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AN OVERVIEW OF STOCHASTIC PROCESSES AFFECTING SUSPENDED SEDIMENT PRODUCTION IN ENVIRONMENTAL ENGINEERING; CASE OF NORTHERN GHANA

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ABSTRACT

Modelling of environmental processes within a basin is an important aspect of environmental engineering. Since most of these processes are stochastic in nature, the application of a stochastic model for an input-output process like the production levels of suspended sediment in a river basin resulting from some amount of rainfall is very important. The problem was therefore to establish a model relating daily rainfall depth with suspended sediment concentration (SSC). The Nasia River Basin was therefore used as a study area as it presents a typical northern savanna situation. The vastness of the river basin, the remote location of the rainguage and the complexity associated with runoff process of SSC informed the model being stochastic in nature. Data which formed important aspect of the study included rainfall and suspended sediment concentration levels in the river channel over a period of 90 days in the year 2007. A black-box type model with model parameters of $\lambda = 0.9829$, $\kappa = 0.7117$, $\alpha = 2.846$, $\gamma = 0.7404$ and $\sigma = 0.6098$ were identified for the period understudy. The black-box therefore serves as a reasonable model for the processes involved in the production of suspended sediment. The stochastic process C has a tendency to revert to a value depending on log S. Therefore, the system perturbed from the equilibrium state of equation 13 and equation 15 becomes

Further investigations especially on SSC and rainfall are recommended especially for the study area.

Key words: Environmental, Stochastic, Basin, Suspended, Sediment Concentration.

1.0 INTRODUCTION

In environmental engineering there are many situations where stochastic processes appear. For example, the simple population growth model is written as

$$\frac{dN}{dt} = a(t)N(t), \quad N(0) = N_0$$

Equ. 1

where $N(t)$ is the size of population at time t, $a(t)$ the relative rate of growth at time t, and N_0 is a given initial value. In most cases, $a(t)$ contains some random component whose exact behaviour is not known. Therefore, $a(t)$ should be "identified" in a statistic sense, and then equation (1) should be "solved" in the context of stochastic differential equations, which are essentially different from classical deterministic differential equations.

A review of stochastic processes according to the works of Ɜksendal (2005) has been considered in relation to this work. An approach is then presented to apply another stochastic model to time series data of rainfall and suspended sediment concentration (SSC) in the basin. The model represents an input-output relationship between rainfall and SSC, and the model parameters are identified from observed data via a linear regression model. The transfer function from an input to an output can be used in assessment problems, as in Unami and Kawachi (2005).

1.2 Stochastic Processes

A stochastic process is a parameterized collection of random variables $\{X_t\}_{t\in T}$, where T is the parameter space. An important example of stochastic process is the Brownian motion. The n-dimensional Brownian motion $\{B_t\}_{t\geq 0}$ starting at x_0 is the stochastic process whose finite-dimensional distributions for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$ are given by

$$P^{x_0}\left(B_{t_1} \in F_1, \cdots, B_{t_k} \in F_k\right) = \int_{F_1\times\cdots\times F_k} p(t_1, x_0, x_1)\, p(t_2 - t_1, x_1, x_2) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k)\, dx_1 \cdots dx_k$$

Equ. 2

where $F_1, \cdots, F_k$ denote Borel sets in $\mathbb{R}^n$ and

$$p(t, x_i, x_j) = (2\pi t)^{-n/2} \exp\left(-\frac{|x_i - x_j|^2}{2t}\right)$$ Equ. 3

for $t > 0$ and $p(0, x_i, x_j)dx_j = \delta_{x_i}(x_j)$, the unit point mass at x_i. Some basic properties of Brownian motion are

$$E^x[B_t] = x, \quad E^x[(B_t - x)(B_s - x)] = ns, \quad E^x\left[(B_t - B_s)^2\right] = n(t - s)$$ Equ. 4

where $t \geq s \geq 0$ and E^x denotes expectation with respect to P^x.

An *n*-dimensional Ito diffusion is the solution **X** of a stochastic differential equation of the form

$$d\mathbf{X} = \mathbf{u}(t, \mathbf{X})dt + v(t, \mathbf{X})d\mathbf{B}$$ Equation 5

where **u** is the *n*-dimensional drift coefficient vector, *v* the *n*Ǹ*m* diffusion coefficient matrix, and **B** is the *m*-dimensional Brownian motion. When a smooth function $g(t, \mathbf{x})$ is applied to **x** = **X**, the Ito formula states that

$$dg(t, \mathbf{X}) = \frac{\partial g}{\partial t}(t, \mathbf{X})dt + \frac{\partial g}{\partial \mathbf{x}}(t, \mathbf{X})d\mathbf{X} + \frac{1}{2}\frac{\partial^2 g}{\partial \mathbf{x}^2}(t, \mathbf{X})d\mathbf{X}^2$$ Equ. 6

with the Ito rules

$$dt^2 = 0, \quad dtd\mathbf{B} = d\mathbf{B}dt = \mathbf{0}, \quad d\mathbf{B}^2 = Idt$$ Equ. 7

where *I* is the *m*Ǹ*m* unit matrix.

2.0 RESULTS

2.1 Modeling and Parameter Identification of SSC in the Nasia River

Nasia River Basin, whose total area is 5,339km^2, is in Northern Region of Ghana. From April 1 (Julian day 91) through June 30 (Julian day 181), 2007, runoff water from the basin was sampled on daily basis, and its Suspended Sediment Concentration (SSC) measured in the laboratory. A raingauge of 0.2 mm tipping bucket type is located at the Gung site of Bontanga River Basin, which is outside of Nasia River Basin, and a pulse logger is connected to it so that time series data of rainfall intensity is obtained at the maximum resolution. The rainfall data and the SSC data were therefore obtained to help model and identify the various parameters relating two. The problem was therefore to establish a model relating daily rainfall depth with daily SSC produced for the river. However, it is easily imagined that the model should be stochastic because of the vastness of the basin, the remoteness of the raingauge, and complexity of runoff process of SSC.

The cumulative rainfall depth S_t (mm) with a decay effect is recursively defined as;

$$S_{t+1} = R_t + \lambda S_t \qquad \text{Equ. 8}$$

where;

R_t is the rainfall depth at the tth day, and λ is the decay coefficient. The SSC observed on the tth day is denoted by C_t (mg/L), and its logarithm is denoted by X_t. A linear regression model with residuals ε_t

$$X_{t+1} = f_0 + f_1 X_t + f_2 R_t + f_3 \log S_t + \varepsilon_t \qquad \text{Equ. 9}$$

where;

f_i (i = 0, 1, 2, 3) and of which are constant parameters and considered to relate rainfall with SSC.

Using the least squares minimizing $\frac{1}{2}\sum_t \left(f_0 + f_1 X_t + f_2 R_t + f_3 \log S_t - X_{t+1}\right)^2$, the constant parameters f_i are estimated as the solution to;

$$\begin{pmatrix} \sum_t 1 & \sum_t X_t & \sum_t R_t & \sum_t \log(S_t) \\ \sum_t X_t & \sum_t X_t^2 & \sum_t X_t R_t & \sum_t X_t \log(S_t) \\ \sum_t R_t & \sum_t R_t X_t & \sum_t R_t^2 & \sum_t R_t \log(S_t) \\ \sum_t \log(S_t) & \sum_t \log(S_t) X_t & \sum_t \log(S_t) R_t & \sum_t (\log(S_t))^2 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \sum_t X_{t+1} \\ \sum_t X_t X_{t+1} \\ \sum_t R_t X_{t+1} \\ \sum_t \log(S_t) X_{t+1} \end{pmatrix}.$$

Equation 10

The decay coefficient λ is chosen so that the constant parameter f_2 vanishes, and equation 9 results in

$$X_{t+1} = X_t + \kappa(\beta + \gamma \log S_t - X_t)\Delta t + \varepsilon_t \qquad \text{Equ 11}$$

where;

Δt is the sampling interval of the time series (1 day), and κ, β, and γ are constants.

Equating the right hand sides of equation 9 and 11, these constants are obtained as;

$\kappa = \dfrac{1-f_1}{\Delta t}$, $\beta = \dfrac{f_0}{1-f_1}$, and $\gamma = \dfrac{f_3}{1-f_1}$.

When the residuals ε_t obey to the normal distribution, equation 11 is regarded as a discretized version of the stochastic differential equation;

$$dX = \kappa(\beta + \gamma \log S - X)dt + \sigma dB \qquad \text{Equ. 12}$$

where σ is the standard deviation of ε_t, and B is the Brownian motion. Therefore, fitness of the residuals ε_t to the normal distribution is an essential criterion for appropriateness of equation 12. The Ito formula transfers equation 12 into the geometric stochastic differential equation

$$dC = \kappa(\alpha + \gamma \log S - \log C)Cdt + \sigma CdB \qquad \text{Equ. 13}$$

where

$$\alpha = \beta + \frac{\sigma^2}{2\kappa}. \qquad \text{Equ. 14}$$

The continuous version of equation 8 is given as;

$$dS = \left(\frac{\lambda - 1}{\Delta t}S + \frac{1}{\Delta t}R\right)dt \qquad \text{Equ. 15}$$

and equation 13 constitute a system whose input and output are R and C, respectively.

Figure 1 shows the observed and generated data of rainfall and SSC from April 1 through December 31, 2007. The daily rainfall depths are available since September 2005, and there was no rain from November 6, 2006 through March 20, 2007. Therefore, S_t on March 21, 2007 is set as 0, and S_t generated by equation 8 is depicted as the black line in the Figure. The dots represent observed SSCs, from which the model parameters are identified as in Table 1. The chi square test proves fitness of the residuals ε_t to the normal distribution. The grey line in the Figure describes the nominal SSCs generated by equation 13 and 15.

Figure 1: Time series data of rainfall and SSC

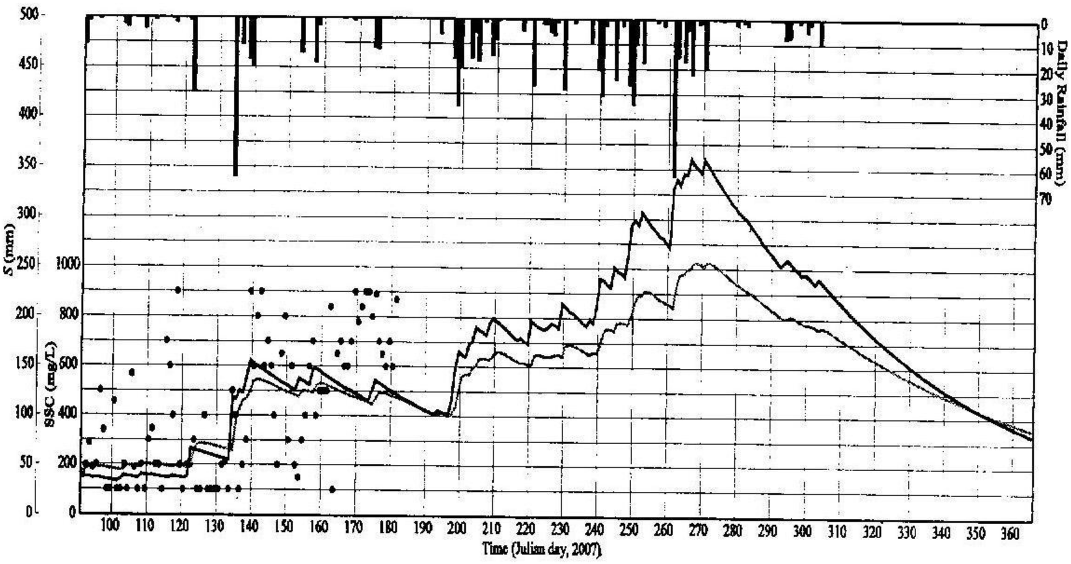

Table 1: Identified model parameters

λ	κ	α	γ	σ
0.9829	0.7117	2.846	0.7404	0.6098

3.0 DISCUSSIONS AND CONCLUSIONS

The population growth model presented as equation 1 is now rewritten as;

$$dN = b(t)N(t)dt + c(t)N(t)dB, \quad N(0) = N_0$$

Equ. 16

when $a(t)dt$ is split into the deterministic part $b(t)dt$ and the stochastic part $c(t)dB$.

The geometric stochastic differential equation for SSC of equation 13 is a special case of equation 16 where $N = C$, $b(t) = \kappa(\alpha + \gamma \log S - \log C)$, and $c(t) = \sigma$. Though the model does not fully explain physical process of SSC runoff in the basin, it serves as a reasonable black-box type model for the processes. However, it is not appropriate to extrapolate the ranges of *S* and *C*, as have been considered in this case. Perennial acquisition of data is imperative for further consideration in the modeling.

The stochastic process *C* has a tendency to revert to a value depending on log *S*. Therefore, the system perturbed from the equilibrium state of equation 13 and equation 15 becomes

$$\begin{cases} d\delta S = \left(-\frac{1-\lambda}{\Delta t}\delta S + \frac{1}{\Delta t}\delta R \right) dt \\ d\delta C = \kappa\left(\gamma\frac{\delta S}{S_0} - \frac{\delta C}{C_0} \right) C_0 dt + \sigma C_0 dB \end{cases}$$

Equ. 17

where prefix δ and subscript 0 represent perturbed and equilibrium values, respectively.

The transfer function *P* from δR to δC is

$$P = \frac{\kappa\gamma C_0}{\Delta t S_0 \left(s + \frac{1-\lambda}{\Delta t} \right)(s + \kappa)}$$

Equation 18

where *s* is the frequency. Since $0 < \lambda < 1$, $0 < \kappa$ and $0 < \gamma$, the transfer function equation 18 is strictly proper and stable.

For rigorous researchers in environmental engineering, treating uncertain phenomena in the context of stochastic processes is inevitable and necessary.

From the results obtained however it is suggested that further investigations are highly recommended.

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