velocity, temperature and concentration profiles. The results revealed that the effect of transverse magnetic field in the boundary layer increases the skin friction coefficient which reduces the velocity profiles. Furthermore, the rate of heat and mass transfers in the boundary layer was significantly influenced by the transverse magnetic field and chemical reactions.
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DEDICATION

To My Family
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NOMENCLATURE

x, y  Cartesian coordinate axes
u, v  Velocity components in the x- and y- directions
n    Arrhenius reaction exponent
Nu   Nusselt number
Re   Reynolds number
\vec{q}   Velocity vector
\vec{g}  Acceleration due to gravity
D    Mass diffusivity
T_\infty  Free stream temperature
T    Fluid temperature
Gr   Thermal Grashof number
Gc   Solutal Grashof number
P    Fluid pressure
H_0  Magnetic field parameter
C_0  Wall surface concentration
T_0  Wall surface temperature
Sc  Schmidt number
Sh  Sherwood number
Ec  Eckert number
Pr  Prandtl number
N  Uniform heat sink/source parameter
f  Dimensionless stream function
c_p  Specific heat at constant pressure
C_\infty  Free stream species concentration
e  Internal energy per unit mass
t  Time
GREEK SYMBOLS

θ  Dimensionless temperature
φ  Dimensionless concentration
ψ  Stream function
βₜ  Thermal expansion coefficient
λ  Thermal conductivity
βₑ  Concentration expansion coefficient
ρ  Fluid density
ν  Fluid kinematic viscosity
σ  Electrical conductivity
γ  Chemical reaction rate parameter
μ  Dynamic viscosity
β  Reaction rate parameter
ε  Activation energy parameter
τ  Skin friction parameter
α  Thermal diffusivity
η  Dimensionless span-wise coordinate.
CHAPTER

INTRODUCTION AND BACKGROUND STUDIES

1.0 Introduction

Fluid refers to any substance that has the tendency to flow under the influence of a force. It is observed to lack the ability to offer sustained resistance to deforming forces and deforms continuously under the action of a force or its own weight. Fluid flow is a common phenomenon in many industrial and engineering processes. It therefore poses greater challenges to engineers at all stages of design. Research works in fluid dynamics have been on the increase due to its important applications in industry. Fluid flows can be found in many areas such as flow past wings and structures, flow in pipes and channels, circulation of blood in vascular tumors, as well as flow of water in rivers and streams among others. In the design of ships, aircrafts, submarines, automobile, nuclear plants and energy systems, the effect of fluid flow is very critical. Industrial processes involving coating, spraying and cooling all involve the flow of fluid from one point to another. The flow of conducting fluids in the presence of transverse magnetic field and chemical reactions are emerging areas of current research.

When fluid flows past objects, it generates aerodynamic forces near the surfaces of the objects. These forces are created due to the interaction between the object
and the fluid molecules. Fluid molecules touching the surfaces of objects stick to them. Molecules located away from objects are slowed down in their collision with those sticking, such that, at far away locations called the free stream, fluid molecules are least disturbed. As a result of these variations in disturbance between molecules, a thin layer is created near the surface of the object in which the velocities of molecules vary from zero at the surface to the free stream value. Engineers called this the ‘boundary layer’ because it occurs near the boundary of the object.

This study investigates the characteristics of the boundary layer on flat plates. In the rest of this chapter, the concepts of boundary layer have been discussed and the objectives of the study outlined. Mathematical models have been proposed for investigation and some background studies conducted. The expected contributions of the study conclude the chapter.

1.1 The Concepts of Boundary Layer

Boundary layer refers to the region formed in the vicinity of a surface bounding an object. The phenomenon arises when fluid flows on solid surfaces or when solid surfaces move in fluid such as in the interior of pipes, in irrigation channels, around aeroplane wings, around ships and submarines and in river beds or inside blood veins. When fluid flows round a solid body, it exerts hydrostatic pressure which acts normal to the surface generating tangential stresses along the surface, John (2005), fig 1.1.
Many research works in fluid mechanics over the years are based on the boundary layer concepts introduced by Prandtl (1904). Prandtl’s boundary layer theory stipulates that, due to the ‘no-slip’ condition near the surfaces of bodies, the velocity of fluid molecules at the surface of a stationary body will be zero, but the velocity given by inviscid flow theory is reached within a thin layer close to the surface called the ‘boundary layer’. These concepts led to simplifications of equations describing many flow problems. Within the boundary layer region, viscosity is considered dominant and majority of drag experienced by bodies are created. Outside the boundary layer, viscosity is neglected and the flow is considered invicid. Prandtl’s concepts allowed closed-form solutions for flow in both regions and simplifies the solution of the full Navier-Stokes equations. The majority of heat transfer cases occur within the boundary layer which allowed
simplifications of the equations in the flow field outside the boundary layer. Fox and McDonald (1973) observed that the frictional resistance offered by objects occurs inside the boundary layer.

The boundary layer thickness is important when determining the effects of fluid flow on objects. It is defined as the distance measured from the solid surface at which the flow velocity is 99% of the free-stream value. Alternatively, in terms of displacement thickness, it represents a deficit in mass flow compared to invicid case with slip at the wall. The details of flow within the boundary layer are important in design and provide significant information for engineers in the design of high-performance sailplanes and commercial transport aircrafts. In studying boundary layer, two main effects must be noted.

i. The boundary layer adds to the effective thickness of the body leading to increasing pressure drag.

ii. Shear forces at the surface of objects create skin friction drag which tends to oppose the motion of objects.

1.2 Aerodynamic Forces

When objects move in fluid they experience four fundamental aerodynamic forces as illustrated for an aircraft in figure 1.2. To fly, an aircraft requires the thrust force which is provided by its engine. During the flight, drag forces are generated opposite the thrust force which tends to slow the motion of craft. Air movement around the wings of aircrafts generate lift forces which act to counter their weights. For example, the remarkable achievement made in the aerospace industry
is attributed to the huge successes made in reducing the drag effects due to boundary layer. The historical largest aircraft known as the concord and the unveiling of the dreamer aircraft have all been possible due to new developments in the area.

Source: NASA, Glenn Research Center

Figure 1.2 Component Forces Acting on Aircraft in Motion

The Thrust:

Thrust is the force which propels bodies to move in their desired directions. Thrust is provided by propellers or engines and must always be greater than the drag offered by the fluid to maintain its desired direction.
The drag:

When objects move in fluid media, molecules of the fluid tends to oppose the motion of the object in a phenomenon called drag. In the field of engineering, frictional drag is the most predominant. The magnitude and effect of frictional drag depend on the shape and design of the object, the velocity of flow as well as the viscosity or stickiness of the fluid. Reducing the frictional effects is important to enhance the performance of moving objects. The drag equation attributed to Lord Rayleigh is often used to compute the drag force experienced by a moving object as:

\[ F_d = \frac{1}{2} \rho v^2 C_d A, \]  

where \( F_d \) represents the drag force, \( \rho \) is the fluid density, \( v \) is the velocity, \( A \) is the reference area, and \( C_d \) is the drag coefficient.

The power required to overcome the drag force is computed from;

\[ P_d = F_d \times v, \]  

Hence,

\[ P_d = \frac{1}{2} \rho v^3 C_d, \]  

It is observed that the power required to overcome the drag force acting on a body is proportional to the cube of the velocity.
The Weight:

Weight describes the gravitational effect on bodies which act through their centre of gravities. It represents the attraction between objects and the earth. It is a vector quantity always directed downwards. Weight is defined as the product of a body’s mass and the acceleration due to gravity with units of force in Newton (N) or kilogram meters per square second (kgm/s²). Thus;

\[ \text{Weight} (W) = \text{Mass} (m) \times \text{Acceleration due to gravity} (g) \]

\[ W = mg. \]

The Lift:

Lift is generated when objects move in air. It is a component of aerodynamic forces which acts to oppose the weight of objects. The magnitude of the lift force largely depends on the shape of the object, the size and the velocity with which the object moves. Lift force acts through the center of pressure which is defined similar to the center of gravity, but using the pressure distribution around the body.
1.3 The Boundary Layer Problem

Boundary layer is a phenomenon which occurs frequently in nature particularly with viscous fluids. When fluid moves in pipes, channels, and surfaces, boundary layer problems will occur. When objects move in fluid such as ships, aircrafts, submarines, and space shuttles, boundary layer will occur. It is also common in the operations of nuclear plants and energy systems involving heat exchangers. In nuclear plants, heat is transferred from one loop to the other through heat exchangers. As fluid flows in the loops, boundary layer problems arise. To extract energy from geothermal sources, water is pumped through injection wells into the core of rocks. The water returns as superheated steam through production wells suitable for direct heating or turning turbines to generate electric power, Carlo (2009).

1.4 Flow Regimes

The flow of real fluids is considered much more complex than that of ideal fluids, owing to the existence of viscosity. Viscosity introduces resistance to fluid motion and causes shear and frictional effects between the fluid particles and the boundary walls. For flow to take place, work must be done against resistance forces, and in the process energy is converted to heat. The inclusion of viscosity allows the possibility of two physically different flow regimes. The effects of viscosity on the velocity profile renders invalid the assumptions of uniform velocity distribution. Although the Euler equations may be altered to include shear
stresses of a real fluid, the result is a set of partial differential equations to which no general solution is known.

Viscous effects are exhibited when real fluids flow. As fluid molecules stick to surfaces, molecules above glide over those sticking. Newton experiments with different fluids led him to establish a proportional relationship between the shear stress and the velocity gradient as:

\[ \tau \propto \frac{du}{dy} \]  

(1.4)

For fluids obeying Newton’s law of viscosity, (1.4) becomes:

\[ \tau = \mu \frac{du}{dy} \]  

(1.5)

where the constant of proportionality (\( \mu \)) is called the coefficient of viscosity or the dynamic viscosity. For non-Newtonian fluids, the dynamic viscosity (\( \mu \)) varies with the stress and the velocity gradient. This makes analysis of such fluids very complex. In this study, Newtonian fluids are assumed in all cases. Equation (1.5) shows that the shear stress (\( \tau \)) is directly proportional to the velocity gradient - the rate of change of velocity across the fluid path.

In real fluids, the presence of viscosity leads to the formation of two different flow conditions as laminar and turbulent flows. Osbourne Reynolds first demonstrated the characteristics of these flow by using the apparatus similar to that shown in Fig 1.3. In the experiment, water flows from a tank through a bell-
mouthed glass pipe controlled by a valve. A thin tube leading from a reservoir of dye had its opening within the entrance of the glass pipe. Reynolds discovered that, for low velocities of flow in the glass pipe, a thin filament of dye issuing from the tube did not diffuse but was maintained intact throughout the pipe, forming a thin straight line parallel to the axis of the pipe. As the valve was opened, the velocity of flow increased and the dye filament wavered and broke, eventually diffusing through the flowing water in the glass pipe. Reynolds established a constant value relating to:

$$\frac{\rho ud}{\mu}$$

(1.6)

where $\rho$ represent the density of the fluid, $u$ the mean velocity of flow, $d$ the diameter of pipe, and $\mu$, the dynamic viscosity of the fluid.

(Source: http://images.aechumphon.multiply.multiplycontent.com)

Figure 1.3 Osbourne Reynolds Experimental Setup
The magnitude of the expression (1.6) is attributed to Osbourne Reynolds and is use to classify fluid flow as laminar, turbulent or as transitional flow.

\[ Re = \frac{\rho ud}{\mu}. \]  

(1.7)

In laminar flow, Reynolds number is less than 2000, \((Re < 2000)\). In turbulent flow, the Reynolds number is greater than 4000 \((Re > 4000)\). Between the two extremes lies the transitional zone, i.e. \(2000 < Re < 4000\).

**1.4.1 Laminar Flow**

Laminar flow occurs at low velocities and is common with viscous fluids. During laminar flow, layers of fluid move orderly without mixing with one another figure 1.4. In the Reynolds experiment, the injected dye produces a thin filament which is parallel to the surface of the wall.

**1.4.2 Transition Zone**

The instability of laminar flow as the Reynolds number increases cause disruptions of the laminar pattern of the fluid motion. Transition sets in when the thin filament of dye begins to waver due to the interference of layers of fluid with one another, see figure 1.4. This occurs at low to medium velocities.
1.4.3 Turbulent Flow

Turbulence is characterized by the irregular, chaotic motion of fluid particles. The magnitude of the Reynolds number during turbulence is greater than 4,000 for pipe flow which occurs at high velocities, figure 1.4.

(Source: http://www-mdp.eng.cam.ac.uk/web/library/enginfo/aerothermal_dvd_only/aero/fprops/pipeflow/node8.html)

Figure 1.4 Flow Regimes – Laminar, Transition and Turbulent
Prandtl (1914) observed that the flow in the boundary layer region became turbulent when triggered by surface roughness resulting in high skin friction drag. For a streamline body like airfoil whose drag is mainly due to skin friction, delaying transition to turbulence reduced the drag force. Beyond a certain Reynolds number, laminar boundary layers become sensitive to small disturbances.

Tollemien (1929) and Schlichting (1933) theoretically calculated the critical Reynolds numbers (see Schlichting & Gersten (2000)) for flat plate boundary layers. The effect of pressure gradient on boundary layer transition was also reported by Schlichting and Ulrich (1940) when they observed that transition to turbulence was not possible with favorable pressure gradient. However, transition was hastened by adverse pressure gradient and thus delayed on smooth surfaces. These findings led to the development of laminar flow airfoils by Jacob (1939) and others at NASA laboratories.

1.5 Solving the Boundary Layer Problem

The general fluid dynamic equations have been known for many years. However, solutions to these equations did not properly describe observed flow effects until Prandtl (1904) discovered his concepts. Prandtl concepts stipulate that the relative magnitude of inertia and viscous forces changed from layers very close to the surface to regions far away. Exact solutions to boundary layer problems are difficult especially for turbulent conditions. Attempts to solve turbulent flow models using laminar solvers have often resulted in time-unsteady solutions which
failed to converge appropriately. Approximate solutions are obtained using time-averaged equations such as the Reynolds-Average Navier-Stokes (RANS) equations supplemented with turbulent models in practical Computational Fluid Dynamics (CFD). The development of boundary layer on a surface is observed more generally by a simplified flow diagram in figure 1.5.

![Figure 1.5 Simplified diagram of a boundary layer problem](http://www.scribd.com/doc/15320337/Boundary-Layers)

Towards the end of the nineteenth century, researchers in fluid mechanics were divided into two broad categories, viz., those engaged in the study of hydrodynamics (the study of inviscid flows) and those studying hydraulics. Hydraulics, though mathematically elegant, was unable to predict the drag experienced by bodies in fluid. This is known as D’Alembert’s paradox, Rao et al (1917). Hydraulics on the other hand, offered solutions to practical problems but was based purely on empirical data. Though Prandtl originally developed his concepts for laminar flow models, they soon extended to turbulent flows and gained acceptance after some years. The boundary layer concepts are now applied
in almost all branches of engineering. Prandtl's theory led to the development of mathematical tools like method of matched asymptotic expansions.

The character of the boundary layer changed as it developed along the airfoil. Generally, starting out as laminar flow, the boundary layer thickens, undergoes transition to turbulence and continued to develop along the surface of the body, possibly separating from the surface under certain conditions. The development of the boundary layer over an airfoil is illustrated in figure 1.6.

Figure 1.6 Development of Boundary Layer over Airfoil
1.6 The Fluid Properties Relevant to the Study

Although the properties of fluid generally arise from its molecular structure, engineering problems are often concerned with the bulk behavior of fluid. In practical fluid dynamics, fluid is considered as a continuum. Whether in gaseous or liquid state, fluid cannot resist externally applied forces without deformation. Fluids are distinguished from one another by its properties. Some fundamental properties associated with fluid flow relevant to this study are discussed below.

1.6.1 Nonlinearity

When fluid flows, it exhibits non-linear behaviors which are modeled mathematically using non-linear differential equations. These equations are best modeled as partial derivatives to adequately describe the behavior of real systems. Exact solutions to partial differential equations are difficult particularly for fluid flow problems. This is mostly due to the convective accelerations associated with changes in velocity over position. Thus, for any convective flow, whether turbulent or not, nonlinearities will arise.

1.6.2 Density of Fluid

Density is defined as the quantity of matter contained in a unit volume of a substance and is measured in $kg/m^3$. That is;
\[ \rho = \frac{m}{v}. \] (1.7)

In liquids, density is nearly constant.

For compressible fluids, density varies with pressure and

\[ \frac{\partial \rho}{\partial t} \neq 0. \] (1.8)

For incompressible fluids, density is constant and

\[ \frac{\partial \rho}{\partial t} = 0. \] (1.9)

Density can also be expressed as specific weight or relative density. Specific weight is defined as the weight per unit volume of a substance with units of N/m³ while relative density is the ratio of the mass density of a substance to some standard mass density. For solids and liquids, the standard mass density is taken to be the maximum density of water (1000 kg/m³), which occurs at a temperature of 4°C and a pressure of 101.4 KPa, Douglas et al (1996).

### 1.6.3 Dynamic Viscosity

Fluid cannot resist the induced shearing effects of forces. For fluids obeying Newton’s law of viscosity, the x - coordinate axis is taken to act along the direction of flow with velocity \( u \), and the shearing stress acting between two layers of fluid located a distance \( y \) apart is given by Douglas et al (1996) as:
\[ \tau_s = \mu \frac{du}{dy}, \]  
(1.10)

where \( \mu \) is the co-efficient of dynamic viscosity defined as the shear force per unit area required to drag one layer of fluid with unit velocity past another layer a unit distance away. The SI unit for dynamic viscosity is the Newton-second per meter squared (Nsm\(^{-2}\)).

### 1.6.4 Kinematic Viscosity

Kinematic viscosity is defined as the ratio of the dynamic viscosity to the mass density expressed mathematically as;

\[ \nu = \frac{\mu}{\rho}, \]  
(1.20)

where \( \mu \) is the dynamic viscosity and \( \rho \) is the mass density. Kinematic viscosity has units of square meters per second (m\(^2\)s\(^{-1}\)).

### 1.6.5 Internal Energy

The internal energy (\( \tilde{u} \)) refers to the energy stored in a system due to its molecular interaction or bonding between its molecules. A given mass of viscous fluid may be viewed as a thermodynamic system that stores various forms of energies. The internal energy in a fluid element consists of the energy due to
translation, rotation and vibration as well as energy of molecular dissociation and electronic excitation. Internal energy has units of J mol\(^{-1}\).

### 1.6.6 Thermal Conductivity

Thermal conductivity relates the vector rate of heat transfer per unit area to the vector gradient of temperature, Kay et al (1993). For solids and liquids, the Fourier’s law of heat conduction applies:

\[
q = -k \nabla T, \tag{1.11}
\]

where the negative sign (-) indicates that heat flux is positive in the direction of decreasing temperature.

### 1.6.7 Vapour Pressure

The intermolecular attraction between molecules in a liquid are weaker compared to those in solids. Molecules in liquids are in constant agitation and easily escape from the surface. For exposed liquids, molecules escape freely to the atmosphere. For confined surfaces, molecules are not able to escape occupies the space above the free surface. The pressure exerted by the freed molecules on the liquid is called the vapour pressure.
1.7 Statement of the Problem

Though several boundary layer problems have been tackled in the literature, there are many areas in which research is continuing, especially with respect to heat and mass transport phenomena. The mathematical analyses proposed by Schlichting and Gerstein (2000) found great use in MHD generators, plasma studies, nuclear technology, continuous coating, and extrusion in manufacturing processes. The boundary layer along a film in condensation process and aerodynamic extrusion of plastic sheets are areas of active research. The combine effects of transverse magnetic field and chemical reaction on boundary layer flow control is an important research problem.

1.8 Aim of Study

This study aims at theoretically investigating some nonlinear problems arising from hydromagnetic boundary layer flow over flat surfaces.

1.8.1 Specific Objectives of Study

The study will achieve the following specific objectives:

- Develop nonlinear mathematical model for hydromagnetic boundary layer flow over a stretching sheet with chemical reaction and uniform heat source.
Develop nonlinear mathematical model for the natural convection boundary layer flow past a vertical plate with suction and chemical reaction in the presence of transverse magnetic field.

Develop a theoretical framework to predict the effects of heat and mass transfer towards a wall on transport phenomena in industrial and engineering systems.

Provide numerical results for nonlinear systems of differential equations modelling boundary layer flow over flat surfaces.

### 1.8.2 Significance of Study

Boundary layer has pronounced effects on objects immersed and moving in fluid. Drag experienced by ships, aircrafts, submarines and friction in pipes are common manifestations of boundary layer. Magnetohydrodynamic (MHD) boundary layer flow with heat and mass transfer in various media has been reported in the literature. The findings of these researches are used mostly in the chemical processing industries such as in fiber drawing, crystal pulling from melts and polymer processing. Magnetoconvection is applied in magnetic control of molten iron in the steel industry, liquid metal cooling in nuclear reactors and magnetic suppression of molten semi-conducting materials. Other areas of applications include sustained plasma confinement, thermonuclear fusion, electromagnetic casting of metals, electromagnetic pumps, controlled fusion research, crystal growing, plasma jets and chemical synthesis. Understandably, boundary layer research has become very important in the study of fluid dynamics.
1.9 Proposed Mathematical Models

In this research, non-linear mathematical models have been proposed to investigate the laminar hydromagnetic boundary layer flow interaction with heat and mass transfer over flat surfaces. The positive x-coordinate axis is taken along the direction of the moving flat plate with the slot as the origin while the y-coordinate axis measured normal to the surface. Steady, laminar, incompressible, and electrically conducting fluid past semi-infinite flat surfaces are investigated.

The following assumptions were made in the study:

- The magnetic field strength $H_0$ is applied parallel to the y-axis and perpendicular to the plate.
- The induced magnetic field produced by the motion of the electrically conducting fluid is negligible.
- The fluid physical properties such as the viscosity and thermal conductivity are constants, while Boussinesq approximation invoked for the density variation in the body force term of the momentum equation.
- A chemical species diffused into the ambient fluid initiates Arrhenius irreversible chemical reaction.

The following specific flow problems have been investigated:

Model 1: Hydromagnetic boundary layer flow of a chemically reacting fluid over a stretching sheet with uniform heat source, mathematically modelled as:
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \]  
\[ (1.12) \]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma H_0^2}{\rho} u, \]  
\[ (1.13) \]

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \nu \frac{\partial u}{\partial y} + \frac{\sigma H_0^2}{\rho c_p} u^2 + Q(T - T_\infty), \]  
\[ (1.14) \]

\[ u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} - \gamma (C - C_\infty), \]  
\[ (1.15) \]

The boundary conditions for this model are;

\[ u = U(x), v = 0, T = T_w(x), C = C_w(x) \text{ on } y = 0, \]

\[ u = 0, v = 0, T = T_\infty, C = C_\infty \text{ as } y \to \infty. \]  
\[ (1.16) \]

Equations (1.12) to (1.15) represent the continuity, momentum, energy and concentration equations respectively while equation (1.16) represents the boundary conditions of the problem.

**Model 2:** Chemically reacting MHD boundary layer flow with suction and Ohmic heating past a vertical porous plate. This is mathematically modelled as;

\[ \frac{\partial v}{\partial y} = 0, \]  
\[ (1.17) \]

\[ \nu \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma H_0^2}{\rho} u + g \beta_T (T - T_\infty) + g \beta_C (C - C_\infty), \]  
\[ (1.18) \]
with boundary conditions obtained as:

\[ u = 0, \ n = -V, \ T = T_w(x), \ C = C_w(x) \text{ at } y = 0, \]
\[ u = 0, \ n = 0, \ T = T_\infty, \ C = C_\infty \text{ as } y \rightarrow \infty. \]  
(1.21)

In this model, equations (1.17) to (1.20) represent the continuity, momentum, energy and concentration equations respectively while equation (1.21) defines the boundary conditions.

**Model 3:** Chemically reacting MHD boundary layer flow past a vertically moving low-heat-resistant sheet.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.22}
\]
\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma H_0^2}{\rho} u + g \beta_T (T - T_\infty) + g \beta_C (C - C_\infty), \tag{1.23}
\]
\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\sigma H_0^2}{\rho c_p} u^2, \tag{1.24}
\]
\[
u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2}, \tag{1.25}
\]

subject to the boundary conditions,
\[ u = U, v = 0, T = T_w = A(x/L)^{\frac{1}{2}} + T_{\infty}, C = C_w = B(x/L)^{\frac{3}{2}} + C_{\infty} \text{ at } y = 0, \]

\[ u \to 0, T \to T_{\infty}, C \to C_{\infty} \text{ as } y \to \infty. \] (1.26)

Equations (1.22) to (1.25) represent the continuity, momentum, energy and concentration equations respectively while equation (1.26) defines the boundary conditions.

1.10 Computational Approach

Mathematically speaking, the differential equations modeling MHD boundary layer flow interaction with heat and mass transfer over flat surfaces constitute a nonlinear problem in the unbounded computational domain. The theory of nonlinear differential equations is quite elaborate and their solutions remain extremely important in industry. Approximate solutions for nonlinear systems of differential equations modeling boundary layer flow are constructed using the fourth order Runge-Kutta integration scheme coupled with numerical shooting techniques, Roche (1998).

1.10.1 The Fourth-Order Runge-Kutta Method

The fourth-order Runge-Kutta method is given in equation (1.27). It requires four evaluations at the right hand side per step, \( h \), equation (1.28). The method is
derived from taking weighted averages of the \( f(x, y) \) at certain points in \([x_n, x_{n+1}]\) with an error proportional to \( h^5 \).

The Runge-Kutta method does not require computations of partial derivatives. It is a good combination of accuracy and simplicity as a numerical equation solver.

\[
y_{n+1} = y_n + \frac{h}{6} [A_n + 2B_n + 2C_n + D_n],
\]

where
\[
A_n = f(x_n, y_n), \\
B_n = f(x_n + \frac{1}{2} h, y_n + \frac{1}{2} hA_n), \\
C_n = f(x_n + \frac{1}{2} h, y_n + \frac{1}{2} hB_n), \\
D_n = f(x_n + h, y_n + hC_n).
\]

1.10.2 The Newton-Raphson Algorithm

In the Newton-Raphson method, the solution to the function \( f(x) = 0 \) is found by making an initial guess for \( x_0 \) and compute for \( n = 0,1,2,..., \)

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

Now, compute \( x_1 \) from \( x_0, x_2 \) from \( x_1, \) and so on, until some \( x_{n+1} \) and \( x_n \) agree to the number of decimal places being used. The value \( x_N \) is then the approximate solution. It is often useful to replace \( f'(x_n) \) by \( \frac{f(x_n + h) - f(x_n)}{h} \), with \( h \) small.

In practice \( h = 0.001 \) usually gives a good accuracy, Noble (1964).
1.11 Analysis

Analytical and numerical methods are employed to investigate the above proposed mathematical models. Results are presented graphically to illustrate the velocity, temperature and concentration profiles whilst quantitative data is obtained for the skin-friction coefficient, the rate of heat and mass transfers at the plate surface using a computer software package called MAPLE (Heck, 2003), which is already available in our discipline.

1.12 Background Studies

Research into boundary layer problems became popular after Prandtl’s (1904) discoveries and has attracted the attention of many scientists over the last century. Flow in pipes, channels, rivers as well as heat and mass transfer in transport phenomena are areas where research is continuing. The character of the boundary layer developed on stationary or moving surfaces in the presence of transverse magnetic field is a basic problem in fluid dynamics. It is encountered in various technical systems employing liquid metals and finds application in rolling, wire drawing, metal and polymer extrusion, metal spinning, flow meter design, piping and casting systems. Some background studies on boundary layer research are presented below.
1.12.1 Boundary Layer Research on Horizontal Surface

The boundary layer formed on horizontal surfaces is a common phenomenon in most industrial processes. It occurs when fluid flows over stationary surfaces or when flat plates move in quiescent fluid. The ‘no-slip’ condition requires that the velocities of fluid particles that stick to stationary surfaces are zeros. Thus, the molecules of fluid in contact with a surface will have the same velocity as the surface itself.

After Prandtl’s (1904) paper on boundary layer concepts, only six papers appeared in the period 1904 to 1914. Blasius (1908) first successfully computed the flow over flat plates as an explicit solution of the Prandtl equations (but not of Navier – Stokes). Prandtl (1914) further applied the boundary layer concepts to heat transfer problems. Heimenz (1911) studied the development of boundary layer over circular cylinders with experimental pressure distributions and Prandtl (1925) again explained the reduction in drag of a sphere after a certain Reynolds number due to the transition of flow in the boundary layer from laminar to turbulent. It should be noted that transition to turbulence in channels and pipes was known earlier through the experiment of Reynolds in 1883, Schlichting and Gerstein (2000). Bibliographic details of some of the boundary layer investigations can be found in Tani (1977).

In the pioneering works of Sakiadis (1961), the boundary layer flows over continuous solid surfaces moving with constant speed were investigated. Crane (1970) extended the work of Sakiadis to include extensible surfaces and presented
analytical solutions for the boundary layer flow of incompressible liquids. Tsou and Sparrow (1967) presented both analytical and experimental results for the flow and heat transfer aspects arising from stretching sheet. Gupta and Gupta (1977) then investigated the heat and mass transfer in MHD fluid flowing over isothermal stretching sheets with suction/blowing effects. Chen and Char (1988) extended the works of Gupta and Gupta to that of non-isothermal stretching sheets. Grubka and Bobba (1985) then investigated the heat transfer characteristics of a continuous stretching surface with variable temperature whilst Vajravelu and Nayfeh (1993) later analyzed the flow and heat transfer characteristics introducing the temperature dependent heat source and sink. Kendoush (1996, 2009) conducted theoretical analysis of convective heat and mass transfer with fluids flowing normal to and across flat plates.

Sayers and Law (1996) investigated the boundary layer growth on flat circular plates whilst mixed convection boundary layer flow of viscoelastic fluid over horizontal circular cylinders was conducted by Ilyana et al (2008). Similarly, the boundary layer flow of micropolar fluid on continuously moving or fixed permeable surfaces were reported by Ishak et al (2007) whilst Laplace and Arquis (1998) analyzed the boundary layer flow over slotted plates due to their importance in filtration and air conditioning systems. David and Wang (2000) analyzed the effects of Prandtl number for Marangoni convection over flat surfaces and concluded that for liquid pool heated at both ends, the surface temperature gradient were linear for very low Prandtl number fluids. Heat transfer in MHD viscoelastic boundary layer flow over stretching sheets with thermal
radiation and non-uniform heat source/sink have been reported by Mahantesh et al (2011) while Abass et al (2010) investigated the unsteady MHD flow of heat transfer on stretching sheets in rotating fluid.

Sajid et al (2007) investigated the unsteady flow and heat transfer of second grade fluid over stretching sheets while Wang (2007) analyzed the viscous flow due to stretching sheets with surface slip and suction. Hsiao et al (2008) studied the conjugate heat transfer of convection for visco-elastic fluid past horizontal flat plate fin whilst Tiegang (2008) analyzed the unsteady boundary layer flow over flat plates and observed that the leading edge accretion or ablation affects the fluid motion and its heat transfer characteristics. Tsai et al (2008) analyzed the heat transfer over unsteady stretching surfaces with non-uniform heat sources using the Chebyschev finite difference method.

The analysis of flow and heat transfer of viscoelastic fluid over moving semi-infinite horizontal flat surfaces were reported by Rafel (2008) whilst Orhan and Kaya (2005) presented results for laminar boundary layer flow over horizontally permeable flat plates. Suction was observed to enhance heat transfer coefficient while injection had a reverse effect. Recently, Anuar et al (2008) investigated the magnetohydodynamic flow and heat transfer due to a stretching.

It should be noted that MHD power-law fluid flow and heat transfer over non-isothermal stretching sheets and the diffusion of chemically reactive species of a non-Newtonian fluid immersed in porous media were investigated by Prasad et al (2003, 2008) while Tenzer-Sezgin and Bozkaya (2008) presented results for boundary element of MHD flow in infinite regions. Furthermore, Subhas et al

Furthermore, Samuel and Joubert (1975) investigated the development of boundary layers with increasing adverse pressure gradient whilst Siddappa and Subhas (1985) and Rajagopal et al (1984) investigated the flow of viscoelastic fluid over stretching sheets. Rao (1992) analysed the flow of a second grade fluid over a stretching sheet whilst Anderson (1992) examined the influence of uniform
magnetic field on the motion of electrically conducting fluid past flat and impermeable elastic sheets and obtained closed form solutions of the momentum boundary layer equations. Cortell (2006) analyzed the effects of transverse magnetic field on the flow of heat and mass transfer in electrically conducting second grade fluid over stretching sheets subjected to suction whilst Elbashbeshy et al (2010) then investigated the heat transfer over unsteady porous stretching surfaces embedded in porous media with variable heat flux in the presence of heat source/sink. Tulapurkara (2005) outlined some important aspects of boundary layer research after a century of its discovery. Boundary layer investigations have excited the interest of many scientists due to their extensive use in aeronautical and other branches of engineering.

1.12.2 Boundary Layer Research across Vertical Plates

The last three decades have seen tremendous interest in MHD free or mixed convection from vertical plates embedded in porous media due to their important applications in industry involving heat exchanger design, petroleum production, filtration, chemical catalytic reactors, and MHD generators. A number of analytical and numerical results explaining various aspects of boundary layer flow with heat and mass transfer across vertical plates are available in literature. References may be made to interesting works of Soundalgekar et al (1979), Chen and Armaly (1987), Bestman (1990), and Pop and Ingham (2001). However, only a limited attention has been given to the combined effect of Arrhenius chemical
reaction and Ohmic heating on MHD boundary layer flow past vertical porous plates. The effect of Ohmic heating on the MHD free convective heat transfer was reported by Hossain (1992) for a Newtonian fluid. Makinde and Sibanda (2008) investigated MHD mixed convection flow of heat and mass transfer past vertical flat plates in porous media with constant wall suction. Das et al (1994) analyzed the effects of homogeneous first-order chemical reaction on flow past impulsively started infinite vertical plates with uniform heat flux and mass transfer. Kandasamy and Perisamy (2005) later investigated the heat and mass transfer effects along wedges with suction and injection taking into account the first-order chemical reaction. Muthucumarswamy and Ganesan (2002) investigated the diffusion for zero and first-order chemical reaction on impulsively started infinite vertical plates with variable temperature while Prasad et al (2003) analyzed the reaction rate of chemically reactive species in laminar, non-Newtonian fluid immersed in porous media over stretching sheets. Most of these researchers observed that chemical reaction was more effective for zero and first-order reactions than second and third order reactions. Ghaly and Seddeek (2004) investigated the effects of chemical reaction, heat and mass transfer on laminar flow along a semi-infinite horizontal plate with temperature depended viscosity. The problem of chemically reactive species of non-Newtonian fluid in porous media over stretching sheets was investigated by Alkyildiz et al (2006).

Some researchers including Makinde et al (2007), Anuar et al. (2008), Makinde and Ogulu (2008), and Ibrahim and Makinde (2010) have included various physical aspects of the problem of combined heat and mass transfer. Furthermore,
Ahmed et al (2007) theoretically analyzed the steady MHD free convective and mass transfer boundary layer of electrically conducting fluid through porous media. This was an extension of the earlier works by Raptis and Kafousias (1982) who investigated the chemically reacting MHD boundary layer flow of heat and mass transfer over a moving vertical plate with suction.

The free convective flow with thermal radiation and mass transfer past moving vertical porous plates were investigated by Makinde (2005) while the effect of thermal radiation on heat and mass transfer of variable viscosity fluid past vertical porous plates permeated by transverse magnetic field was reported by Makinde and Ogulu (2008). Ibrahim and Makinde (2011) recently investigated the radiation effect on chemically reacting MHD boundary layer flow of heat and mass transfer past a porous vertical flat plate whilst Makinde and Sibanda (2008) investigated the boundary layer problem of MHD mixed convective flow with heat and mass transfer past vertical plates in porous media with constant wall suction. Brouwers (1991) earlier analyzed the film applied to free convection over vertical plates with blowing or suction.

Network numerical study of laminar free convection flow from a continuously moving vertical surface in thermally-stratified non-Darcian high-porosity medium was presented by Ilyana et al (2008). Ali et al (2010) analyzed the melting and radiation effects on mixed convection from a vertical surface embedded in a non-Newtonian fluid saturated non-Darcy porous medium for aiding and opposing external flows. Mulolani and Rahman (2000) conducted similarity analysis for natural convection from vertical plates with distributed wall concentration while
Chang and Lee (2008) investigated the free convection on vertical plates with uniform heat flux in a thermally stratified micropolar fluid and concluded that increasing the stratification parameter reduced the wall temperature, the skin friction parameter and the wall couple stress.

The local non-similarity solution for vertical free convection boundary layer was presented by Lok and Amin (2002) whilst group analysis for natural convection from a vertical plate was conducted by Rashed and Kassem (2007). The influence of radiation on MHD free convection from a vertical flat plate embedded in porous media with thermophoretic deposition of particles was analyzed by Rashad (2007). Abdelkhalek (2008) presented numerical results on heat and mass transfer in MHD free convection from a moving permeable vertical surface by perturbation techniques. The effects of joule heating and viscous dissipation of mixed convection MHD flow in vertical channels were reported by Barletta and Celli (2007) while Alam and Rahman (2006) presented the Dufour and Soret effects on mixed convection flow past vertical porous flat plates with variable suction.

The unsteady hydromagnetic free convection flow of a dissipative and radiating fluid past vertical plates with constant heat flux was investigated by Ogulu and Makinde (2009) while very recently, Elbashbeshy and Aldawody (2011) examined the effects of thermal radiation and magnetic field on unsteady mixed convection flow and heat transfer over porous stretching surfaces in the presence of internal heat generation/absorption. Yih (1999) earlier investigated the free convection effect on MHD coupled heat and mass transfer of a moving permeable vertical surface. A comprehensive review on the subject of heat resistant sheets moving in
transverse magnetic field has been reported by many authors including Sakiadis (1961), Bejan and Khair (1985), Nield and Bejan (1999), Ingham and Pop (1998, 2002). Chamkha et al (2010) analysed the effect of thermally stratified ambient fluid on MHD convective flow along a moving non-isothermal vertical plate.

Anghel et al (2000) studied the Dufour and Soret effects on free convection boundary layer flow over vertical surfaces embedded in porous media while Postelnicu (2004) numerically analyzed the influence of magnetic field on heat and mass transfer by natural convection from vertical surfaces in porous media considering the Soret and Dufour effects. Alam et al (2006) extended the works of Anghel et al to include mixed convection flow past vertical porous plates with variable suction and concluded that for fluids with medium molecular weight (H₂, air), Dufour and Soret effects could not be neglected. Free convection across vertical plates with uniform and constant heat flux in thermally stratified micropolar fluid was reported by Chang and Lee (2008).

Vajravelu and Nayfeh (1993) studied the convective heat transfer past stretching sheets earlier investigated by Crane (1970). Makinde (2010) later analyzed the similarity solution of hydromagnetic heat and mass transfer over a vertical plate with convective surface boundary conditions whilst Gupta and Gupta (1977), and Kays and Crawford (1993) analyzed the heat and mass transfer on stretching sheets with suction or blowing.
1.13. Thesis Outline

The thesis is organized into six chapters. Chapter one presents the introduction and background studies while the basic fluid dynamic equations of continuity, momentum, energy and concentration are derived in chapter two. In chapter three, an analysis of a MHD boundary layer flow of a chemically reacting fluid with heat and mass transfer past a stretching sheet is presented. Chapter four investigates the model for a chemically reacting MHD boundary layer flow with suction and Ohmic heating whilst chapter five examines the cooling of a low-heat-resistant sheet moving vertically downwards in a chemically reacting fluid permeated by transverse magnetic field. Chapter six concludes the thesis and makes recommendations for future work.
CHAPTER

THE BASIC FLUID DYNAMIC EQUATIONS

2.0 Introduction

Boundary layer problems can be described by mathematical models based on the basic conservation laws of nature. Such mathematical representation enables a better understanding of fluid motion and helps in predicting the behavior of the fluid under varied conditions. These equations, when solved provide valuable insights to engineers for design purposes. In this chapter, the basic fluid dynamic equations of continuity, momentum, energy and concentration are derived by the method of control-volume approach.

2.1 The Fluid Mechanic Equations

The four basic equations required to adequately describe a flow problem are the continuity, momentum, energy and concentration. These equations are derived from the conservation laws of nature; mass, momentum, energy and concentration. The control volume analysis approach is preferred by engineers because it provides better understanding for engineering analysis, White (1994). By this method, an arbitrary state of variable fluid defined by its geometry, the boundary conditions, the laws of mechanics and the particular fluid properties are important.
2.1.1 The Continuity Equation

In a continuous body of fluid in motion, if we consider an S fixed in space containing a volume V, it is observed that the increase in the mass of the fluid flowing into the system is the same as the mass of fluid flowing out of the system.

The mass of the fluid within the surface is given as

\[ m = \int \rho \, dv. \]  

(2.1)

The rate of increase of the mass within the surface is given as

\[ \frac{\partial m}{\partial t} = \frac{\partial}{\partial t} \int \rho \, dv = \int \frac{\partial \rho}{\partial t} \, dv. \]  

(2.2)

\[ \hat{n} \]

\[ \hat{q} \]

Fluid flowing in

Fluid flowing out

Figure 2.1 Flow Continuity Diagram
Consider the diagram in figure 2.1 which depicts the continuity of flow, the net amount of fluid within the surface is constant assuming there are no sources of fluid within the volume \( V \). Since the volume \( V \) does not vary with time. The rate of flow of \( S \) is given as

\[
\int_S (\vec{q} \cdot \vec{n}) \rho \, ds = \int_T \nabla \cdot (\rho \vec{q}) \, dv,
\]

(2.3)

\[
\int_T \frac{\partial \rho}{\partial t} \, dv = -\int_T \nabla \cdot (\rho \vec{q}) \, dv,
\]

(2.4)

\[
\int_T \frac{\partial \rho}{\partial t} \, dv + \int_T \nabla \cdot (\rho \vec{q}) \, dv = 0,
\]

(2.5)

\[
\int_T \left( \frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{q}) \right) \, dv = 0.
\]

(2.6)

Since the choice of \( V \) was arbitrary, the integral vanishes and equation (2.6) becomes;

\[
\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{q}) = 0.
\]

(2.7)

Equation (2.7) is the continuity equation for any fluid flow. It can be written as;

\[
\frac{\partial \rho}{\partial t} + \rho \text{div} \vec{q} + (\vec{q} \cdot \nabla) \rho = 0,
\]

(2.8)

i.e.

\[
\frac{D}{Dt} + \rho \text{div} \vec{q} = 0
\]

(2.9)
where, \( \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{\bar{q}} \cdot \nabla \),

is the material derivative, which represents differentiation following the fluid motion. For incompressible flow, density is constant and equation (2.9) becomes

\[
div \mathbf{\bar{q}} = 0.
\]  

(2.10)

If the velocity components in the x, y and z coordinates are \( u, v, \) and \( w \), then we can write

\[
\mathbf{\bar{q}} = (u, v, w).
\]

(2.11)

Equation (2.10) can be written in expanded form as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\]  

(2.12)

For a two dimensional, incompressible flow, the continuity equation can be written as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.
\]  

(2.13)

Introducing the stream function \( \psi \), such that;

\[
u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad u = -\frac{\partial \psi}{\partial x},
\]

i.e

\[
-\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0.
\]  

(2.14)
Now if the flow is irrotational, then \[ \nabla \cdot \vec{q} = 0 \]

This implies that \[ \vec{q} = -\nabla \phi. \] (2.15)

In two dimensions, we have \[ u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y}, \]

From (2.13), we obtain

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \] (2.16)

This is called the Laplace equation which must be satisfied for irrotational flow.

### 2.1.2. Linear Momentum Equation

We consider a small parallel piped of volume isolated instantaneously from the fluid whose centre is \((x_1 + x_2 + x_3)\) given as

\[ dV = dx_1 + dx_2 + dx_3, \] (2.17)

We denote by \(\sigma_{ij}\) the stress acting in the \(x_i\) direction on the force whose normal lies in the \(x_j\) direction. The stress components; \(\sigma_{11}, \sigma_{12}, \sigma_{13}, \text{ etc.}\), form a second order Stress tensor \(\sigma_{ij}\)

\[ \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \] (2.18)
Equation (2.18) is called the stress matrix. The components \( \sigma_{ij(i=j)} \) are the normal stresses while the components \( \sigma_{ij(i\neq j)} \) are shear stresses. The stress tensor and the corresponding matrix are symmetric, i.e. \( \sigma_{ij} = \sigma_{ji} \).

In relation to a surface at right angles to the axis \( x_i \), the stress per unit area at \( (x_1 + x_2 + x_3) \) are \( \sigma_{11}, \sigma_{12}, \text{ and } \sigma_{13} \). The corresponding stresses at the centre of the forces are given as:

\[
\sigma_{11}(x_1 + \frac{dx_1}{2}, x_2, x_3),
\]

\[
\sigma_{12}(x_1 + \frac{dx_1}{2}, x_2, x_3),
\]

\[
\sigma_{13} = (x_1 + \frac{dx_1}{2}, x_2, x_3).
\] (2.19)

Using Taylor’s expansion in the form

\[
y(x + h) = y(x) + hy'(x) + \frac{h^2}{2} y''(x) + \frac{h^3}{6} y'''(x) + ..., \]

(2.20)

We write

\[
\sigma_{11}(x_1 + \frac{1}{2} dx_1, x_2, x_3) = \sigma_{11}(x_1, x_2, x_3) + \frac{1}{2} \frac{\partial \sigma_{11}}{\partial x_1} dx_1 + \frac{1}{4} \frac{\partial^2 \sigma_{11}}{\partial x_1^2} dx_1^2 + .... \]

(2.21)

Approximating to the second term, we obtain
\[ \sigma_{11}(x_1 + \frac{1}{2} dx_1, x_2, x_3) = \sigma_{11}(x_1, x_2, x_3) + \frac{1}{2} \frac{\partial \sigma_{11}}{\partial x_1} dx_1, \]

Similarly

\[ \sigma_{12}(x_1 + \frac{1}{2} dx_1, x_2, x_3) = \sigma_{12}(x_1, x_2, x_3) + \frac{1}{2} \frac{\partial \sigma_{12}}{\partial x_1} dx_1, \]

\[ \sigma_{13}(x_1 + \frac{1}{2} dx_1, x_2, x_3) = \sigma_{13}(x_1, x_2, x_3) + \frac{1}{2} \frac{\partial \sigma_{13}}{\partial x_1} dx_1, \] (2.22)

At the centre of the opposite force, the corresponding stresses are respectively;

\[ \sigma_{11}(x_1, x_2, x_3) - \frac{1}{2} \frac{\partial \sigma_{11}}{\partial x_1} dx_1, \]

\[ \sigma_{12}(x_1, x_2, x_3) - \frac{1}{2} \frac{\partial \sigma_{12}}{\partial x_1} dx_1, \] (2.23)

\[ \sigma_{13}(x_1, x_2, x_3) - \frac{1}{2} \frac{\partial \sigma_{13}}{\partial x_1} dx_1, \]

acting on the fluid in parallel compound. The stresses on a pair of opposite forces may be compounded into

\[ \frac{\partial \sigma_{11}}{\partial x_1} dx_1 dx_2 dx_3, \]

\[ \frac{\partial \sigma_{12}}{\partial x_1} dx_1 dx_2 dx_3, \] (2.24)

\[ \frac{\partial \sigma_{13}}{\partial x_1} dx_1 dx_2 dx_3, \]
acting at \((x_1 + x_2 + x_3)\) parallel to \((0x_1 + 0x_2 + 0x_3)\). The stress on the other two pair of opposite forces may be compounded into similar forces at \((x_1 + x_2 + x_3)\). The resultant force in \(x_i\) direction becomes;

\[
\left( \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} \right) dx_1 dx_2 dx_3.
\] (2.25)

If X, Y, and Z are the forces per unit area due to variation of stress along \(0x_1 + 0x_2 + 0x_3\), then

\[
X = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} = \frac{\partial \sigma_{a1}}{\partial x_a},
\]

\[
Y = \frac{\partial \sigma_{22}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} = \frac{\partial \sigma_{a2}}{\partial x_a},
\] (2.26)

\[
Z = \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = \frac{\partial \sigma_{a3}}{\partial x_a}, \alpha = 1, 2, 3.
\]

Generally, we denote the body force by \(F = (F_1, F_2, F_3) = F_i, i = 1, 2, 3\), then the equation of motion is given by

\[
\rho \frac{\partial u_i}{\partial t} = \frac{\partial \sigma_{yi}}{\partial x_i} + F_i.
\] (2.27)

In tensor notation, we write the velocity vector as

\[
\bar{q} = (U_1, U_2, U_3) = U_i, i = 1, 2, 3.
\] (2.28)
It is convenient to regard the stress $\sigma_{ij}$ as the sum of inviscid part $-p\sigma_{ij}\delta_{ij}$ and a viscous part $\sigma_{ij}$ where $p$ is the average of the three normal stresses for any orthogonal set of axes

$$p = \frac{\sigma_{ij}\delta_{ij}}{3} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

(2.29)

where $\delta_{ij}$ is the substitution tensor. Now separate the second tensor $\frac{\partial U_i}{\partial x_j}$ as follow,

$$\frac{\partial U_i}{\partial x_j} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_i}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_i}{\partial x_i} \right)$$

$$= \frac{1}{2} e_{ij} + \frac{1}{2} \gamma_{ij}.$$  

(2.30)

where $e_{ij}$ is the symmetric tensor while $\gamma_{ij}$ is the anti-symmetric rotational tensor defining the velocity of the motion. In uniform media, viscous stresses are induced only by deformation and not by rotation. $e_{ij}$ is called rate of strain tensor. It represents the rate of change in the size and shape of a fluid element. The linear momentum equation then becomes;

$$\rho \frac{dq}{dt} = \rho \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} = -\nabla P + \nabla \cdot \sigma + \rho F.$$  

(2.31)

In Cartesian coordinate system, we write
\[ \nabla \cdot \sigma = \left[ \mu \left( \frac{\partial^2 U_i}{\partial x_i \partial x_j} + \frac{\partial^2 U_j}{\partial x_j \partial x_i} \right) + \partial U_i \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + \lambda \frac{\partial^2 U_i}{\partial x_j \partial x_j} + \frac{\partial \lambda}{\partial x_i} \frac{\partial U_i}{\partial x_i} \right] \] (2.32)

Any additional simplification of \( \nabla \cdot \sigma \) would require further assumptions or approximations.

### 2.1.3 The Energy Equation

To derive the energy content of a system, the zeroth and the first law of thermodynamics in addition to the Fourier conduction are required.

#### The Zeroth Law of Thermodynamics

The Zeroth Law of Thermodynamics states that there exists the temperature \( T \) such that when two systems in contact are in thermal equilibrium, then \( T \) is the same in both systems. \( T \) is called the absolute temperature.

#### The First Law of Thermodynamics

The first law of thermodynamics expressed the principle of energy conservation which states that there exists a variable of state \( E \), such that if a system is transformed from one state of equilibrium to another by the process in which an amount of work \( W \) is done on a system from the surrounding, and the amount of
heat Q is added to a system from the surrounding, then the difference between the initial and final values of energy of the systems $E_i$ and $E_f$ is given by

$$E_f - E_i = Q + W,$$  \hfill (2.33)

In differential form we have

$$dE = dQ + dW,$$  \hfill (2.34)

Or

$$\frac{dE}{dt} = \frac{dQ}{dt} + \frac{dW}{dt}.$$  \hfill (2.35)

To determine the work, we consider the first contribution from the component $\sigma_y$ of stress. The work in unit time is given by

$$\frac{\partial (W\sigma_{xx})}{\partial t} = dydz\left\{-U\sigma_{xx} + \left(U + \frac{\partial U}{\partial x}\right)\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \right\},$$  \hfill (2.36)

where,

$$\nabla V = dxdydz.$$  \hfill (2.37)

The total work done by the stress per unit mass on deforming elements of fluid $\nabla V$ is given by

$$\frac{1}{\rho} \frac{\partial (\sigma_{yy} U_j)}{\partial x_i} = \frac{1}{2} \left( \frac{\partial \sigma_{yy}}{\partial x_j} + \sigma_{yy} \frac{\partial U_j}{\partial x_i} \right).$$  \hfill (2.38)
where \( \sigma_{ij} \) is the stress acting in the \( x_i \) on the force whose normal lies in the \( x_j \) direction. From the equation of fluid motion, we have

\[
\rho \frac{\partial U_i}{\partial t} = \frac{\partial (\sigma_{ij})}{\partial x_i},
\]

(2.39)

and

\[
U_j \frac{\partial (\sigma_{ij})}{\partial x_i} = \rho U_j \frac{\partial U_i}{\partial t} = \frac{\rho}{2} \frac{d(U_iU_j)}{dt}.
\]

(2.40)

This is clearly the change in kinematic energy of the fluid element following the motion. The remaining term in equation (2.38) represents the rate of dissipation of energy per unit mass. Substituting \( \sigma_{ij} = -p \delta_{ij} + \mu e_{ij} \).

We get

\[
\frac{dW}{dt} = \frac{1}{2} \frac{\partial (U_iU_j)}{\partial t} - \frac{\rho}{\rho} \sigma_{ij} \frac{\partial U_i}{\partial x_i} + \nu \left( \frac{\partial U_i}{\partial x_i} + \frac{\partial U_j}{\partial x_j} \right),
\]

(2.41)

But \( \delta_{ij} \frac{\partial U_j}{\partial x_i} = \frac{\partial U_j}{\partial x_i} = 0 \)

It is clear that only viscous force and not pressure force contribute to energy dissipation. Defining the energy dissipation function \( \Phi \) by

\[
\Phi = \frac{U_j}{\partial x_i} \left( \frac{\partial U_i}{\partial x_i} + \frac{\partial U_j}{\partial x_j} \right) = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_i} + \frac{\partial U_j}{\partial x_j} \right)^2,
\]

(2.42)

Which shows that \( \Phi \geq 0 \) and we have,

\[
\frac{dW}{dt} = \frac{1}{2} \frac{d}{dt} (U_iU_j) + \nu \Phi.
\]

(2.43)
Similarly the heat transferred to the system from the surrounding is $Q$. Neglecting the transfer of heat by radiation and consider only that by conduction. If we consider the element of the volume, $dv = dx dy dz$ of the mass $\rho dv$ and the change in kinetic energy of an amount

$$d\left\{\frac{1}{2} \rho dv (U_1^2 + U_2^2 + U_3^2)\right\}, \quad (2.44)$$

and neglecting changes in potential energy, we have

$$\frac{dE}{dt} = \rho dv \left\{ \frac{de}{dt} + \frac{1}{2} \frac{d(U_1^2 + U_2^2 + U_3^2)}{dt} \right\}, \quad (2.45)$$

where $e$ is equal to the thermal energy per unit mass.

**The Fourier Conduction**

The Fourier law of conduction states that the heat flux $q$ per unit area and time is proportional to the temperature gradient. That is:

$$q = \frac{1}{A} \frac{\partial Q}{\partial t} = -k \frac{\partial T}{\partial n}, \quad (2.46)$$

or

$$q = -k \nabla T,$$

where $k$ is the thermal conductivity. The negative sign signifies that the heat flux is reckoned as positive in the direction of temperature gradient (i.e. heat flux in the
direction of decreasing temperature). Hence the amount of heat transferred into the volume $dv$ through surface elements which are normal to the $x$ direction is equal to

$$
(-k \frac{\partial T}{\partial n}) dydz, \quad (2.47)
$$

The amount of heat leaving the volume is given by

$$
\left[ k \frac{\partial T}{\partial x} + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) \frac{dx}{dy} \right] dydz, \quad (2.48)
$$

Thus the amount of heat added by conduction in the direction during time $dt$ to volume $dv$ is

$$
dt \cdot dv \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right),
$$

Hence, the total amount of heat added in all directions is given by

$$
\frac{\partial Q}{\partial t} = dv \left[ \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) \right]. \quad (2.49)
$$

Substituting equations (2.43), (2.45) and (2.49) into (2.35), we obtain

$$
\rho \frac{\partial e}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + q \Phi \quad (2.50)
$$

where,

$$
\Phi = 2 \left( \left( \frac{\partial U_1}{\partial x} \right)^2 + \left( \frac{\partial U_2}{\partial y} \right)^2 + \left( \frac{\partial U_3}{\partial z} \right)^2 \right) + \left( \frac{\partial U_1}{\partial y} + \frac{\partial U_2}{\partial x} \right)^2 + \left( \frac{\partial U_3}{\partial x} + \frac{\partial U_1}{\partial z} \right)^2 + \left( \frac{\partial U_3}{\partial z} + \frac{\partial U_2}{\partial x} \right)^2.
$$
Equation (2.50) holds for incompressible fluids with constant thermal conductivity $k$. For a perfect fluid,

$$\frac{de}{dt} = C_v \frac{dT}{dt}, (de = C_v dT)$$

Equation (2.50) now takes the form $\rho C_v \frac{dT}{dt} = k \nabla^2 T + \nu \Phi$. (2.52)

This is the energy equation where $C_v$ is the specific heat at constant volume.

Equation (2.52) can be written as:

$$\rho C_v \left( \frac{\partial T}{\partial t} + (\bar{\mathbf{q}}, \nabla) T \right) = k \nabla^2 T + \nu \Phi,$$ (2.53)

where $\Phi$ is small relative to $\rho C_v \frac{dT}{dt}$, equation (2.53) becomes;

$$\frac{dT}{dt} = \frac{k}{\rho C_v} \nabla^2 T = \nu \frac{\nabla^2 T}{\sigma},$$ (2.54)

where $\sigma = \frac{\mu C_v}{k}$ is the Prandtl number

i.e. $\frac{\partial T}{\partial t} + U_1 \frac{\partial T}{\partial x} + U_2 \frac{\partial T}{\partial y} + U_3 \frac{\partial T}{\partial z} = \nu \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$. (2.55)

For steady two-dimensional flow, $\frac{\partial T}{\partial t} = 0, U_3 = 0, z = 0$, and taking $\alpha = \frac{\nu}{\sigma}$

We get $U_1 \frac{\partial T}{\partial x} + U_2 \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$. (2.56)
2.1.4 Species Concentration Equation

When chemical species are introduced into a fluid medium, the concentration of the chemical species in the fluid is derived by considering the material derivative of the species. At any point in the fluid, the concentration can be expressed as a function of position and time, $C(x, y, z, t)$. Thus, changes in concentration between two points can be expressed as:

$$dC = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} \frac{dx}{dt} + \frac{\partial C}{\partial y} \frac{dy}{dt} + \frac{\partial C}{\partial z} \frac{dz}{dt},$$

(2.57)

The rate of change in concentration is:

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} \frac{dx}{dt} + \frac{\partial C}{\partial y} \frac{dy}{dt} + \frac{\partial C}{\partial z} \frac{dz}{dt},$$

(2.58)

By noting $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, $\frac{dz}{dt} = \omega$,

We write,

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + \omega \frac{\partial C}{\partial z},$$

(2.59)

From the Fick’s law of mass diffusivity, the change in mass flux of the reacting species together with the type of chemical kinetics must balance the time rate of change of the reacting species in the flow system.

Thus,

$$\frac{dC}{dt} = D\nabla^2 C \pm R(C),$$

(2.60)
where $D$ is the molecular diffusivity coefficient and $R(C)$ is the destructive chemical kinetics when it is negative and generative chemical kinetics when it is positive. The reacting species in the fluid transport equation is then given as

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + \omega \frac{\partial C}{\partial z} = D \nabla^2 C \pm R(C). \quad (2.61)$$

2.2 Summary

The basic fluid dynamic equations of continuity, momentum, energy and concentration have been derived. The conservation laws of nature were used in conjunction with the Newton’s second of motion and the Fick’s law of mass diffusivity. The equations depict partial differential forms which represents the non-linear characteristics embedded in every physical system.
CHAPTER

CHEMICALLY REACTING MHD BOUNDARY LAYER FLOW
PAST A STRETCHING SHEET

3.0 Introduction

The magnetohydrodynamic boundary layer flow of a chemically reacting fluid with heat and mass transfer past a stretching sheet is investigated. The mathematical model of the flow problem is transformed from partial equations to ordinary differential equations of higher order which were reduced to systems of first order differential equations. The transformed equations were coupled and solved numerically using the forth-order Runge-Kutta integration scheme along with the Newton-Raphson algorism. Results of various flow parameters on the overall flow structure were presented numerically and discussed.

3.1 Formulation of the Problem

We considered steady 2-dimensional flow of an incompressible, viscous and electrically conducting fluid which flowed past a flat and impermeable stretching sheet with heat generation or absorption, figure 3.1. The $x$-coordinate axis was taken to act in the direction along which the stretching sheet moved while the $y$-axis taken perpendicular to the flow direction. The flow was generated by the
action of two equal and opposite forces acting along the $x$-axis and the sheet was stretched in such a way that the velocity at any instant was proportional to the distance from the origin ($x = 0$). The flow field was exposed to the influence of external transverse magnetic field of strength $H_0$ and the induced magnetic field was negligibly small. The cooling fluid had weak electrical conductivity so that any charge generated during the process got accumulated on the extrusion.

A chemical species diffused into the ambient fluid initiates a first-order irreversible chemical reaction. With these assumptions the boundary layer equations governing the flow were modeled mathematically as follows:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}
\]

\[
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma H_0^2}{\rho} u, \tag{3.2}
\]
\[
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{\partial y} \left( \frac{\partial u}{\partial y} \right) ^2 + \frac{\sigma H_0^2}{\rho c_p} u^2 + \frac{Q}{\rho c_p} (T - T_a), \tag{3.3}
\]

\[
\frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = \frac{D}{\partial y^2} - \gamma (C - C_\infty). \tag{3.4}
\]

Equations (3.1) to (3.4) represent the continuity, momentum, energy and concentration equations respectively.

where \( u \) and \( v \) are the velocity components in the \( x \) and \( y \) directions respectively, \( \rho \) is the density of the liquid, \( \nu \) is the kinematic viscosity, \( H_0 \) is the strength of the applied magnetic field, \( \sigma \) is the electrical conductivity of the fluid, \( C \) is the species concentration, \( T \) is the fluid temperature, \( D \) is the mass diffusivity, \( \gamma \) chemical reaction coefficient, \( \alpha \) is the fluid thermal diffusivity and \( c_p \) represent the specific heat at constant pressure. Here, we make a note that the case \( Q > 0 \) corresponds to internal heat generation and \( Q < 0 \) corresponds to internal heat absorption.

The boundary conditions for the flow problem were given as:

\[
u = \alpha l, \nu = 0, \quad T = T_w(x) = T_\infty + A \left( \frac{x}{l} \right)^{\lambda_1}, \quad C = C_w(x) = C_\infty + B \left( \frac{x}{l} \right)^{\lambda_2} \quad \text{on} \quad y = 0,
\]

\[
u \to 0, T \to T_w, C \to C_\infty \quad \text{as} \quad y \to \infty, \tag{3.5}
\]

where \( A \) and \( B \) are constants, \( l \) is the characteristic length, \( T_w \) and \( C_w \) are the sheet surface temperature and concentration respectively, \( T_\infty \) and \( C_\infty \) are the temperature and chemical species concentration of the fluid far away from the sheet surface,
while $\lambda_1$ and $\lambda_2$ are the variable wall temperature and concentration parameters respectively.

### 3.2 Analysis of the Problem

Equations (3.1) - (3.4) are partial differential equations which admit self-similar solutions of the form;

$$
\begin{align*}
\frac{\partial u}{\partial x} &= a f', \\
\frac{\partial v}{\partial y} &= -\sqrt{1 - \nu} f' \sqrt{\frac{a}{\nu}} = -a f'.
\end{align*}
$$

The continuity equation (3.1) is simultaneously satisfied.

Furthermore, from (3.6), the following derivatives are obtained;

$$
\begin{align*}
\frac{\partial u}{\partial x} &= af', \\
\frac{\partial u}{\partial y} &= a f' \sqrt{\frac{a}{\nu}} = x \sqrt{\frac{a^3}{\nu}} f''.
\end{align*}
$$
\[
\frac{\partial^2 u}{\partial y^2} = x \sqrt{\frac{a^3}{v}} f'' \left( \frac{a}{v} \right) = \frac{x a^2}{v} f'' ,
\]

Substituting into the momentum equation (3.2), we have;

\[
axf' . af'' + - \sqrt{a v} f x a^2 \frac{f'}{v} = \nu a^2 x f'' - \frac{\sigma H_0^2}{\rho} axf',
\]

\[
a^2 xf''^2 - a^2 xff'' = a^2 xf''' - \frac{\sigma H_0^2 a x f'}{\rho},
\]

\[
f''' - f'' + ff'' = R f',
\]

(3.7)

where \( R = \frac{\sigma H_0^2}{\rho a} \).

Furthermore, the following derivatives are obtained from equations in (3.6) as;

\[
\frac{\partial T}{\partial x} = A \lambda_1 x^{(\lambda_1 - 1)} \frac{A \lambda_1 x^{(\lambda_1 - 1)}}{l^{\lambda_1}},
\]

\[
\frac{\partial T}{\partial y} = (T_w - T) \theta' \frac{a}{\sqrt{v}},
\]

\[
\frac{\partial^2 T}{\partial y^2} = (T_w - T) \theta'' a \frac{a}{\nu}.
\]

(3.8)

The energy equation (3.3) transforms as;

\[
axf'. A \lambda_1 x^{-1} \left( \frac{x}{l} \right)^{\lambda_1} - \sqrt{a v} f \left( (T_w - T) \theta' \frac{a}{\sqrt{v}} \right)
\]

\[
= \frac{\alpha}{\nu} \ a(T_w - T) \theta'' + \frac{v}{c_p} (x \sqrt{\frac{a^3}{v}} f')^2 + \frac{\sigma H_0^2}{\rho c_p} a^2 \ x^2 f''^2 + \frac{Q}{\rho c_p} (T_w - T) \theta.
\]

- 59 -
Simplifying;

\[ aA\lambda_1 \left( \frac{x}{l} \right)^{\lambda_1} f' - a(T_w - T_\infty) f\theta' = \]

\[ \frac{\alpha}{\nu} a(T_w - T_\infty) \theta'' + \frac{\nu}{c_p} \frac{x^2 a^3}{\nu} f'' + \frac{\sigma H_0^2}{\rho c_p} a^2 x^2 f'' + \frac{Q}{\rho c_p} (T_w - T_\infty) \theta. \]

Dividing through by

\[ \frac{\alpha}{\nu} a(T_w - T_\infty), \]

we get

\[ \frac{\nu}{\alpha} \frac{a A \lambda_1}{(T_w - T_\infty)} \left( \frac{x}{l} \right)^{\lambda_1} f' - \frac{v}{\alpha} f\theta' = \]

\[ \theta'' + \frac{\nu}{\alpha c_p a(T_w - T_\infty)} \frac{x^2 a^3}{\nu} f'' + \frac{\sigma H_0^2}{\rho c_p} \frac{a^2 x^2}{\alpha} f'' + \frac{Q}{\rho c_p} \frac{v}{\alpha} \theta. \]

This implies

\[ \frac{\nu}{\alpha} \frac{A \lambda_1 (x/l)^{\lambda_1}}{T_w - T_\infty} f' - \frac{v}{\alpha} f\theta' = \]

\[ \theta'' + \frac{\nu}{\alpha c_p (T_w - T_\infty)} \frac{a^2 x^2}{\alpha} f'' + \frac{\sigma H_0^2}{\rho c_p} \frac{a^2 x^2}{\alpha} f'' + \frac{Q}{\rho c_p} \frac{v}{\alpha} \theta. \]

Recall that;

\[ \text{Pr} = \frac{\nu}{\alpha}, Ec = \frac{a^2 x^2}{c_p (T_w - T_\infty)}, N = \frac{Q}{\rho c_p a}, R = \frac{\sigma H_0^2}{\rho c_p a}, \left( \frac{l}{x} \right)^{\lambda_1} = \frac{A}{T_w - T_\infty}, \]

- 60 -
By substituting and rearranging terms, we get;

$$\theta'' + Pr f\theta' - Pr \lambda_{1} f' + Pr Ec f' n_{2} + Pr REcf' r_{2} + N Pr \theta = 0. \quad (3.9)$$

Factoring Pr and Ec in terms they appear, we get;

$$\theta'' + Pr N \theta - Pr(\lambda_{1} f' - f\theta') + Pr Ec(f' n_{2} + Rf' r_{2}) = 0. \quad (3.10)$$

Finally, the concentration equation given as (3.14) is transformed by noting that;

$$\frac{\partial C}{\partial x} = \frac{B\lambda_{2} x^{(\lambda_{2} - 1)}}{l^{\lambda_{2}}} ,$$

$$\frac{\partial C}{\partial y} = (C_{w} - C_{\infty}) \phi \frac{a}{\sqrt{v}} ,$$

$$\frac{\partial^{2} C}{\partial y^{2}} = (C_{w} - C_{\infty}) \phi^{*} \frac{a}{v} .$$

Thus,

$$ax^{f} B\lambda_{2} \left( \frac{x}{l} \right)^{\lambda_{2}} + \sqrt{a v f .(C_{w} - C_{\infty})} \frac{a}{\sqrt{v}} \phi = D(C_{w} - C_{\infty}) \frac{a}{\sqrt{v}} \phi^{*} - \gamma(C_{w} - C_{\infty}) \phi .$$

$$aB\lambda_{2} \left( \frac{x}{l} \right)^{\lambda_{2}} f' - a(C_{w} - C_{\infty}) f\phi' = D \frac{a}{\sqrt{v}} (C_{w} - C_{\infty}) \phi^{*} - \gamma(C_{w} - C_{\infty}) \phi .$$

Or

$$\frac{aB\lambda_{2} (x/l)^{\lambda_{2}} v}{a(C_{w} - C_{\infty}) D} f' - \frac{a(C_{w} - C_{\infty}) f\phi'}{D} = \phi^{*} - \frac{V}{D} \frac{\gamma(C_{w} - C_{\infty})}{a(C_{w} - C_{\infty})} \phi .$$

- 61 -
Recall that

\[ Sc = \frac{\nu}{D}, \beta = \frac{\gamma}{D}, \left( \frac{L}{x} \right)^{\frac{1}{2}} = \frac{B}{(C_w - C_\infty)}. \]

That is,

\[ \lambda_2 Sc \phi' - Sc \phi' = \phi'' - Sc \beta \phi \]  \hspace{1cm} (3.11)

Or

\[ \phi'' + Sc(\phi' - \lambda_2 f' - \beta \phi) = 0. \]  \hspace{1cm} (3.12)

The primes in equations (3.7), (3.10) and (3.12) denote differentiation with respect to \( \eta \). The terms \( R, Pr, N, Sc, Ec \) and \( \beta \) represent the Chandrasekhar number, Prandtl number, uniform heat source/sink parameter, Schmidt number, Eckert number and the chemical reaction parameter respectively. The boundary conditions given in equation (3.5) were rendered dimensionless noting that on the surface of the stretching sheet, \( y \) was zero. This implied that \( \eta = 0 \) and the velocity components led to the boundary conditions of \( f'(0) \) and \( f(0) \) as follows;

\[ u = a \nu f(\eta = 0) = a \nu, \]

Hence, \( f'(0) = 1. \)

\[ v = -\sqrt{a \nu f(\eta = 0)} = 0 \Rightarrow f(0) = 0. \]
The temperature and concentration of the fluid at the surface of the plate, $T$ and $C$ are respectively equal to the temperature and concentration of the plate, $T_w$ and $C_w$ at $y = 0$.

Thus, $\theta(\eta) = 1, \phi(\eta) = 1$ when $\eta = 0$.

At distances located away from the plate, $y$ approaches infinity (i.e. $y \to \infty$), when $\eta$ approaches infinity $\eta \to \infty$. The boundary conditions then became;

$$f'(\eta) \to 1, f(\eta) = 0, \theta(\eta) = 1, \phi(\eta) = 1 \text{ at } \eta = 0,$$

$$f'(\eta) \to 0, \theta(\eta) \to 0, \phi(\eta) \to 0, \text{ as } \eta \to \infty.$$ (3.13)

Chang (1989) and Rao (1992) obtained closed form solutions of equation (3.7) which clearly revealed that the solution was not unique, and the appropriate solution among them was chosen. Using the realistic solution, $f(\eta) = \frac{1-e^{-m\eta}}{m}$, the velocity components are given by

$$u = axe^{-m\eta}, v = -\sqrt{av\left(\frac{1-e^{-m\eta}}{m}\right)},$$ (3.14)

where $m=\sqrt{1+R}$ represent the local skin-friction coefficient or the frictional drag coefficient given by

$$C_f = \frac{\tau_w}{p\sqrt{\alpha u}} = m.$$ (3.15)
The other physical quantities of interest in the problem namely the local Nusselt number \( (Nu) \) and the Sherwood number \( (Sh) \) were easily computed. In dimensionless terms, they are defined as:

\[
Nu = -\theta'(0) \quad \text{and} \quad Sh = -\phi'(0).
\] (3.16)

### 3.3 Computational method

The ordinary differential equations (3.7), (3.10) and (3.12) are observed to be higher order and can be reduced to first order differential equations. The equations were solved along with the boundary conditions given in (3.13) using the Newton–Raphson shooting method and the fourth-order Runge-Kutta integration algorithm.

To transform the equations, we let; \( \theta = x_1, \ \theta' = x_2, \ \phi = x_3, \ \phi' = x_4 \). (3.17)

The resulting first order differential equations are:

\[
x_1' = x_2, \\
x_2' = -Pr N_x + Pr(\lambda_4 x_1 e^{-\eta \gamma} - \frac{x_2 (1 - e^{-\eta \gamma})}{m}) - Pr Ec (m^2 e^{-2m\gamma} + Re e^{-2m\eta}), \\
x_3' = x_4, \\
x_4' = \beta Sc x_3 + Sc (\lambda_2 x_2 e^{-\eta \gamma} - \frac{x_4 (1 - e^{-\eta \gamma})}{m}),
\] (3.18)

subject to the following initial conditions,

\[
x_1(0) = 1, \ x_2(0) = s_1, \ x_3(0) = 1, \ x_4(0) = s_2.
\] (3.19)
The unspecified initial conditions; $s_1$ and $s_2$ are guessed systematically and equation (3.18) integrated numerically as an initial valued problem to a given terminal point. The procedure is repeated until results up to the desired degree of accuracy are obtained: namely $10^{-7}$. A code was written in MAPLE package, Heck (2003) and solutions presented graphically. The value of $\eta_\infty$ was found to each iteration loop by the assignment statement $\eta_\infty = \eta_\infty + \Delta \eta$. The maximum value of $\eta_\infty$ to each group of parameters $R$, $Pr$, $N$, $\beta$, $Sc$ and $Ec$ were determined when the values of unknown boundary conditions at $\eta = 0$ not changed to successful loop with error less than $10^{-7}$. Results showing the rate of mass and heat transfers on the plate surface are obtained and quantitatively discussed whiles graphical results for velocity, temperature and concentration profiles are illustrated in the boundary layer region.

3.4. Numerical Results and Discussions

Numerical computations were conducted to study the effect of various physical parameters such as Chandrasekhar number ($R$), Prandtl number ($Pr$), Schmidt ($Sc$), uniform heat source/sink parameter ($N$) and the power law variable surface temperature and concentration parameters ($\lambda_1$ and $\lambda_2$) on the boundary layer.

The value of $Pr$ was taken to be 0.71 which corresponds to air and the values of $Sc$ chosen in such a way that it represents the diffusing chemical species of most common interest in air like $H_2$, $H_2O$, $NH_3$ and Propyl Benzene whose $Sc$ values
are 0.24, 0.6, 0.78 and 2.62 respectively. Results for wall temperature and concentration gradients are shown in table 3.1

Table 3.1 Computations showing the rate of heat and mass transfers at the wall (N = 0.1, Pr = 0.71, Sc = 0.6)

<table>
<thead>
<tr>
<th>R</th>
<th>Ec</th>
<th>β</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>(-\theta'(0))</th>
<th>(-\phi'(0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.448975</td>
<td>1.10530</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.386606</td>
<td>1.10020</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.162847</td>
<td>1.08281</td>
</tr>
<tr>
<td>0.1</td>
<td>1.5</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.220909</td>
<td>1.10020</td>
</tr>
<tr>
<td>0.1</td>
<td>2.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.055212</td>
<td>1.10020</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
<td>1.5</td>
<td>1.0</td>
<td>1.0</td>
<td>0.386606</td>
<td>1.23894</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
<td>2.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.386606</td>
<td>1.36128</td>
</tr>
<tr>
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<td>1.0</td>
<td>2.0</td>
<td>1.0</td>
<td>0.711738</td>
<td>1.10020</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
<td>1.0</td>
<td>3.0</td>
<td>1.0</td>
<td>0.980962</td>
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</tr>
<tr>
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<td>1.0</td>
<td>1.0</td>
<td>2.0</td>
<td>0.386606</td>
<td>1.28456</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>3.0</td>
<td>0.386606</td>
<td>1.45380</td>
</tr>
</tbody>
</table>
Analyzing the table, it is inferred that the wall heat flux decreases with increasing values of $R$ but increases with increasing $\lambda_1$. Furthermore, the rate of mass transfer at the sheet surface decreases with increasing magnetic parameter ($R$) but increases with increasing values of $\lambda_2$.

Table 3.2 Comparison of wall temperature gradient $-\theta(0)$ for various values of Pr and R when $N = Sc = Ec = L_1 = L_2 = \beta = 0$

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.72</td>
<td>0</td>
<td>0.45255</td>
<td>0.46314</td>
<td>0.46359</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>0.5820</td>
<td>0.59988</td>
<td>0.58197</td>
<td>0.58201</td>
</tr>
<tr>
<td>10.0</td>
<td>0</td>
<td>-</td>
<td>2.29589</td>
<td>2.30801</td>
<td>2.30800</td>
</tr>
<tr>
<td>0.72</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.39133</td>
</tr>
<tr>
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<td>5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.27893</td>
</tr>
<tr>
<td>1.0</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.50175</td>
</tr>
<tr>
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<td>-</td>
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<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.21756</td>
</tr>
<tr>
<td>10.0</td>
<td>5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1.99160</td>
</tr>
</tbody>
</table>
Table 3.2 compares results of heat transfer rates at the wall of a surface from the study to earlier published works for Newtonian fluids. It is observed that with Prandtl number, (Pr) values of 0.72, 1.0, and 10 without a magnetic parameter (R = 0), the results agreed well with Gupta and Gupta (1977), Ali (1995), Abel and Mahesha (2008). However, introducing the magnetic parameter R = 1, and R = 5, a considerable reduction in the wall temperature gradient, −θ(0), was recorded due to the Lorenz force produced by the transverse magnetic field which caused a reduction in the heat transfer rate at the plate surface.

3.5 Graphical Results and Discussions

The effects of the transverse magnetic field on velocity and temperature profiles are depicted in figures 3.2 and 3.3 respectively. It is observed that the transverse magnetic field causes the reduction in the velocity profiles and the thickening of the thermal boundary layer. This is evident from the fact that applied transverse magnetic field produces a body force, the Lorentz force, which opposes the motion. The resistance offered to the flow was responsible in enhancing the temperature profile.
Figure 3.2 Variation of dimensionless velocity profiles with increasing magnetic field strength

Figure 3.3 Variation of dimensionless temperature profiles with increasing magnetic field strength when $\lambda = Ec = 1$, $Pr = 0.71$, $N = 0.1$
Figure 3.4 Variation of dimensionless temperature profiles with increasing Eckert number when $\lambda_1 = 1, \text{Pr} = 0.71, N = R = 0.1$

Figure 3.5 Variation of dimensionless temperature profiles with increasing wall temperature exponent $\lambda_1$ when $N = 0.1, R = 0.1, \text{Pr} = 0.71, Ec = 1$. 
Figure 3.4 illustrates the effect of Eckert (Ec) on the fluid temperature. It was evident from this plot that large values of Ec number due to the increasing viscous dissipation resulted in thickening of the thermal boundary layer. Furthermore, the fluid temperature increases when the surface temperature increased as illustrated in figure 3.5.

The chemical concentration profiles are depicted in figures 3.6, 3.7 and 3.8. A general exponential decrease is observed in the concentration profiles. It is interesting to note further that the concentration boundary layer decreases with increasing values of Schmidt number (Sc), heat absorption parameter (N) and reaction rate parameter (β).

![Graph showing variation of concentration profiles with increasing reaction parameter](image)

**Figure 3.6** Variation of dimensionless concentration profiles with increasing reaction parameter when Sc = 0.6, R = 0.1, λ₂ = 1.
Figure 3.8 Variation of dimensionless concentration profiles with increasing wall concentration exponent parameter when \( \text{Sc} = 0.6, \beta = 0.5, R = 0.1 \)

3.6 Summary

The effects of MHD boundary layer flow of a chemically reacting fluid with heat and mass transfers past a stretching sheet has been presented. Numerical results were obtained for the heat and mass transfer rates at wall of the plates using the fourth order Runge-Kutta integration scheme and the shooting technique. Results of various parameter variations controlling the velocity, temperature and concentration profiles have been illustrated graphically and discussed. A comparison with previously published results gives a good agreement. The study revealed that the rate of cooling of objects moving in electrically conducting fluids can be controlled by applying transverse magnetic field to the flow direction. Furthermore, the presence of chemical reactive species in the fluid influences the rate of cooling which can be altered to achieve desired product characteristics.
CHAPTER 4

CHEMICALLY REACTING MHD BOUNDARY LAYER FLOW WITH OHMIC HEATING PAST A VERTICAL POROUS PLATE

4.0 Introduction

MHD boundary layer flow with simultaneous heat and mass transfer under the influence of chemical reaction is a common phenomenon in many transport processes. Practical areas of application include the petrochemical industry, cooling systems of nuclear reactors and in MHD generators. In this chapter, an analysis is conducted to investigate the combine effect of Arrhenius chemical reaction and Ohmic heating on MHD flow past a vertical porous plate. The problem is modeled as partial differential equations, reduced, coupled and solved both numerically and graphically. The forth order Runge–Kutta integration scheme with a modified version of the Newton–Raphson shooting technique is employed to solve the systems of ordinary differential equations describing the flow. The effect of various physical parameters on the velocity, temperature and concentration profiles is presented graphically and discussed. Numerical results were obtained for the local skin-friction coefficient and local Nusselt numbers with the variation of various physical parameters.
4.1 Problem Formulation

Consider the steady flow of incompressible, viscous, electrically conducting and chemically reacting fluid past a semi-infinite vertical porous plate, figure 4.1. A uniform transverse magnetic field of magnitude $H_0$ applied in the presence of thermal and solutal buoyancy effects is directed along the $y$–axis. The magnetic Reynolds number was assumed small so that the induced magnetic field and the Hall effects were neglected. The wall temperature and concentration, $T_0$ and $C_0$ respectively, were maintained constant and higher than the ambient temperature and concentration, $T_\infty$ and $C_\infty$ respectively. Furthermore, a homogeneous Arrhenius chemical reaction between the diffusing species and the fluid was assumed.

![Figure 4.1 Flow configuration and coordinate system](image-url)
The governing equations for the problem is based on similar models of Chen and Armaly (1987), Bestman (1990), as well as Pop and Ingham (2001) given as;

\[ \frac{\partial v}{\partial y} = 0, \quad (4.1) \]

\[ -v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma H_0^2}{\rho} u + g \beta_f (T - T_\infty) + g \beta_e (C - C_\infty), \quad (4.2) \]

\[ -v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\sigma H_0^2}{\rho c_p} u^2, \quad (4.3) \]

\[ -v \frac{\partial C}{\partial y} = \frac{D}{\partial} \frac{\partial^2 C}{\partial y^2} - \gamma (T - T_\infty)^n e^{\frac{-E}{R(T - T_\infty)}} (C - C_\infty), \quad (4.4) \]

The boundary conditions describing the flow condition is;

\[ u = 0, \quad v = V, \quad T = T_0, \quad C = C_0, \quad \text{on} \quad y = 0, \quad (4.5) \]

\[ T \rightarrow T_\infty, \quad C \rightarrow C_\infty \quad \text{as} \quad y \rightarrow \infty. \]

4.2 Analysis of the Problem

Introducing the following dimensionless variables, equations (4.1) - (4.5) are transformed to their corresponding dimensionless forms as:
Integrating equation (4.1) yields $v = -V$, where the negative sign indicates suction on the surface of the plate.

Also:

$$\frac{\partial \eta}{\partial y} = \frac{V}{\nu}, \quad u = Vw(\eta),$$

$$\frac{\partial u}{\partial y} = \nu \frac{\partial}{\partial \eta} \frac{\partial w}{\partial y} = \frac{V^2 w'}{\nu}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{V^2}{\nu} \frac{\partial w'}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{V^3}{\nu^2} w''.$$ 

$$T = (T_0 - T_\infty) \theta + T_\infty,$$

$$\frac{\partial T}{\partial y} = (T_0 - T_\infty) \theta' \frac{V}{\nu}, \quad \frac{\partial^2 T}{\partial y^2} = (T_0 - T_\infty) \theta'' \frac{V^2}{\nu^2},$$

$$C = (C_0 - C_\infty) \phi + C_\infty,$$

$$\frac{\partial C}{\partial y} = (C_0 - C_\infty) \phi' \frac{V}{\nu},$$

$$\frac{\partial^2 C}{\partial y^2} = (C_0 - C_\infty) \phi'' \frac{V^2}{\nu^2}.$$
Substituting relevant terms into equation (4.2) as;

\[ -V \frac{V^2 w'}{V^2} = \frac{V^3}{V^2} w' - \frac{\sigma H_o^2}{\rho} V w + g\beta_r (T_o - T_\infty) \theta + g\beta_c (C_o - C_\infty) \phi \]

\[ w'' + w' - \frac{\sigma B_0^2 \theta}{\rho V^2} w = -\frac{g\beta_r v (T_o - T_\infty)}{V^3} \theta - \frac{g\beta_c v (C_o - C_\infty)}{V^3} \phi \]

\[ w'' + w' - Haw = -Gr \theta - Gc \phi . \quad (4.8) \]

Also, equation (4.3) is transformed as:

\[ -V (T_o - T_\infty) \theta \frac{V}{V} = \alpha (T_o - T_\infty) \theta' \frac{V^2}{V^2} + \frac{V}{c_p} \left( \frac{V^2 w'}{V} \right)^2 + \frac{\sigma H_o^2}{\rho c_p} \frac{V^2}{c_p} (Vw)^2, \]

\[ \theta'' + \frac{\theta'}{\alpha} = -\frac{V^2}{c_p (T_0 - T_\infty) \alpha} \frac{\theta'}{w''} - \frac{\sigma H_o^2}{\rho V^2 c_p (T_0 - T_\infty) \alpha} \frac{V^2}{\theta} w'. \]

Thus, substituting relevant terms and parameters lead to:

\[ \theta'' + \text{Pr} \theta' + \text{Pr} Ec w'^2 + \text{Pr} EcHaw^2 = 0. \quad (4.9) \]

Whilst, equation (4.4) is transformed as:

\[ -V (C_o - C_\infty) \phi \frac{V}{V} = D (C_o - C_\infty) \phi' \frac{V^2}{V^2} - \gamma (T_o - T_\infty) \theta e^{\frac{E}{\alpha (T_0 - T_\infty) \theta}} (C_o - C_\infty) \phi, \]

\[ \phi'' + \frac{\phi'}{D} - \frac{\phi'}{V} (T_0 - T_\infty) \theta e^{\frac{E}{\alpha (T_0 - T_\infty) \theta}} \phi = 0, \]

Substituting relevant terms and parameters result in:
\[ \phi'' + Sc \phi' - Sc \beta \phi'' e^{\frac{-\xi}{\beta}} = 0. \] (4.10)

The boundary conditions were also transformed by noting that:

\[ w = 0, \ \theta = \phi = 1 \text{ and } \eta = 0 \text{ on } y = 0, \] (4.11)

\[ w = \theta = \phi = 0 \text{ and } \eta \to \infty \text{ at } y \to \infty, \]

Other physical quantities of interest in the problem namely; the local skin-friction parameter (\( \tau \)), the Nusselt number (Nu) and the Sherwood number (Sh) were easily computed. These quantities are defined in dimensionless terms as:

\[ \tau = w'(0), \ \text{Nu} = -\theta'(0) \text{ and } \text{Sh} = -\phi'(0). \] (4.12)

### 4.3 Numerical Procedure

The higher ordinary differential equations given in (4.8) - (4.10) were reduced to first order differential equations by letting;

\[ w = x_1, \ w' = x_2, \ \theta = x_3, \ \theta' = x_4, \ \phi = x_5, \ \phi' = x_6. \] (4.13)

Hence,
\[ x'_1 = x_2, \]
\[ x'_2 = -x_2 + Hax_1 - Grx_3 - Gcx_5, \]
\[ x'_3 = x_4, \]
\[ x'_4 = -Pr x_4 - Pr Ecx_2^2 - HaEc Pr x_1^2, \]
\[ x'_5 = x_6, \]
\[ x'_6 = -Scx_6 + \beta Scx_5 x_5^{(\frac{\varepsilon}{x_5})} e^{\frac{(-\varepsilon)}{x_5}}, \]

subject to the initial conditions;

\[ x_1(0) = 0, \quad x_2(0) = s_1, \quad x_3(0) = 1, \quad x_4(0) = s_2, \quad x_5(0) = 1, \quad x_6(0) = s_3. \]  

(4.15)

In the shooting method, the unspecified initial conditions; \( s_1, s_2 \) and \( s_3 \) were assumed and equation (4.14) integrated numerically as an initial valued problem to a given terminal point. The accuracy of the assumed missing initial conditions was checked by comparing the calculated value of the dependent variable at the terminal point with its given value there. Improved values of the missing initial conditions were obtained and the process repeated when differences existed. The computations were done by a written program which uses a symbolic and computational computer language MAPLE, Heck (2003). A step size of \( \Delta \eta = 0.001 \) was selected to be satisfactory for a convergence criterion of \( 10^{-7} \) in nearly all cases. The maximum value of \( \eta_\infty \), to each group of parameters \( Sc, Ha, Gr, Gc, Pr, Ec, \varepsilon, n \) and \( \beta \) was determined when the values of unknown boundary conditions at \( \eta = 0 \) not changed to successful loop with error less than \( 10^{-7} \).
4.4 Results and Discussion

Results were computed for a variety of physical parameters and tabulated and graphically illustrated. The results demonstrate the influence of thermal and concentration Grashof numbers (Gr, Gc), Prandtl number (Pr), magnetic field parameter (Ha), Schmidt number (Sc), Eckert number (Ec), chemical reaction parameter (β), activation energy parameter (ε), and Arrhenius reaction exponent (n) on the velocity, temperature and concentration profiles.

In the computations, the value of Pr was taken as 0.71 which corresponded to air while Sc was chosen in such a way that it represented the diffusing chemical species of most common interest in air like H₂, H₂O, NH₃ and Propyl Benzene. The Sc values for these chemical species are 0.24, 0.6, 0.78 and 2.62 respectively.

From table 4.1, it is observed that the skin-friction coefficient at the plate surface increases with increasing values of the Eckert number (Ec), the reaction exponent (n) but decreases with increasing values of the magnetic field parameter (Ha), the Schmidt number (Sc), and the reaction parameter (β). The rate of heat transfer $-\theta'(0)$ at the plate surface decreases with increasing values of the Ec, and n but increases with increasing values Ha, Sc, and β. Moreover, the rate of mass transfer at the plate surface increases with increasing values of Ec, Sc, and β but decreases with increasing values of Ha, and n. It was noted that the effect of the magnetic field on the heat transfer rate was augmented by the inclusion of Ohmic heating into the governing energy balance equation.
Table 4.1 Computations showing local skin friction, wall heat flux and mass transfer rate when ($Gr = 0.5$, $Gc = 0.1$, $Pr = 0.71$, $\varepsilon = 0.1$)

<table>
<thead>
<tr>
<th>$Ha$</th>
<th>$Ec$</th>
<th>$\beta$</th>
<th>$Sc$</th>
<th>$n$</th>
<th>$w'(0)$</th>
<th>$-\theta'(0)$</th>
<th>$-\phi'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>1.0</td>
<td>0.60</td>
<td>0.5</td>
<td>0.73129237975</td>
<td>0.69882387815</td>
<td>0.99309061742</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>1.0</td>
<td>0.60</td>
<td>0.5</td>
<td>0.54627778459</td>
<td>0.70296831665</td>
<td>0.99290333487</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1</td>
<td>1.0</td>
<td>0.60</td>
<td>0.5</td>
<td>0.44323034454</td>
<td>0.70508451207</td>
<td>0.99273073965</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5</td>
<td>1.0</td>
<td>0.60</td>
<td>0.5</td>
<td>0.75018667397</td>
<td>0.64785361914</td>
<td>0.99734820265</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
<td>1.0</td>
<td>0.60</td>
<td>0.5</td>
<td>0.77919326229</td>
<td>0.57251504194</td>
<td>1.00353942892</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>2.0</td>
<td>0.60</td>
<td>0.5</td>
<td>0.71097779030</td>
<td>0.69961768031</td>
<td>1.27324857159</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>3.0</td>
<td>0.60</td>
<td>0.5</td>
<td>0.698299969002</td>
<td>0.70008860370</td>
<td>1.49888448523</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>1.0</td>
<td>0.78</td>
<td>0.5</td>
<td>0.70984328441</td>
<td>0.69967529299</td>
<td>1.22087293049</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>1.0</td>
<td>2.62</td>
<td>0.5</td>
<td>0.65487958961</td>
<td>0.70148539207</td>
<td>3.27024187597</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>1.0</td>
<td>0.60</td>
<td>1.0</td>
<td>0.74196390680</td>
<td>0.69837480642</td>
<td>0.93015947272</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>1.0</td>
<td>0.60</td>
<td>1.5</td>
<td>0.74938849081</td>
<td>0.69805926787</td>
<td>0.88044648531</td>
</tr>
</tbody>
</table>
4.5 Graphical Results

The effects of varying various parameters on velocity profiles in the boundary layer are depicted in figures 4.2 – 4.12. From these figures, it is observed in all cases that the velocity starts from a minimum value of zero at the surface of the plate and increases to a peak value before reducing to a free stream value far away from the plate for all parameter values. Furthermore, increasing the magnetic field parameter (Ha) reduces the velocity profiles throughout the boundary layer. The magnetic field effects were more prominent at the point of peak values, (i.e. the peak value drastically decreased with increasing magnetic field strength). This has been attributed to the Lorentz force produced by the magnetic field which reduces the velocity profiles as shown in figure 4.2.

![Graphical Results]

Figure 4.2 Variation of dimensionless velocity profiles with increasing magnetic field strength when Ec = Gc = ε = 0.1 Gr = 0.5, Pr = 0.71, Sc = 0.6, β = n = 1.
Figure 4.3 Variation of dimensionless velocity profiles with increasing Eckert number when $Ha = Ge = e = 0.1$, $Gr = 0.5$, $Pr = 0.71$, $Sc = 0.6$, $\beta = n = 1$.

Furthermore, it is observed that increasing the Eckert number ($Ec$) increases the velocity profiles, figure 4.3. Increases of the Eckert number ($Ec$) due to viscous dissipation led to increases of the boundary layer thickness and a reduction in the heat transfer rate in the presence of thermal and solutal buoyancy forces. Figures 4.4 and 4.5 illustrate the influence of thermal buoyancy force parameter ($Gr$) and solutal buoyancy force parameter ($Gc$), respectively. The buoyancy forces enhance the fluid velocity and increase the boundary layer thickness.
Figure 4.4 Variation of dimensionless velocity profiles with increasing Gr when \( Ha = 0.1, Ec = 0.1, Gr = 0.1, Pr = 0.71, Sc = 0.6, \varepsilon = 0.1, \beta = 1, n = 1. \)

Figure 4.5 Variation of dimensionless velocity profiles with increasing Gc when \( Ha = 0.1, Ec = 0.1, Gr = 0.5, Pr = 0.71, Sc = 0.6, \varepsilon = 0.1, \beta = 1, n = 1. \)
Figure 4.6 Variation of dimensionless velocity profiles with increasing reaction rate when $Ha = \varepsilon = Ec = Gc = 0.1$, $Gr = 0.5$, $Pr = 0.71$, $Sc = 0.6$, $n = 1$.

Figure 4.7 Variation of dimensionless velocity profiles with increasing reaction exponent when $Ha = Gc = \varepsilon = Ec = 0.1$, $Gr = 0.5$, $Pr = 0.71$, $Sc = 0.6$, $\beta = 1$
Figures 4.6 and 4.7 illustrate the influence of chemical reaction rate parameter \((\beta)\) and the reaction exponent \((n)\) on the velocity profiles. The momentum boundary layer thickness decreased with increasing values of \(\beta\) and \(n\).

![Graph showing velocity profiles with increasing Schmidt number](image)

**Figure 4.8 Dimensionless velocity profiles with increasing Schmidt number \((Sc)\) when \(Ha = Ge = Ec = \varepsilon = 0.1, Gr = 0.5, Pr = 0.71, \beta = n = 1\).**

It is observed from figure 4.8 that at low values of Schmidt number \((Sc = 0.24)\), the peak velocity near the plate surface increases, whereas for higher values of Schmidt number \((Sc = 2.62)\), the peak velocity decreases. Furthermore, the momentum boundary layer thickness decreased with increasing values of \(Sc\).

Figure 4.9 depicts the variation of temperature profile against spanwise coordinate \((\eta)\) for various values of Eckert number \((Ec)\) when the other physical parameters are maintained constant.
Generally, the fluid temperature attains a maximum value at the plate surface and decreased exponentially to the free stream value of zero away from the plate. It is observed that increasing the Ec increases the temperature distribution throughout the boundary layer due to the increasing boundary layer thickness. The effect of reaction rate parameter $\beta$ on species concentration profiles for Arrhenius chemical reaction is shown in figure 4.10. It is observed that the species concentration is maximum at the start of the boundary layer and decreased slowly to a minimum value of zero at the end. This trend was observed for all the values of reaction rate parameters. Furthermore, increasing the value $\beta$ decreases the concentration of the species in the boundary layer. This is due mainly to the fact

Figure 4.9 Variation of temperature profiles with increasing Eckert number when $Ha = Gr = \varepsilon = 0.1$, $Pr = 0.71$, $Sc = 0.6$, $\beta = n = 1$. 

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that, destructive chemical reaction reduces the solutal boundary layer thickness and increases the mass transfer rate.

\[ \beta = 0.1 \]
\[ n = 0.1 \]
\[ n = 0.5 \]
\[ n = 1.0 \]
\[ n = 1.5 \]

Figure 4.10 Variation of concentration profiles with increasing reaction rate when \( Ha = \varepsilon = Ec = Gc = 0.1, Gr = 0.5, Pr = 0.71, Sc = 0.6, n = 1. \)

Figure 4.11 Variation of concentration profiles with increasing reaction exponent when \( Ha = \varepsilon = Ec = Gc = 0.1, Gr = 0.5, Pr = 0.71, Sc = 0.6, \beta = 1. \)
Figure 4.12 Variation of the concentration profiles with increasing Schmidt number (Sc) when \( Ha = g = Gc = Ec = 0.1, Gr = 0.5, Pr = 0.71, \beta = n = 1 \).

In figure 4.11, the concentration profiles are plotted against the span-wise coordinate (\( \eta \)) for various values of reaction index (\( n \)). It is observed that the concentration of the species increases with increasing reaction index (\( n \)) in the solutal boundary layer. Figure 4.12 illustrates the influence of Schmidt number (Sc) on the solutal boundary layer. Increasing the Sc decreases the concentration boundary layer thickness which is associated with the reduction in the concentration profiles. Physically, an increase in Sc meant a decrease in the molecular diffusion.
4.6 Summary

The combined effects of Arrhenius chemical reaction and magnetic field strength with Ohmic heating on a steady flow of electrically conducting fluid across a vertical porous plate have been investigated. The governing differential equations were rendered dimensionless and reduced to first order differential equations. These equations were coupled and solved numerically using the fourth order Runge-Kutta integration scheme with the shooting method. The results for the skin friction co-efficient, the surface heat flux, and the rate of mass transfer at the plate surface were analyzed. The velocity, temperature and concentration profiles were illustrated graphically. It was observed that the velocity and concentration profiles decreased as the chemical reaction parameter increases. Similar observations were made for the Schmidt number and the magnetic parameter variations. The reverse trend was observed when the Eckert number, the thermal Grashof number and the solutal Grashof number were increased.
CHAPTER 5

CHEMICALLY REACTING MHD BOUNDARY LAYER FLOW PAST A MOVING LOW - HEAT - RESISTANT SHEET

5.0 Introduction

An investigation is presented for the cooling of a low - heat - resistant sheet moving vertically downwards in a fluid permeated by transverse magnetic field. The shooting technique based on the Newton-Raphson algorism and the modified version of Runge-Kutta integration scheme were employed to obtain numerical results for the skin friction coefficient, and the heat and mass transfer rates at the plate surface. The effects of varying various physical parameters on the velocity, temperature and concentration profiles have been illustrated graphically and discussed quantitatively.

5.1 Problem Formulation

Consider the steady flow of a viscous, incompressible and chemically reacting fluid past a semi-infinite vertically moving flat plate, figure 5.1. A uniform transverse magnetic field \( (H_0) \) is applied in the presence of thermal buoyancy effects in the direction of y – axis. Assuming a small magnetic Reynolds number, the induced magnetic field and the Hall effects can be neglected. The wall
temperature and concentration $T$ and $C$ are assumed linear but varied along the plate with $T_\infty$ and $C_\infty$ as the ambient temperature and concentration respectively.

The governing differential equations modeling the flow structure are based on the conservation laws of mass, linear momentum, energy and concentration as:

\begin{align}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma H_0^2}{\rho} u + g \beta_T (T - T_\infty) + g \beta_C (C - C_\infty), \\
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\sigma H_0^2}{\rho c_p} u^2, \\
\frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} &= D \frac{\partial^2 C}{\partial y^2} - \gamma (C - C_\infty).
\end{align}
The boundary conditions for the flow problem are:

\[ u = Bx, v = 0, T = T_w = ax + T_\infty, C = C_w = bx + C_\infty \text{ at } y = 0, \]

\[ u \to 0, T \to T_\infty, C \to C_\infty \text{ as } y \to \infty, \quad (5.5) \]

where \( u \) and \( v \) represent the velocity components in the \( x \) and \( y \) directions respectively. \( T, T_w, \) and \( T_\infty \) represent the fluid temperature, wall temperature and the ambient temperature respectively. \( C, C_w, \) and \( C_\infty \) represent the fluid concentration, wall concentration, and the ambient concentration at a distant location from the surface. \( \beta, \nu, \alpha \) and \( D \) represent the thermal expansion coefficient, kinematic viscosity, thermal diffusivity and the mass diffusivity while \( a \) and \( b \) are constants. In this study, the surface temperature \( (T_w(x)) \) and concentration \( (C_w(x)) \) are considered greater than the ambient temperature \( (T_\infty) \) and concentration \( (C_\infty) \) respectively. Under these conditions, the free convective motion of the fluid is upward along the plate, as shown in figure 5.1. In a reverse scenario, where the surface is colder than the ambient temperature and concentration, \( (T_w(x) < T_\infty \text{ and } C_w(x) < C_\infty) \) the boundary layer profile remained the same but the direction will reverse, Yang et al (1982). That is, the fluid flow will appear downwards.
5.2 Analysis of the Problem

The partial differential equations (5.1) to (5.4) admit self similar solutions of the form;

\[ \eta = \gamma \sqrt{\frac{B}{v}}, \quad \psi = x \sqrt{vB} f(\eta), \quad \theta(\eta) = \frac{T - T_w}{T_w - T_\infty}, \quad \phi(\eta) = \frac{C - C_w}{C_w - C_\infty}, \quad (5.6) \]

where \( \eta, \psi, \theta, \) and \( \phi \) are the dimensionless variable, the dimensionless stream function, the dimensionless temperature and the dimensionless concentration respectively. The stream function relates to the velocity components in the usual way as;

\[ u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \quad (5.7) \]

This satisfies the continuity equation (5.1) automatically. That is;

\[ u = \frac{\partial \psi}{\partial y} = x \sqrt{vB} f' \sqrt{\frac{B}{v}} = xBf', \]

and

\[ \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) = Bf', \quad (5.8) \]

\[ v = -\frac{\partial \psi}{\partial x} = -\sqrt{B} f, \]

and

\[ \frac{\partial v}{\partial y} = -\sqrt{B} f' \sqrt{\frac{B}{v}} = -Bf'. \]

Thus

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = Bf' - Bf' = 0. \]
By noting that:

\[ u = \frac{\partial u}{\partial y} = xB f', \quad \frac{\partial u}{\partial y} = xB f'' \sqrt{\frac{B}{v}} = x \sqrt{\frac{B^3}{v}} f'' = \sqrt{\frac{B^3}{v}} f'' = \frac{xB^2}{v} f''. \]  

Equation (5.2) becomes;

\[ xB f' f' + v \frac{B}{v} f' = v \frac{B^2}{v} f'' - \frac{\sigma H_0^2}{\rho} xB f' + g \beta_r (T_w - T_\infty) \theta + g \beta_c (C_w - C_\infty) \phi. \]  

This implies,

\[ xB^2 f'' + xB^2 f''' = xB^2 f'' - \frac{\sigma H_0^2}{\rho} xB f' + g \beta_r (T_w - T_\infty) \theta + g \beta_c (C_w - C_\infty) \phi. \]  

Dividing through by xB^2 and re-arranging

\[ f''' + f'' - f'' = - \frac{\sigma H_0^2}{\rho B} f' + \frac{g \beta_r (T_w - T_\infty)}{xB^2} \theta + \frac{g \beta_c (C_w - C_\infty)}{xB^2} \phi. \]  

Hence,

\[ f''' + f'' - f'' = - \frac{\sigma H_0^2}{\rho B} f' + \frac{g \beta_r (T_w - T_\infty)}{xB^2} \theta + \frac{g \beta_c (C_w - C_\infty)}{xB^2} \phi = 0, \]  

where \( Ha = \frac{\sigma H_0^2}{\rho B} \), \( G = \frac{g \beta_r (T_w - T_\infty)}{xB^2} \), and \( G_c = \frac{g \beta_c (C_w - C_\infty)}{xB^2} \) represent the Hartmann number, the thermal Grashof number and the concentration Grashof number respectively.

Furthermore, by noting that;

\[ T = T_\infty + (T_w - T_\infty) \theta, \quad T = T_w = T_\infty + ax, \quad \frac{\partial T}{\partial x} = a, \]  

\[ \frac{\partial T}{\partial y} = (T_w - T_\infty) \theta' \sqrt{\frac{B}{v}}, \quad \frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty) \theta' \sqrt{\frac{B}{v}}. \]
Equation (5.3) becomes;

\[ xBf' - (\sqrt{B \nu f}) (T_w - T_\infty) \theta' \sqrt{\frac{B}{\nu}} = \alpha (T_w - T_\infty) \frac{\theta' B}{\nu} + \frac{\sigma H^2}{\rho c} x^2 B^2 f'^2, \]

\[ xBaf' - B(T_w - T_\infty) f \theta' = \frac{\alpha}{\nu} B(T_w - T_\infty) \theta'' + \frac{\sigma H^2}{\rho c} x^2 B^2 f'^2, \]

Dividing through by

\[ \frac{\alpha}{\nu} B(T_w - T_\infty), \]

and rearranging,

\[ \theta'' + \frac{\nu}{\alpha} f \theta' - \frac{\nu}{\alpha (T_w - T_\infty)} f' + \frac{\nu}{\alpha} \frac{\sigma H^2}{\rho c} \frac{x^2 B^2}{c_p (T_w - T_\infty)} f'^2 = 0. \]

From the boundary condition,

\[ \alpha x = T_w - T_\infty, \]

\[ \theta'' + \text{Pr} f \theta' - \text{Pr} f' + \text{HaPr} E \text{cf}'' = 0, \quad (5.13) \]

where

\[ \text{Pr} = \frac{\nu}{\alpha}, \quad \text{Ha} = \frac{\sigma H^2}{\rho B} \quad \text{and} \quad \text{Ec} = \frac{x^2 B^2}{c_p (T_w - T_\infty)}, \]

represent the Prandtl number the Hartmann number and the Eckert number respectively.

The concentration equation (5.4) is transformed to dimensionless form by noting that;

\[ C = C_\infty + (C_w - C_\infty) \phi, \quad C = C_w = C_\infty + bx, \quad (5.13) \]

\[ \frac{\partial C}{\partial x} = b, \quad \frac{\partial C}{\partial y} = (C_w - C_\infty) \phi' \sqrt{\frac{B}{\nu}}, \quad \frac{\partial^2 C}{\partial y^2} = (C_w - C_\infty) \phi'' \frac{B}{\nu}. \]
Thus;

\[ xBf' - \sqrt{B\nu} f(C_w - C_\infty)\phi' \sqrt{\frac{B}{\nu}} = D(C_w - C_\infty)\phi'' \frac{B}{\nu} - \gamma(C_w - C_\infty)\phi, \]

\[ xBf' - B(C_w - C_\infty) f\phi' = DB \frac{D}{\nu} (C_w - C_\infty)\phi'' - \gamma(C_w - C_\infty), \]

\[ \phi'' + Scf\phi' - Scf' - Sc\beta \phi = 0, \quad (5.14) \]

where, \( Sc = \frac{\nu}{D} \) and \( \beta = \frac{\gamma}{B} \) represent the Schmidt number and the reaction rate parameter respectively.

The primes in equations (5.10), (5.12) and (5.14) indicate differentiation with respect to \( \eta \). The boundary conditions (5.5) are also transformed as:

\[ f'(0) = 1, f(0) = 0, \theta(0) = 1, \phi(0) = 1, \text{ on } \eta = 0, \]

\[ f'(\eta) \to 0, \theta(\eta) \to 0, \phi(\eta) \to 0 \text{ as } \eta \to \infty. \quad (5.15) \]

The other physical quantities of interest in the problem namely; the skin friction parameter \( (\tau) \), the Nusselt number \( (Nu) \) and the Sherwood number are easily computed. These quantities are defined in dimensionless terms as:

\[ \tau = f''(0), \quad Nu = -\theta'(0), \quad Sh = -\phi'(0), \quad (5.16) \]
5.3 Numerical Procedure

The systems of ordinary differential equations (5.10), (5.12) and (5.14) obtained as higher order were reduced to first order differential equations. These equations were then solved numerically using the forth order Runge–Kutta integration scheme with a modified version of the Newton–Raphson shooting technique.

Letting:

\[ f = x_1, \quad f' = x_2, \quad f'' = x_3, \quad \theta = x_4, \quad \theta' = x_5, \quad \phi = x_6, \quad \text{and} \quad \phi' = x_7. \]  

(5.17)

The corresponding first order differential equations were obtained as

\[
\begin{align*}
x_1' &= x_2, \\
x_2' &= x_3, \\
x_3' &= -x_1 x_3 + x_2^2 + H a x_2 - G r x_4 - G c x_6, \\
x_4' &= x_5, \\
x_5' &= -P r x_1 x_3 + P r x_2 + P r H a E c x_2^2, \\
x_6' &= x_7, \\
x_7' &= -S c x_1 x_7 + S c x_2 + S c f x_6,
\end{align*}
\]

subject to the following initial conditions

\[ x_1(0) = 0, \quad x_2(0) = 1, \quad x_3(0) = s_1, \quad x_4(0) = 1, \quad x_5(0) = s_2, \quad x_6(0) = 1, \quad x_7(0) = s_3. \]

(5.19)

In the shooting method, the unspecified initial conditions \( s_1, \ s_2 \) and \( s_3 \) were assumed and equation (5.18) integrated numerically as an initial valued problem to a given terminal point. The accuracy of the assumed missing initial conditions was checked by comparing the calculated value of the dependent variable at the terminal point with its given value there. If a difference existed, improved values of the missing initial conditions must be obtained and the process was repeated.
The computations were done by a written program which uses a symbolic and computational computer language MAPLE, Heck (2003). A step size of $\Delta \eta = 0.001$ was selected to be satisfactory for a convergence criterion of $10^{-7}$ in nearly all cases. The maximum value of $\eta_0$ to each group of parameters $Ha$, $Ec$, $Gr$, $Gc$ and $Pr$ were determined when the values of unknown boundary conditions at $\eta = 0$ not changed to successful loop with error less than $10^{-7}$.

5.4. Results and Discussion

Numerical and graphical results were obtained for the heat and mass transfer rates of the chemically reacting fluid pasts a moving vertical plate in the presence of transverse magnetic field. The results illustrated the influence of thermal Grashof number ($Gr$), Prandtl number ($Pr$), solutal Grashof number ($Gc$), magnetic field parameter ($Ha$), Schmidt number ($Sc$), and Eckert number ($Ec$), on the velocity, temperature and the concentration profiles.

5.4.1 Computational Results

Table 5.1 shows the numerical results for various parameter variations. The value of the Prandtl ($Pr$) was taken to be 0.71 and 7.1 which corresponds to air and water respectively. It is observed that the magnitude of the skin friction coefficient at the surface of the plate increases with increasing values of the Hartmann number ($Ha$).
Table 5.1 Computations showing the variation in local skin friction, Nusselt number and Sherwood number for $Sc = 0.24$

<table>
<thead>
<tr>
<th>Ha</th>
<th>Gr</th>
<th>Gc</th>
<th>Pr</th>
<th>Ec</th>
<th>$\beta$</th>
<th>$f''(0)$</th>
<th>$-\theta'(0)$</th>
<th>$-\phi'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>0.1</td>
<td>-0.867231515</td>
<td>0.845372426</td>
<td>0.453884876</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1</td>
<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>0.1</td>
<td>-1.304041183</td>
<td>0.769606228</td>
<td>0.396845801</td>
</tr>
<tr>
<td>2.0</td>
<td>0.1</td>
<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>0.1</td>
<td>-1.636322453</td>
<td>0.715256170</td>
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</tr>
<tr>
<td>3.0</td>
<td>0.1</td>
<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>0.1</td>
<td>-1.914170216</td>
<td>0.674830166</td>
<td>0.341732155</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>0.1</td>
<td>-0.383965869</td>
<td>0.952761742</td>
<td>0.526868457</td>
</tr>
<tr>
<td>0.1</td>
<td>2.0</td>
<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>0.1</td>
<td>0.117635777</td>
<td>1.023730668</td>
<td>0.573999041</td>
</tr>
<tr>
<td>0.1</td>
<td>3.0</td>
<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>0.1</td>
<td>0.575405411</td>
<td>1.076218586</td>
<td>0.608106764</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>1.0</td>
<td>0.71</td>
<td>0.1</td>
<td>0.1</td>
<td>-0.503692401</td>
<td>0.910555760</td>
<td>0.490199921</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>2.0</td>
<td>0.71</td>
<td>0.1</td>
<td>0.1</td>
<td>-0.096333584</td>
<td>0.964542811</td>
<td>0.521869527</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>3.0</td>
<td>0.71</td>
<td>0.1</td>
<td>0.1</td>
<td>0.280567922</td>
<td>1.006308898</td>
<td>0.546212748</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>1.00</td>
<td>0.1</td>
<td>0.1</td>
<td>-0.924967648</td>
<td>1.031442438</td>
<td>0.444296214</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>5.00</td>
<td>0.1</td>
<td>0.1</td>
<td>-0.946942273</td>
<td>2.585623333</td>
<td>0.440220191</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>7.10</td>
<td>0.1</td>
<td>0.1</td>
<td>-0.950260912</td>
<td>3.126407697</td>
<td>0.439933962</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.71</td>
<td>1.0</td>
<td>0.1</td>
<td>-0.951456214</td>
<td>3.282065838</td>
<td>0.439783615</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.71</td>
<td>2.0</td>
<td>0.1</td>
<td>-0.952782046</td>
<td>3.454820684</td>
<td>0.439616724</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.71</td>
<td>3.0</td>
<td>0.1</td>
<td>-0.954105483</td>
<td>3.627367015</td>
<td>0.439450002</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>1</td>
<td>-0.966443205</td>
<td>3.120700638</td>
<td>0.668349553</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.71</td>
<td>0.1</td>
<td>3</td>
<td>-0.979117864</td>
<td>3.116315582</td>
<td>0.979307236</td>
</tr>
</tbody>
</table>
However, both the Nusselt and Sherwood numbers, which represent the rates of heat and mass transfer at the plate surface, respectively decreased with increasing values of Hartmann number (Ha).

Furthermore, increasing the thermal and solutal Grashof numbers increase the skin friction coefficient and the Nusselt and Sherwood numbers. Similarly, increasing the Prandtl number (Pr) increases the local skin friction coefficient and the Nusselt number but decreases the Sherwood number. It is further observed that increasing the Eckert number increases the skin friction co-efficient and the Nusselt number but decreases the Sherwood number. Finally, the reaction rate parameter was observed to increase the skin friction parameter and the Sherwood numbers while causing a reduction in the Nusselt numbers for obvious reasons.

Table 5.2 Comparison of non-dimensional wall velocity gradient $F'(0)$ for values of the Hartmann number (Ha) when Pr = Sc = Ec = Gr = Gc = 0

<table>
<thead>
<tr>
<th>Ha</th>
<th>0.0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Takhar et al (1986)</td>
<td>-1.0</td>
<td>-1.22</td>
<td>-1.41</td>
<td>-1.58</td>
<td>-1.73</td>
</tr>
<tr>
<td>Yih (1999)</td>
<td>-1.0</td>
<td>-1.2247</td>
<td>-1.4142</td>
<td>-1.5811</td>
<td>-1.7321</td>
</tr>
<tr>
<td>Abdelkalek (2009)</td>
<td>-1.0</td>
<td>-1.2356</td>
<td>-1.4156032</td>
<td>-1.58212</td>
<td>-1.73342</td>
</tr>
<tr>
<td>Present study</td>
<td>-1.0</td>
<td>-1.22474</td>
<td>-1.4142136</td>
<td>-1.5811388</td>
<td>-1.7320508</td>
</tr>
</tbody>
</table>
Table 5.3 Comparison of non-dimensional wall temperature gradient for various values of the Prandtl number for $\text{Ha} = \text{Sc} = \text{Ec} = \text{Gr} = \text{Gc} = 0$

<table>
<thead>
<tr>
<th>Pr</th>
<th>0.72</th>
<th>1</th>
<th>3</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grubka et al (1985)</td>
<td>0.8086</td>
<td>1.0</td>
<td>1.9237</td>
<td>3.7207</td>
<td>12.294</td>
</tr>
<tr>
<td>Ali (1994)</td>
<td>0.8058</td>
<td>0.9961</td>
<td>1.9144</td>
<td>3.7006</td>
<td>-</td>
</tr>
<tr>
<td>Yih (1999)</td>
<td>0.8086</td>
<td>1.0</td>
<td>1.9237</td>
<td>3.7207</td>
<td>12.294</td>
</tr>
<tr>
<td>Abdelkalek (2009)</td>
<td>0.80865</td>
<td>1.0</td>
<td>1.9246</td>
<td>3.7216</td>
<td>12.29453</td>
</tr>
<tr>
<td>Present study</td>
<td>0.808834</td>
<td>1.0</td>
<td>1.923679</td>
<td>3.7206712</td>
<td>12.294081</td>
</tr>
</tbody>
</table>

Tables 5.2 and 5.3 compare the results of the present study to previously published works in the literature. It is observed to agree with earlier works of Grubka et al (1985), Ali (1994), Yih (1999), and Abdelkalek (2009) under similar conditions.

5.4.2 Effect of parameter variation on velocity profiles

The effect of various parameters on velocity profiles in the boundary layer region was depicted in figures 5.2 to 5.6. It was observed that the velocity started from a maximum value of 1 at the plate surface and decreased exponentially to a free stream value of zero, satisfying the far field boundary conditions for all parameter
values. Moreover, it was noted in figure 5.2 that the effect of increasing the magnetic field parameter (Ha) was to decrease the velocity profiles throughout the boundary layer region. A resistive force called the Lorentz force was introduced into the electrically conducting fluid due to the transverse magnetic field.

Figure 5.2 Velocity profile for Pr = 0.71, Gr = 1, Gc = 1, Sc = 0.24, Ec = 1
Figure 5.3 Velocity profile for Pr = 0.71, Ha = 0.1, Gc = 1, Sc = 0.24, Ec = 1.

Figure 5.4 Velocity profile for Gr = 1, Pr = 0.71, Sc = 0.24, Ec = 0.1, Ha = 0.1.
In figures 5.3 and 5.4, the buoyancy force parameters $Gr$ and $Gc$ were noted to enhance the fluid velocity resulting in increased boundary layer thickness. The velocity profiles started at one on the plate surface and increased to a peak value before reducing to the free stream value of zero.

Figure 5.5 Velocity Profile for $Ha = 0.1$, $Gr = 1$, $Gc = 1$, $Ec = 1$, $Pr = 0.71$. 

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Figures 5.5 and 5.6 showed the influence of increasing the Schmidt number and Prandtl number on the velocity fields. Increasing the Schmidt number caused a reduction in the velocity profile whereas increasing the Prandtl number decreased the velocity profile.

5.4.3 Effect of Parameter Variation on Temperature Profiles

It was observed from figures 5.7 to 5.11 that, the fluid temperature attained a maximum value at the plate surface and decreased exponentially to the free stream zero value away from the plate. In figure 5.7, the thermal boundary layer increased when the strength of the applied magnetic field was increased.
Figure 5.7 Temperature profile for $Pr = 0.71$, $Gr = 1$, $Gc = 1$, $Sc = 0.24$, $Ec = 0.1$

Figure 5.8 Temperature profile for $Gr = Ha = 0.1$, $Pr = 0.71$, $Ec = 1$, $Sc = 0.24$. 

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Figure 5.9 Temperature profile for Sc = 0.24, Ha = 0.1, Gc = Ec = 1, Pr = 0.71

Figures 5.8 and 5.9 showed that the thermal boundary layer thickness decreased when the buoyancy force parameters, Gr and Gc increased.

The influence of Prandtl number (Pr) and the Schmidt number were illustrated in figures 5.10 and 5.11. The temperature profiles were observed to decrease when the Prandtl number was increased but increased when the Schmidt number increased.
Figure 5.10 Temperature profile for $Gr = Gc = Ec = 1$, $Ha = 0.1$, $Sc = 0.24$

Figure 5.11 Temperature profile for $Gr = Gc = Ec = 1$, $Ha = 0.1$, $Pr = 0.71$
5.4.4 Effect of Parameter Variation on Concentration Profiles

Generally, the species concentration in the fluid had a maximum value at the plate surface and decreased exponentially to the free stream value of zero away from the plate. These were observed in figures 5.12 - 5.16. In figure 5.12, the solutal boundary layer was observed to increase with increasing magnetic field strength whilst in figures 5.13 and 5.14, it was observed to decrease with increasing buoyancy force parameters, Gr and Gc.

Fig 5.12 Concentration Profile for Gr = 1, Gc = 1, Pr = 0.71, Sc = 0.24, Ec = 1

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Figure 5.13 Concentration Profile for $Ha = 0.1, Pr = 0.71, Gr=Ec=1, Sc=0.24$.

Figure 5.14 Concentration Profile for $Ha = 0.1, Pr = 0.71, Gr=Ec=1, Sc=0.24$. 
Figure 5.15 Concentration Profile for $Ha = 0.1$, $Gr = Ge = 1$, $Ec = 0.1$, $Pr = 0.71$

Figure 5.16 Concentration Profile for $Ha = 0.1$, $Gr = Ge = 1$, $Ec = 0.1$, $Sc = 0.24$
It was further observed that increasing the Schmidt number reduced the concentration boundary layer as shown in figure 5.15 whilst changes in Prantl numbers caused a slight variation in the concentration boundary layer thickness as illustrated in figure 5.16.

5.5 Summary

In this chapter, the governing differential equations modeling the effect of transverse magnetic field on a low – heat – resistant sheet have been coupled and solved numerically using the forth order Runge-Kutta shooting method. Numerical and graphical results were obtained for various parameter values. It was found that, increasing buoyancy force parameters reduced the resistance offered by the fluid and hence increases the velocity field. This also resulted in a reduction of the thermal boundary layer due to convective cooling.
CONCLUSIONS AND RECOMMENDATIONS

6.0 Introduction

This chapter presents the conclusions of the study and the specific contributions made. It also outlines some recommendations for future work.

6.1 Conclusions

It is concluded by this study that both transverse magnetic field and chemical concentration in the boundary layer region significantly affects the cooling rate of a heated surfaces. The transverse magnetic field increases the skin-friction coefficient at the surface thereby increasing the resistance to flow leading to reductions of velocity profiles. As a result, the rate of mass and heat transfers decrease at the plate surface. Furthermore, the profiles depicting the velocity, temperature and concentration boundary layers converged away from the plate satisfying the free stream conditions. The presence of destructive chemical species in the boundary layer region further reduces the resistance offered by the flow to the object leading to a reduction in the skin-friction co-efficient. Furthermore, suction allows cross boundary flow and therefore enhances temperature distribution within the boundary layer which reduces the resistance to flow.
6.2 Major Contributions

The thesis has made three main contributions to boundary layer research. The first contribution is the development of appropriate mathematical models to describe boundary layer flows. The second contribution is the rigorous analysis of the nonlinear differential equations modeling boundary layer flow. The third is the use of these models to analyze the effects of transverse magnetic field and chemical reaction on the boundary layer flow for vertical and horizontal surfaces.

The specific contributions made include:

i) The design of non-linear mathematical models for boundary layer flow.

ii) The analysis of non-linear mathematical model for a chemically reacting MHD boundary layer flow past a stretching sheet.

iii) The analysis of non-linear mathematical model for a chemically reacting MHD boundary layer flow pasts a low-heat-resistant sheet.

iv) The analysis of non-linear mathematical model for a chemically reacting MHD boundary layer flow with Ohmic heating across a vertical plate.

v) The development of a theoretical framework to predict the transfer of heat and mass towards a wall in transport phenomena.

vi) Determination of the effects of various physical parameters on the boundary layer flow control.

vii) The thesis would serve as reference material for future research.
6.3 Future work

The study established the influence of transverse magnetic field and chemical reaction on boundary layer flow control. Future research can be extended to include several engineering and industrial flow systems such as nanofluids (a mixture of nano sized particles and fluid base flow heat transfer enhancement).

Specifically, future research can investigate the following areas of boundary layer.

i. Steady MHD boundary layer flow over inclined planes with chemical reaction.

ii. Hydromagnetic boundary layer flow on vibrating surfaces.

iii. Effects of turbulence on MHD Boundary layer flow.

iv. Unsteady MHD boundary layer flow with chemical reaction.

v. Analyses of three dimensional boundary layer flow.

vi. Boundary layer flow with particle suspensions and heat generation.

vii. Radiation and mass transfer effects on steady MHD free convective fluid flow embedded in porous media with heat absorption.
References:


Das UN, Deka RK, Soundalgekar VM (1994). Effects of mass transfer on flow past an impulsively started infinite vertical plate with constant heat flux and chemical reaction. Forschung im-Ingenieurwesen, 60: 284-287


Elbashbeshy EMA, Aldawody DA (2011), Effects of thermal radiation and magnetic field on unsteady mixed convection flow and heat transfer over a porous


Prandtl L (1904). Über Flussigkeits bewegung bei sehr kleiner Reibung Verhaldlg III Int. Math. Kong. pp 484–491; Also available in translation as: Motion of fluids with very little viscosity, NACATM (1928).452.


Prandtl L (1925). Bericht über untersuchungen zur ausgebildete Turbulenz ZAMM 5: 136 –139.


Appendix I: Publications


Appendix II: Conservation laws

In all of physics there are only six conservation laws, each describing a conserved quantity, i.e. the total amount is the same before and after something occurs.

These laws have the restriction that the system is closed, that is, the system is not affected by anything outside it. There are:

Conservation of charge  Conservation of angular momentum
Conservation of momentum  Conservation of baryons
Conservation of mass/energy  Conservation of leptons

Conservation of Charge

Used in chemistry. The total charge in the system is conserved.

Conservation of Momentum

Momentum, \( p \), (a vector) equals mass, \( m \), (a scalar) times velocity, \( v \), (a vector).

\[
(p = m v)
\]

Conservation of Energy/Mass

\[
KE_i + RME_i = KE_f + RME_f.
\]
Appendix III: Fick’s law of mass diffusivity

1. The general expression for Fick’s first law is stated as:

\[ \mathbf{J} = D \nabla c \]

Where \( \mathbf{J} \) is the flux vector (units: number \( m^{-2}s^{-1} \)), \( D \) is the diffusivity tensor (units: \( m^2s^{-1} \)), and \( \nabla c \) is the concentration gradient (units: number \( m^{-4} \)). Note that instead of “number” the units could involve mass, e.g., kg.

2. The expression for Fick’s second law is:

\[ \frac{\partial c}{\partial t} = -\nabla \cdot \mathbf{J} \]

Where \( c \) is the concentration (units: number \( m^{-3} \)), \( t \) is the time (units: s), \( \nabla \cdot \mathbf{J} \) is the divergence of the flux (units: number \( m^{-3}s^{-1} \)). Fick’s second law is a statement of the conservation of mass (with no allowance for production).
Appendix IV: Maple code for graphical results of model 1

# Please note the followings:
# u(y) represents velocity profile
# theta(y) represents temperature profile

with(plots):
N:=0.1:E1:=1:Pr:=0.71:Sc:=0.6:beta:=0.5:R:=0.1:m:=sqrt(1+R):L1:=1:L2:=1:
fcns:={theta(y),phi(y)}:
f:=(1-exp(-m*y))/m:
f1:=diff(f,y):
f11:=diff(f1,y):

sys:=diff(theta(y),y$2)+N*Pr*theta(y)+Pr*E1*(f11^2+R*f1^2)-Pr*(L1*theta(y)*f1-f*diff(theta(y),y))=0,
diff(phi(y),y$2)-beta*Sc*phi(y)-Sc*(L2*phi(y)*f1-f*diff(phi(y),y))=0:

p1:=dsolve({sys,phi(0)=1,theta(0)=1,phi(10)=0,theta(10)=0},fcns,type=numeric,
method=bvp,abserr=1e-10):

plt:=odeplot(p1,[y,theta(y)],0..10,numpoints=50,labels=["y","Temperature"],style
=line,color=black):
plc:=odeplot(p1,[y,phi(y)],0..10,numpoints=50,labels=["y","Concentration"],style
=line,color=black):

with(plots):
N:=0.1:E1:=1:Pr:=0.71:Sc:=0.6:beta:=0.5:R:=0.5:
m:=sqrt(1+R):L1:=1:L2:=1:
fcns:={theta(y),phi(y)}:
f:=(1-exp(-m*y))/m:
f1:=diff(f,y):
f11:=diff(f1,y):
sys :=diff(theta(y),y$2)+N*Pr*theta(y)+Pr*E1*(f11^2+R*f1^2)-
Pr*(L1*theta(y)*f1-f*diff(theta(y),y))=0,diff(phi(y),y$2)-beta*Sc*phi(y)-
Sc*(L2*phi(y)*f1-f*diff(phi(y),y))=0:
p2:=dsolve({sys,phi(0)=1,theta(0)=1,phi(10)=0,theta(10)=0},fcns,type=numeric,
method=bvp,abserr=1e-10):
p2t:=odeplot(p2,[y,theta(y)],0..10,numpoints=50,labels=["y","Temperature"],style
=point,symbol=circle,color=black):
p2c:=odeplot(p2,[y,phi(y)],0..10,numpoints=50,labels=["y","Concentration"],style
=point,symbol=circle,color=black):

with(plots):
N:=0.1:E1:=1:Pr:=0.71:Sc:=0.6:beta:=0.5:R:=1:m:=sqrt(1+R):L1:=1:L2:=1:
fcns:={theta(y),phi(y)}:
f:=(1-exp(-m*y))/m:f1:=diff(f,y):f11:=diff(f1,y):

sys:=diff(theta(y),y$2)+N*Pr*theta(y)+Pr*E1*(f11^2+R*f1^2)-
Pr*(L1*theta(y)*f1-f*diff(theta(y),y))=0,diff(phi(y),y$2)-beta*Sc*phi(y)-
Sc*(L2*phi(y)*f1-f*diff(phi(y),y))=0:
p3:=dsolve({sys,phi(0)=1,theta(0)=1,phi(10)=0,theta(10)=0},fcns,type=numeric,
method=bvp,abserr=1e-10):
p3t:=odeplot(p3,[y,theta(y)],0..10,numpoints=50,labels=["y","Temperature"],style
=point,symbol=cross,color=black):
p3c:=odeplot(p3,[y,phi(y)],0..10,numpoints=50,labels=["y","Concentration"],style
=point,symbol=cross,color=black):
with(plots):
N:=0.1:E1:=1:Pr:=0.71:Sc:=0.6:beta:=0.5:R:=1.5:m:=sqrt(1+R):L1:=1:L2:=1:
fens:={theta(y),phi(y)}:
f:=(1-exp(-m*y))/m:
fl:=diff(f,y):
fl1:=diff(fl,y):
sys:=diff(theta(y),y$2)+N*Pr*theta(y)+Pr*E1*(f11^2+R*f1^2)
Pr*(L1*theta(y)*f1-f*diff(theta(y),y))=0,diff(phi(y),y$2)-beta*Sc*phi(y)-
Sc*(L2*phi(y)*f1-f*diff(phi(y),y))=0:
p4:=dsolve({sys,phi(0)=1,theta(0)=1,phi(10)=0,theta(10)=0},fcns,type=numeric,
method=bvp,abserr=1e-10):
p4t:=odeplot(p4,[y,theta(y)],0..10,numpoints=50,labels=["y","Temperature"],style
=point,symbol=point,color=black):
p4c:=odeplot(p4,[y,phi(y)],0..10,numpoints=50,labels=["y","Concentration"],style
=point,symbol=point,color=black):
plots[display]({p1t,p2t,p3t,p4t});
plots[display]({p1c,p2c,p3c,p4c});

Appendix V: Maple code for numerical results for model 1
> N := 1; Pr := 0.71; Sc := 1; beta := 1; Ec := 1; R := 1; L1 := 1; L2 := 1;
fcns := {F(y), phi(y), theta(y)};
sys1:=diff(F(y),y$3)-(diff(F(y),y))^2+F(y)*(diff(F(y),y$2))-R*(diff(F(y),
y))=0,diff(theta(y),y$2)+Pr*F(y)*(diff(theta(y),y))-Pr*L1*theta(y)*(diff(F(y),
y))+Ec*Pr*(diff(F(y),y$2))^2+Pr*R*Ec*(diff(F(y),y))^2+N*Pr*theta(y)=0,
diff(phi(y),y$2)-beta*Sc*phi(y)+Ec*F(y)*(diff(phi(y),y))-

- 137 -
\[
S_c \phi(y) L^2 (\text{diff}(F(y), y)) = 0, \quad F(0) = 0, \quad \phi(0) = 1, \quad \phi(10) = 0, \quad \theta(0) = 1, \\
\theta(10) = 0, \quad (D(F))(0) = 1, \quad (D(F))(10) = 0;
\]

\[
p1 := \text{dsolve} \{\text{sys1}, \ F(0) = 0, \ \phi(0) = 1, \ \phi(10) = 0, \ \theta(0) = 1, \ \theta(10) = 0, \\
(D(F))(0) = 1, \quad (D(F))(10) = 0\}, \text{fcns, type = numeric, method = bvp, abserr = 0.1e-6} ;
\]

# input placeholder

\[
\text{proc(x\_bvp) ... end;}
\]

\[
> \text{dsol1} := \text{dsolve} \{\text{sys1}\}, \text{numeric, output = operator};
\]
Appendix VI: Maple code for Numerical computation of model 2

> Gc := 0.1; Gr := 0.5; Pr := 0.71; Sc := 0.6; beta := 3.0; Ec := 0.1; Ha := 0.1; N := 0.5;
> e := 0; n := 0.1;

fcns := \{phi(y), w(y), theta(y)\};

sys1 := diff(w(y), y$2) + diff(w(y), y) - Ha * w(y) + Gr * theta(y) + Gc * phi(y) = 0,
diff(theta(y), y$2) + Pr * (diff(theta(y), y)) + Pr * Ec * (diff(w(y), y))$2 + Pr * Ec * Ha * w(y)$2 = 0,
diff(phi(y), y$2) + Sc * (diff(phi(y), y)) - Sc * beta * phi(y) * theta(y) * n * e - e / theta(y) = 0,

w(0) = 0, w(10) = 0, phi(0) = 1, phi(10) = 0,
theta(0) = 1, theta(10) = 0;

p1 := dsolve({sys1, phi(0) = 1, phi(10) = 0, w(0) = 0, w(10) = 0, theta(0) = 1, theta(10) = 0},
fcns, type = numeric, method = bvp, abserr = 0.1e-6);

# input placeholder

> dsol1 := dsolve({sys1}, numeric, output = operator);
Appendix VII: Maple code for Numerical computation of model 3

> R := 1; Pr := .72; Sc := 0; Ec := 0; beta := 0; Gc := 3; Gr := 0;
fone := {F(y), phi(y), theta(y)};
sys1 := diff(F(y), y$3) + F(y)*(diff(F(y), y$2)) - R*(diff(F(y), y)) - (diff(F(y), y))^2 + Gr*theta(y) + Gc*phi(y) = 0,
diff(theta(y), y$2) + Pr*F(y)*(diff(theta(y), y)) - Pr*theta(y)*(diff(F(y), y)) -
Ec*Pr*R*(diff(F(y), y))^2 = 0,
diff(phi(y), y$2) + Sc*F(y)*(diff(phi(y), y)) -
Sc*phi(y)*(diff(F(y), y)) - Sc*beta*phi(y) = 0,
F(0) = 0, phi(0) = 1, phi(10) = 0, theta(0) = 1, theta(10) = 0, (D(F))(0) = 1, (D(F))(10) = 0;
pl := dsolve({sys1, F(0) = 0, phi(0) = 1, phi(10) = 0, theta(0) = 1, theta(10) = 0, (D(F))(0) = 1, (D(F))(10) = 0}, fone, type = numeric, method = bvp, abserr = 0.1e-6);
# input placeholder
proc(x_bvp) ... end;
> dsol1 := dsolve({sys1}, numeric, output = operator);
Appendix VIII: Maple code for graphical results of model 3

> with(plots);

R := .1; Pr := .7; Sc := .24; Ec := 1; beta := 1; Gr := 1; Gc := 1;

cns := \{F(y), \phi(y), \theta(y)\};

sys := diff(F(y), y$3) + F(y) * diff(F(y), y$2) - R * (diff(F(y), y)) - (diff(F(y), y))^2 + Gr * \theta(y) + Gc * \phi(y) = 0, diff(\theta(y), y$2) + Pr * F(y) * (diff(\theta(y), y)) - (diff(\theta(y), y))^2 = 0, diff(\phi(y), y$2) + Sc * F(y) * (diff(\phi(y), y)) - Sc * \phi(y) * (diff(F(y), y)) - Sc * beta * \phi(y) = 0;

p1 := dsolve({sys, F(0) = 0, \phi(0) = 1, \phi(10) = 0, \theta(0) = 1, \theta(10) = 0, (D(F))(0) = 1, (D(F))(10) = 0}, cns, type = numeric, method = bvp, abserr = 0.1e-9);

p1t := odeplot(p1, \{y, \theta(y)\}, 0 .. 10, numpoints = 50, labels = ["y", "Temperature"], style = point, symbol = asterisk, color = black);

p1c := odeplot(p1, \{y, \phi(y)\}, 0 .. 10, numpoints = 50, labels = ["y", "Concentration"], style = point, symbol = asterisk, color = black);

p1f := odeplot(p1, \{y, (D(F))(y)\}, 0 .. 10, numpoints = 50, labels = ["y", "velocity"], style = point, symbol = asterisk, color = black);

> with(plots);

R := .1; Pr := 1; Sc := .24; Ec := 1; beta := 1; Gr := 1; Gc := 1;

cns := \{F(y), \phi(y), \theta(y)\};

sys := diff(F(y), y$3) + F(y) * diff(F(y), y$2) - R * (diff(F(y), y)) - (diff(F(y), y))^2 + Gr * \theta(y) + Gc * \phi(y) = 0, diff(\theta(y), y$2) + Pr * F(y) * (diff(\theta(y), y)) - (diff(\theta(y), y))^2 - 141 -
Pr*theta(y)*(diff(F(y), y)) - Ec*Pr*R*(diff(F(y), y))^2 = 0, diff(phi(y), y$2)$
2 + Sc*F(y)*(diff(phi(y), y)) - Sc*phi(y)*(diff(F(y), y)) - Sc*beta*phi(y) = 0;

p2 := dsolve({sys, F(0) = 0, phi(0) = 1, phi(10) = 0, theta(0) = 1, theta(10) = 0,
(D(F))(0)=1,(D(F))(10)=0}, fcns, type = numeric, method = bvp, abserr = 0.1e-9);
p2t := odeplot(p2, [y, theta(y)], 0 .. 10, numpoints = 50, labels = ["y", "Temperature"], style = point, symbol = cross, color = black);
p2c := odeplot(p2, [y, phi(y)], 0 .. 10, numpoints = 50, labels = ["y", "Concentration"], style = point, symbol = cross, color = black);
p2f := odeplot(p2, [y, (D(F))(y)], 0 .. 10, numpoints = 50, labels = ["y", "velocity"], style = point, symbol = cross, color = black);

> with(plots);

R := .1; Pr := 3; Sc := .24; Ec := 1; beta := 1; Gr := 1; Gc := 1;
fCONS := {F(y), phi(y), theta(y)};
sys:=diff(F(y),y$3)+F(y)*(diff(F(y),y$2))-R*(diff(F(y),y))-(diff(F(y), y))$2+Gr*theta(y)+Ge*phi(y) = 0, diff(theta(y), `$`2(y, 2))+Pr*F(y)*(diff(theta(y), y))-Pr*theta(y)*(diff(F(y), y))-Ec*Pr*R*(diff(F(y), y))^2=0,

diff(phi(y), y$2)+Sc*F(y)*(diff(phi(y), y))-Sc*phi(y)*(diff(F(y), y))-Sc*beta*phi(y) = 0;

p3 := dsolve({sys, F(0) = 0, phi(0) = 1, phi(10) = 0, theta(0) = 1, theta(10) = 0,
(D(F))(0)=1,(D(F))(10)=0}, fcns, type = numeric, method = bvp, abserr = 0.1e-9);
p3t := odeplot(p3, [y, theta(y)], 0 .. 10, numpoints = 50, labels = ["y", "Temperature"], style = point, symbol = circle, color = black);
p3c := odeplot(p3, [y, phi(y)], 0 .. 10, numpoints = 50, labels = ["y", "Concentration"], style = point, symbol = circle, color = black);
p3f := odeplot(p3, [y, (D(F))(y)], 0 .. 10, numpoints = 50, labels = ["y", "velocity"], style = point, symbol = circle, color = black);
> with(plots);
R := .1; Pr := 7.1; Sc := .24; Ec := 1; beta := 1; Gr := 1; Gc := 1;
fcns := {F(y), phi(y), theta(y)};
sys := diff(F(y), y$3) + F(y) * (diff(F(y), y$2) - R * (diff(F(y), y)) - (diff(F(y), y))^2 + Gr * theta(y) + Gc * phi(y) = 0, diff(theta(y), y$2) + Pr * F(y) * (diff(theta(y), y)) - Pr * theta(y) * (diff(F(y), y)) - Ec * Pr * R * (diff(F(y), y))^2 = 0, diff(phi(y), y$2) + Sc * F(y) * (diff(phi(y), y)) - Sc * phi(y) * (diff(F(y), y)) - Sc * beta * phi(y) = 0;
p4 := dsolve({sys, F(0) = 0, phi(0) = 1, phi(10) = 0, theta(0) = 1, theta(10) = 0, (D(F))(0) = 1, (D(F))(10) = 0}, fcns, type = numeric, method = bvp, abserr = 0.1e-9);
p4t := odeplot(p4, [y, theta(y)], 0 .. 10, numpoints = 50, labels = ["y", "Temperature"], style = point, symbol = diagonalcross, color = black);
p4c := odeplot(p4, [y, phi(y)], 0 .. 10, numpoints = 50, labels = ["y", "Concentration"], style = point, symbol = diagonalcross, color = black);
p4f := odeplot(p4, [y, (D(F))(y)], 0 .. 10, numpoints = 50, labels = ["y", "velocity"], style = point, symbol = diagonalcross, color = black);
plots[display]([p1t, p2t, p3t, p4t]);
plots[display]([p1c, p2c, p3c, p4c]);
plots[display]([p1f, p2f, p3f, p4f]);