UNIVERSITY FOR DEVELOPMENT STUDIES



# A TWO- PARAMETER GENERALISATION OF THE GAMMA FUNCTION AND ITS PROPERTIES

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BY

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THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, FACULTY OF PHYSICAL SCIENCES, UNIVERSITY FOR DEVELOPMENT STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF MASTER OF PHILOSOPHY IN MATHEMATICS

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# DECLARATION

**Student**

I hereby declare that this thesis is the result of my original work and that no part of it has been presented for another degree in this University or elsewhere:

Candidate

Signature: ............................................... Date : 27-01-2026

Name: Mohammed Hashimu

**Supervisor**

I hereby declare that the preparation and presentation of the thesis were supervised following the guidelines on supervision of thesis laid down by the University for Development Studies.

Signature: ............................................... Date :.......................................................

Name: Prof. Mohammed Muniru Iddrisu

# ABSTRACT

This thesis presents a significant contribution to the field of special functions by introducing and thoroughly analyzing a novel two-parameter generalization of the Gamma function: the ( $\lambda$, v)-generalized Gamma function. The research successfully addressed a critical gap in the literature by unifying previously disparate generalizations, such as the $\lambda$-analogue and $v$-analogue Gamma functions. The thesis meticulously presented the ($\lambda$, v) -generalized Gamma function through a modified integral representation and derived several key properties, including recurrence relations and integral representations. A core strength of the work lies in its exploration of the properties and associated inequalities of the ( $\lambda$, v)-generalized Gamma function. Key analytical tools employed include **Hölder's** inequality and **Young's inequality**, along with techniques of integration and differentiation. The proofs provided are rigorous and clearly presented.

# ACKNOWLEDGEMENT

I would like to express my sincere gratitude to my supervisor, Professor Mohammed Muniru Iddrisu, for his invaluable guidance, support, and insightful feedback throughout this research. His expertise and encouragement were instrumental in the completion of this thesis. I also thank my family and friends for their unwavering support and encouragement.

# DEDICATION

To Allah, the All-Knowing, the All-Wise, I dedicate this humble endeavor. May this work serve as a testament to His boundless blessings and infinite wisdom.

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# CHAPTER ONE

# INTRODUCTION

This chapter serves as an introduction to the Gamma function, detailing its significance in mathematics and its historical development. It outlines the limitations inherent in the classical Gamma function, particularly its restriction to positive numbers, which necessitates the exploration of new generalizations. The objectives of the research are clearly stated, aiming to introduce the $(\lambda, v)$-generalized Gamma function and to derive new inequalities associated with this function. Key research questions guide the inquiry, focusing on the properties and applications of this new function. The scope is limited to the Gamma function defined on the positive real axis, highlighting its relevance in various mathematical contexts. This chapter concludes with a summary of the thesis structure, providing readers with a roadmap for the ensuing discussions.



## 1.1 Background of the Study

The factorial function, a cornerstone of elementary mathematics, finds its roots in the counting of permutations and combinations. Defined initially for non-negative integers, its elegant simplicity contrast with its profound importance across diverse mathematical domains. However, the inherent discreteness of the factorial function limits its direct applicability to continuous mathematical contexts. The need for a continuous analogue, capable of extending the factorial function's properties to the realm of real and complex numbers, became apparent as mathematics progressed beyond its elementary stages. This crucial need spurred the development of the gamma function, a pivotal achievement in mathematical analysis that continues to shape research across numerous fields (Davis, 1959). Introduced by Leonhard Euler in 1729 (Euler, 1729), the gamma function, denoted $\Gamma(z)$, elegantly extends the factorial function to the complex plane, with the exception of non-positive integers. Euler's initial definition, expressed through an integral representation,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \tag{1}$$

provides a powerful tool for bridging the gap between discrete and continuous mathematics. This integral representation, while seemingly simple, encapsulates a wealth of profound mathematical properties and far-reaching consequences. The functional equation $\Gamma(z+1) = z\Gamma(z)$ establishes the fundamental link between the gamma function and the factorial function, highlighting the seamless transition between discrete and continuous frameworks (Artin, 1964). This relationship reinforces the gamma function's importance as a merging notion in mathematics. Later centuries witnessed the determined investigation of the gamma function's characteristics and uses. Mathematicians like Carl Friedrich Gauss, Joseph-Louis Lagrange made an important impact to its theoretical comprehending (Gauss, 1812). Henri Poincar , Augustin-Louis Cauchy also supplemented its uses within integral transforms and complex analysis, uncovering its crucial role in the solution of differential equations and the evaluation of intricate integrals (Cauchy, 1825). The gamma function's leverage broad into the 20th century, becoming a basic element in the progression of statistics and probability concept, especially in the investigation of the gamma distribution and related probability models (Whittaker & Watson, 1927). Its usages permeate fields as diverse as number theory, combinatorics, and the study of special functions, forming its position as, indeed, an elementary mathematical object.

Notwithstanding its peculiar characteristics and broad relevance, the classical gamma function has intrinsic limitations. Its definition specifically excludes non-positive integers from its domain, a restriction that motivates the study of generalizations and extensions. The passion to extend the gamma function's features and applicability to progressively specialized mathematical contexts has pushed the progress of a huge collection of analogues. These concepts, constantly encouraged by specific applications or theoretical studies, have importantly improved our understanding of the gamma function's establishing structure and expanded its potential for applications in various study (Miller, 2006). Noticeable among these generalizations are the $p$-analogue (Euler), the $q$-analogue (Jackson, 1910), the $k$-analogue (Diaz & Pariguan, 2007), the $(p, k)$-analogue (Nantomah, 2017), the $(q, k)$-analogue (Diaz & Teruel, 2005),$(p, q, k)$-analogue (Ege, 2019), the $v$-analogue (Djabang et al., 2020), the $b$-analogue (Chaudhry & Zubair, 1995), and the $\lambda$-analogue (Nantomah & Ege, 2022). Each of these generalizations gives special characteristics and expands the applicability of the gamma function to anoter mathematical domains.

This thesis contributes to this ongoing investigation by presenting a two-parameter generalization of the classical Euler gamma function, the $(\lambda, v)$-generalized gamma function. This generalization decisively connects elements of the prrviously studied $\lambda$-analogue and v-analogue gamma functions, focusing on generating a function with enlightened properties and broader usages than its predecessors. The $(\lambda, v)$-generalized gamma function is defined via a modified integral representation, carefully constructed to extend the core properties of the classical gamma function while simultaneously incorporating the unique characteristics of its $\lambda$- and $v$-analogue counterparts. This reflective unification requires to attain the strengths of both mother functions and to generate another function with enhanced properties and expanded usages. The main focus of this research is to provide an extensive analysis of this new function, as well as the derivation of its key properties, the exploration of its associated inequalities, and the investigation of its relationship to existing gamma function generalizations. This thorough investigation helps to contribute importantly to the continuing development of the theory of generalized gamma functions and to explain their possible usages in a wide cluster of mathematical and scientific applications. The results presented in this thesis are anticipated to have deep implications for both theoretical mathematics and applied fields where the gamma function and its generalizations play a critical role. The investigation of the $(\lambda, v)$-generalized gamma function presents a remarkable opportunity to advance our understanding of this basic mathematical object and its different usages.

## 1.2 Statement of the Problem

The Gamma function $\Gamma(z)$ is an extremely math mechanism. It is like a higher version of the factorial function ($Example, 3! = 3\times2\times1$), but operates on all numbers, not only on whole numbers. It is used everywhere in statistics, figuring out probabilities, and solving complex equations (Artin, 1964; Abramowitz & Stegun, 1964). But the regular Gamma function has limits; it only works well with positive numbers. Scientists have been working on better versions, called "analogues," to make it more useful (Jackson, 1910; Daz & Pariguan, 2007).

These improved Gamma functions are like different versions of the same basic tool, each with its own strengths. Some are good for quantum physics problems (Jackson, 1910), others for specific types of equations (Diaz & Pariguan, 2007), and so on. We have a lot of these different versions, but they are often studied separately. We do not have a good overview of how they all relate to each other (Nantomah, 2017; Djabang et al., 2020; Nantomah & Ege, 2022). This makes it hard to find the best tool for a given problem and limits our understanding of the Gamma function itself. This is a big problem because understanding the connections between these different Gamma functions could lead to even better, more powerful tools. Imagine having a toolbox with lots of hammers, but not knowing which one is best for which nail! We need a better way to organize and understand all these different "hammers". Also, many studies only look at one improved Gamma function at a time. We must examine the wider picture and determine how they are all related. This study directly addresses this issue. By combining the best features of two existing versions, the lambda-analogue and the v-analogue, we are developing an additional, super Gamma function. The lambda-analogue gamma function is defined as (Nantomah & Ege, 2023)

$$\Gamma_\lambda(x) = \int_0^\infty e^{-\lambda t} t^{x-1}\, dt. \tag{2}$$

where $\lambda > 0$ is a parameter that modifies the scale of the function. The v-analogue gamma function, on the other hand, is given by (Djabang et al., 2020)

$$\Gamma_v(\delta) = \int_0^\infty e^{-t} \left(\frac{t}{v}\right)^{\frac{\delta}{v}-1} dt. \tag{3}$$

where $0 < v < 1$ is another parameter that introduces a different type of scaling. By integrating the strengths of these two formulations, as explored in the works of Djabang et al. (2020) and Nantomah & Ege (2023), we aim to establish a more versatile gamma function that retains the desirable properties of both predecessors. This new function has the potential to be significantly more effective and versatile. To provide a more cohesive and potent approach to this significant mathematical tool, this study attempts to explore a novel, two-part gamma function.



## 1.3 Objectives

The objective of this study is to present a two-parameter generalized gamma function. Specifically, the study seeks :

- To define a $(\lambda, v)$-generalized gamma function.
- To derive some properties and inequalities involving the new function.

## 1.4 Research Questions

The research seeks to answer the following questions.

- What new $(\lambda, v)$ gamma function and its properties can be established?
- What new inequalities can be derived from the two-parameter analogue of the Gamma function?

## 1.5 Scope of Study

All works concerning this research is limited to the Gamma function defined on the positive real axis.

## 1.6 Significance of the Study

The research findings have contributed significantly to knowledge in the sense that

- the new function and its analytical properties serve as an advancement in knowledge in the subject area.
- the results serve as a reference material for further research.

## 1.7 Organization of the Study

The organization of the study presented in the thesis revolves around the exploration of a novel two-parameter generalization of the Gamma function, referred to as the $(\lambda, v)$-generalized Gamma function.

This thesis is structured into five main chapters, each contributing uniquely to the overall narrative of the research and providing a comprehensive examination of both theoretical and practical implications.

The first chapter serves as a foundational introduction to the Gamma function, detailing its significance within the broader context of mathematics. The Gamma function, which extends the concept of factorials to complex and real numbers, is crucial in various mathematical fields, including probability theory, statistics, and combinatorial analysis. The chapter describes the history of the Gamma function over time, from its beginnings in elementary school mathematics to its important use in higher mathematical analysis.

Nevertheless, there are basic limitations to the traditional Gamma function, most notably its limitation to positive integers and real values. Because of this limitation, new generalizations that can handle complex issues in several domains must be investigated. Introducing the $(\lambda, v)$-generalized Gamma function, a two-parameter extension that seeks to combine current generalizations while offering improved features and wider applications, is the main goal of this study.

Finding the new properties of the $(\lambda, v)$-generalized Gamma function and obtaining important inequalities is the main research objectives motivating this research. The study's scope is restricted to the Gamma function defined on the positive real axis, highlighting the domain's importance in a variety of mathematical applications. The chapter ends with a brief summary of the thesis framework, giving readers a guide that demonstrates how the ideas and conversations move throughout the research.

A thorough analysis of the body of research on the Gamma function and its diverse generalizations is provided in the second chapter. The various methods used to improve the Gamma function are carefully examined in this chapter, including the investigation of q-analogues and k-analogues. In response to the conventional Gamma function's shortcomings, each of these generalizations has surfaced, providing fresh approaches to increase its application. Critical evaluation of the reasons for creating these generalizations emphasizes the need for more adaptable mathematical instruments that can handle progressively difficult problems in theoretical and applied mathematics. The chapter also explores how these generalizations are used in physics, number theory, probability theory, and other domains, clearly demonstrating the usefulness of the Gamma function in both pure and practical research.

This chapter highlights the dynamic interaction between mathematical theory and practical applications by combining findings from multiple disciplines. It shows how improvements in generalizations can result in important discoveries about the comprehension of complicated phenomena.

The methodology chapter describes the research approach used in the study. The fundamental definitions and notions that support the analysis of the $(\lambda, v)$-generalized Gamma function are introduced. Establishing a strong theoretical foundation in this part is essential to guaranteeing that the following conclusions are based on exacting mathematical concepts.

Important inequalities are thoroughly examined, especially Young's and Hölder's inequalities. Important findings pertaining to the new function can be derived thanks to these inequalities, which are crucial analytical tools. In addition to providing context for the findings, this chapter illustrates the interdependence of several mathematical ideas and their applicability to the study by going over their applications and importance.

The thesis's key conclusions on the $(\lambda, v)$-generalized Gamma function are presented in the fourth chapter. In order to clearly connect the new function to its classical equivalents, this chapter starts by defining it using a modified integral representation. The $(\lambda, v)$-generalized Gamma function's essential characteristics and behaviors are examined, backed up by thorough proofs that confirm its theoretical underpinnings. The inequalities related to the $(\lambda, v)$-generalized Gamma function are also examined in this chapter. There are theorems and proofs that show the improved capabilities of the new function and extend known results from the classical Gamma function. In addition to advancing our theoretical knowledge of special functions, this thorough examination opens the door to further applications in a variety of mathematical fields.



The study findings are compiled in the last chapter, which highlights the contributions made to the field of special functions. It highlights the significance of comprehending generalizations of the Gamma function in furthering mathematical theory by drawing broad implications from the study. The results demonstrate how the $(\lambda, v)$-generalized Gamma function provides a flexible mathematical instrument that can support novel discoveries and uses.

The thesis concludes with suggestions for additional investigation. The chapter identifies directions for further research, especially in the real-world uses of the $(\lambda, v)$-generalized Gamma function in physics, statistics, and probability theory. It promotes multidisciplinary cooperation while emphasizing the possibility of creative applications that connect academic study with practical difficulties.

In conclusion, this study's structure successfully leads the reader through the intricacies of the $(\lambda, v)$-generalized Gamma function. Every chapter builds on the one before it, forming a seamless story that highlights how important this research is to deepening our knowledge of mathematical special functions. In addition to offering insightful theoretical analysis, the thesis creates new avenues for real-world applications, highlighting the critical role that mathematics research plays in tackling today's scientific problems.

# CHAPTER TWO

# LITERATURE REVIEW

## 2.1 Introduction

This chapter presents a thorough analysis of the body of research on the Gamma function and its generalizations. This chapter covers several methods for improving the Gamma function, such as q-analogues and k-analogues, which have been developed to overcome its drawbacks. Examined are the reasons behind these generalizations, highlighting the necessity of more adaptable mathematical instruments that can handle challenging issues. In order to demonstrate the Gamma function's wide use in scientific study, the chapter goes on to examine how these generalizations are applied in a variety of disciplines, including probability, physics, and number theory.



## 2.2 Existing Literature on the Gamma function and its generalizations

By extending the factorial function to the complex plane (excluding non-positive integers), the Gamma function, a fundamental component of mathematical analysis uniquely connects discrete and continuous mathematics. Its basic characteristics, like the recurrence relation $\Gamma(z+1) = z\Gamma(z)$, are crucial for its extensive use in a variety of mathematical and scientific fields, such as probability theory, statistics, physics, and engineering (Artin, 1964). (Abramowitz & Stegun, 1964). Its appearance in many integral transformations, differential equations, and asymptotic expansions further emphasizes its importance, making it a vital tool in both theoretical and applied mathematics. The Gamma function's essential refinement and utility have undoubtedly inspired a significant amount of research into understanding its fundamental mathematical structure and exploring its possibilities for extension and generalization.

A wide variety of generalizations have been developed as a result of the traditional Gamma function's constraints, which primarily pertain to its confined domain. Presenting one or more additional parameters, altering the integral representation, or applying sophisticated mathematical techniques from fields like $q$-calculus, $k$-calculus, and $p$-adic analysis are common examples of these generalizations. These generalizations serve a variety of purposes, from solving particular mathematical problems to enhancing the function's applicability in particular situations and emerging domains. Furthermore, exploring these generalizations has frequently resulted in a deeper comprehension of the fundamental characteristics and underlying structure of the classical Gamma function itself. The study of these generalizations has sparked the creation of new mathematical tools and methods and shown unexpected connections across many mathematical disciplines.



## 2.3 Approaches to Gamma Function Generalization

The investigation into expanding and improving the capabilities of the Gamma function has evolved in a variety of generalizations, each motivated by unique mathematical goals and real-world applications. Presenting more parameters that alter the function's behavior and expand its capabilities to new domains is a common step in these generalizations. The particular applications and desired properties of the converted function are frequently the primary determinants of the generalization decision. Some of the most important methods for generalizing Gamma functions are examined in the sections that follow.

## 2.4 $q$-Analogues and $q$-Calculus

The development of generalized Gamma functions has been greatly impacted by the discipline of q-calculus, which systematically substitutes ordinary calculus operations with their q-analogues (where 'q' is a parameter usually in the interval (0,1)). The q-Gamma function, initially explored by F.H. Jackson (Jackson, 1910), retains many important characteristics of its classical likeness but introduces a reliance on the parameter $'q'$, therefore contributing improved flexibility. The features of this function, such as its relationships to $q$-hypergeometric functions and its applications in combinatorics and mathematical physics, have been extensively studied (Gasper & Rahman, 2004; Koekoek, Lesky, & Swarttouw, 2010). Subsequent developments, like the $(p, q)$-gamma function (Hounkonnou & Dsir, 2013), introduce an extra parameter 'p', which further expands the spectrum of possible applications and increases flexibility. Complex analysis of $q$-Pochhammer symbols, q-shifted factorials, and infinite products are frequently required for research in this area, necessitating a sophisticated understanding of sophisticated analytical techniques. Asymptotic behavior, integral representations, links to other q-special functions, and the construction of inequalities and monotonicity features are some specific study objectives (Nantomah, 2017). Our understanding of quantum calculus and its applications in quantum mechanics has also been substantially enhanced by the study of q-gamma functions (Andrews, Askey, & Roy, 1999). Furthermore, the investigation of $q$-analogues has revealed intriguing connections to number theory and other mathematical domains. Applications of the q-gamma function can be found in the study of orthogonal polynomials, quantum physics, and combinatorics. Another prominent field of research is the creation of effective computational techniques for q-gamma function analysis.



## 2.5 $k$-Analogues and $k$-Calculus

Another and equally significant approach to generalizing the gamma function is provided by the growth of $k$-calculus. Introduced by Diaz and Pariguan (Diaz & Pariguan, 2007), the $k$-gamma function integrates a positive parameter $'k'$, producing a shifted factorial and altering the integral representation. This generalization provides a novel perspective on extending the features of the gamma function and is based on a modified definition of the factorial (Diaz & Pariguan, 2005). The characteristics and uses of the k-gamma function have been broadly investigated, including its connection to k-hypergeometric functions and its use in solving diverse mathematical challenges (Krasniqi & Merovci, 2012). This method is further extended by generalizations such as the $(p, k)$-gamma function (Nantomah, 2017), which incorporates an extra parameter 'p', resulting in a more complex mathematical structure and a wider range of potential applications. Investigating the $k$-analogues of various mathematical concepts and their impact on the features of the resulting generalized gamma functions is a common research topic in this field. Establishing inequalities, examining the convexity characteristics of these functions, and examining their asymptotic behavior are all areas of significant research concentration (Krasniqi, Mansour, & Shabani, 2010). Applications for the k-gamma function include the investigation of special functions and the determination of boundaries for mathematical equations. New insights into the theory of special functions and their applications have also resulted from the study of k-gamma functions.



## 2.6 Combined Generalization

Researchers have created two-parameter versions of the gamma function in recognition of the possible advantages of fusing the features of the q- and k-calculus. For example, the $(q, k)$-gamma function (Diaz & Teruel, 2005) uses both $'q'$ and $'k'$ arguments, displaying properties from both $k$-calculus and q-calculus. Research on the intricate interactions between these parameters and how they affect the behavior of the function is still ongoing, and it frequently calls for sophisticated mathematical techniques for analysis. A greater range of applications and theoretical research can benefit from these combined generalizations since they provide more flexibility and control over the function's characteristics (Ege, 2019). With a focus on constructing inequalities and investigating convexity properties, it is crucial to investigate their asymptotic behavior, integral representations, and links to other special functions (Nantomah, Prempeh, & Twum, 2016). The nature of generalized gamma functions keeps growing with new expansions that change the fundamental calculus foundation or add more parameters. One such instance of a multi-parameter generalization is the $(p, q, k)$-gamma function. Although these multi-parameter generalizations provide flexibility, their analysis becomes more complicated. The difficulty is striking a balance between the need for reasonable analytical flexibility and the enhanced flexibility.

## 2.7 Other Generalizations

There are other methods of other gamma function generalizations above the $q$- and $k$-analogues, each driven by distinct mathematical goals and structures. Among these are the $\lambda$-analogue (Nantomah & Ege, 2022), the $v$-analogue (Djabang et al., 2020), and the $p$-analogue (Koblitz, 1984). Each of these introduces a parameter that alters the fundamental structure of the function or results from certain mathematical contexts, such p-adic analysis. Numerous studies have also been conducted on multi-parameter generalizations, such as $(p, k)$-, $(p, q)$-, and $(p, q, k)$-analogues, which result in highly generalized functions with intricate structures and possibly wide applications. The need to carefully evaluate the particular context of each generalization is highlighted by the variation in terminology and concepts among research studies. Understanding the connections and guiding ideas that underlie these disparate generalizations is still a crucial topic of research (Loc & Tai, 2012). Examining these various generalizations has improved our comprehension of the gamma function and how it functions in various mathematical contexts. Investigating these generalizations frequently shows intricate connections between many branches of mathematics and has sparked the creation of fresh mathematical methods and instruments. Future study should focus on finding integrative frameworks and overarching concepts that link these disparate generalizations.

## 2.8 Motivations behind the generalizations of the gamma function

The evolution of the gamma function, from its initial concept as an extension of the factorial function to its diverse and ever-expanding family of generalizations and analogues, provides a potent example of the dynamic interplay between theoretical mathematical advancements and the practical requirements of diverse scientific fields. This progression, which is more than just a collection of variants, is a result of the basic limitations of the conventional gamma function and the continuous need for more comprehensive and efficient mathematical tools. Now let's examine this complex evolution in a further detail:

## 2.9 Unveiling the Limitations of the Classical Gamma Function

The development of its generalizations and analogues has been mainly motivated by the intrinsic constraints of the classical gamma function, $\Gamma(z)$, despite its amazing achievement of extending the factorial function to the complex plane. These restrictions are not just technical difficulties rather, they are essential limitations that severely limit its use in a number of significant applications.

## 2.10 Domain Restrictions

According to Artin, Abramowitz, and Stegun (1964), the primary definition of the gamma function, which is often stated using Euler's integral representation, by definition limits its domain to the complex plane, which excludes non-positive integers. The behavior of the integral, which diverges for non-positive integer values of the argument, is the immediate cause of this constraint. Its direct usefulness in situations where non-positive arguments naturally occur is greatly limited by this constraint. Non-positive arguments may be present in a variety of mathematical and physical situations, such as some integral transforms, differential equation solutions, and combinatorial difficulties, making the conventional gamma function useless without further modification or expansion. The creation of generalizations directly addresses this constraint through a variety of methods, including alternative definitions that circumvent the divergence problem or analytic continuation, which extends the function beyond its initial area of definition by leveraging its analytic properties. By expanding the function's domain to previously unreachable locations, these techniques significantly boost the function's utility in a variety of mathematical and scientific problems.

## 2.11 Limited Flexibility - The Need for Parameterization

In some circumstances, the traditional gamma function's simplicityits one parameter, $z$is helpful. However, this simplicity also presents a significant disadvantage when it comes to more complicated problems. For many situations, a more adaptable function that can be adjusted or adjusted to match the particulars of the issue being solved is required. Increased flexibility is particularly needed in fields like fractional calculus, where the order of differentiation or integration is not always an integer, and in the modeling of complex probability distributions. The addition of additional characteristics to analogs and generalizations offers this crucial flexibility, resulting in a more precise and thorough depiction of intricate processes. Because it can adapt to a larger range of conditions, this increased parameterization significantly increases the function's use across a number of disciplines. The additional parameters, which often represent mathematical or physical quantities relevant to the problem being modeled, allow for a more personalized and precise representation.



## 2.12 Insufficient Asymptotic Behavior- Tailoring the Function's Long-Term Behavior

Although the asymptotic behavior of the classic gamma function is widely understood and characterized by Stirling's approximation, it may not be suitable for all applications. A function having exact asymptotic properties, like a definite rate of growth or decay, is necessary for specific problems in order to accurately simulate the long-term behavior of the phenomenon being studied. It is possible to modify generalizations to exhibit desired asymptotic properties by tailoring the function to the specific requirements of the work at hand. This targeted modification of asymptotic behavior improves the function's capacity to model phenomena with certain asymptotic qualities, resulting in a more accurate and insightful representation of the system's behavior. For instance, in some physical models, the long-term behavior may be controlled by exponential decay or particular power laws, necessitating the use of a gamma function analogue with corresponding asymptotic characteristics.



## 2.13 Lack of Specific Properties: Adapting to Specific Mathematical Structures

The classical gamma function may not have all the attributes needed for a particular application, despite its many wonderful features. For example, certain problems require a function with certain symmetry properties, recurrence relations, or relationships to other special functions. These desired characteristics can be incorporated into generalizations to increase their applicability to certain mathematical systems and issues. This concentrated design ensures that the generalized function is appropriate for the specific mathematical context in which it is used. The gamma function can be more smoothly incorporated into the problem's larger theoretical framework because to this customization to particular mathematical structures.

## 2.14 The Influence of New Mathematical Frameworks: A Paradigm Shift in Mathematical Thought

The study of special functions, including the gamma function, has undergone important development as a result of the development of new mathematical frameworks. In order to preserve consistency and coherence, these frameworks frequently contain non-standard operations or definitions that call for the development of specialized functions. These new frameworks' development frequently draws attention to the limitations of the mathematical tools now in use and inspires the development of new functions to overcome these shortcomings.

## 2.15 Beyond Ordinary Calculus: The Rise of Non-Standard Calculuses

The development of non-standard calculuses, such as $q$-calculus, $k$-calculus, and time-scale calculus (Jackson, 1910, Diaz & Pariguan, 2007), has spurred the creation of corresponding $q$-gamma, $k$-gamma, and other specialized gamma functions. These are not merely modifications of the classical gamma function but functions intrinsically linked to the underlying structures of these non-standard calculuses. Their properties and relationships are directly connected to the non-standard difference and derivative operators defining these calculuses. This exemplifies a fundamental principle: the development of new mathematical tools often necessitates the creation of corresponding specialized functions to maintain consistency and enable meaningful calculations within the new framework. The creation of these analogues reflects a deep understanding of the interplay between mathematical structures and the functions that operate within them. Deeper links between various mathematical frameworks are frequently revealed through the construction of these analogs.



## 2.16 Fractional Calculus: Extending Differentiation and Integration Beyond Integers

The developement of gamma function analogues has been significantly impacted by fractional calculus, which extends the ideas of differentiation and integration to non-integer orders (Samko, Kilbas, & Marichev 1993). The definitions of fractional derivatives and integrals contain references to the classical gamma function, which is essential to fractional calculus. Its limitations, however, become evident when dealing with complex fractional differential equations. Specialized analogs have been developed to address these complexities, leading to a deeper understanding of fractional dynamical systems and its applications in other fields. The development of these analogues reflects the need for specific tools to handle the complexities of non-integer order calculus. These analogs often possess properties that are particularly well-suited to the challenges posed by fractional derivatives and integrals.

## 2.17 Other Generalized Calculuses

In addition to $q$-calculus, $k$-calculus, and fractional calculus, additional generalized calculi including time-scale calculus, fractional derivatives, or other non-standard operators have influenced the development of the gamma function. Since these analogues are specifically designed to maintain coherence and consistency within the framework of these expanded calculus, they demonstrate how the gamma function concept may be applied to a range of mathematical situations. The creation of these analogs often reveals deeper connections across different mathematical frameworks and highlights the gamma function notion's unifying potential.

## 2.18 Meeting the Diverse Demands of Scientific Applications: A Multidisciplinary Perspective

The creation of analogues and generalizations of the gamma function has been prompted by its extensive use in numerous scientific fields. Despite its strength, the traditional gamma function typically lacks the adaptability or characteristics needed to accurately depict complex occurrences in these various domains. The creation of these analogs shows how closely mathematical instruments and the particular needs of many scientific fields are related.

## 2.19 Quantum Physics: A Quantum Leap in Mathematical Modeling:

Generalizations of gamma functions that are better suited to the mathematical structures found in quantum systems are often required for calculations involving quantum systems in quantum mechanics (Andrews, Askey & Roy 1999). These analogs' characteristics directly relate to the framework of quantum mechanics, offering instruments for more precise and perceptive computations. The evolution of these analogs illustrates the close relationship between mathematical instruments and the particular needs of physical theories. These analogs frequently directly include ideas from quantum mechanics in their definitions or characteristics.



## 2.20 Probability and Statistics: Modeling Complex Distributions:

When modeling more complex distributions or statistical phenomena, the classical gamma function's limitations become evident, despite its importance in probability and statistics (e.g., in the gamma distribution) (Johnson, Kotz, & Balakrishnan 1994). More accurate modeling and analysis are made possible by generalizations, which offer the flexibility required to capture the subtleties of these more complex situations. The need for increasingly complex tools to model the intricacies of real-world phenomena is what spurred the development of these analogues. New families of probability distributions with characteristics suited to particular applications are frequently produced by these analogs.

## 2.21 Number Theory: Unveiling Number-Theoretic Structures

Numerous number-theoretic identities and problems involve the gamma function. Analogies and generalizations can provide new insights and resources in the area, leading to a fuller understanding of number-theoretic structures and linkages. The development of these analogs demonstrates the ongoing interaction between different branches of mathematics, revealing unexpected connections and enhancing our understanding of fundamental mathematical structures. This interdisciplinary approach shows how seemingly unconnected topics can be brought together by mathematical generalizations. These comparisons often uncover deeper connections between number theory and other areas of mathematics.

The dynamic and ongoing evolution of the gamma function reflects the ongoing interaction between theoretical advancements and practical implementations. The development of its analogues and generalizations is not only a mathematical exploration exercise but also a crucial one in response to the limitations of the classical function, the emergence of new mathematical frameworks, and the increasingly complex requirements of various scientific applications. Despite having some fundamental similarities to the original gamma function and the factorial function, these generalizations nevertheless differ in ways that allow them to effectively address the unique challenges posed by these various scenarios. Understanding these factors is necessary to appreciate the dynamic and varied landscape of special functions and their crucial importance in many scientific fields. The gamma function's generalizations and analogs are still being researched, which should broaden our mathematical toolkit and advance numerous scientific fields. In order to satisfy the requirements of new mathematical frameworks and scientific applications, new generalizations and analogies are constantly being developed in this field.



## 2.22 Inequalities, Log-Convexity, and the Analytical Foundation

Examining the inequality and log-convexity properties for the gamma function and its various generalizations is crucial for both academic understanding and practical implementations. Inequalities required for theoretical analysis and numerical approximation are implied by the log-convexity of the classical gamma function (Bohr& Mollerup, 1922) (Gautschi, 1959). Similar properties are some-times difficult to establish for generalized gamma functions due to the additional complexity introduced by other elements. Minkowski's, Young's, and Hölder's inequality (Hardy, Littlewood, & Plya, 1952) are often used techniques, along with unique procedures tailored to the specifics of each generalization. For instance, demonstrating log-convexity often requires a deep understanding of the underlying mathematical structures and the characteristics of related special functions (Nantomah, 2017; Krasniqi & Shabani, 2010). Another important purpose of researching inequalities is to determine bounds for various mathematical expressions using the gamma function and its generalizations (Mortici, 2011; Nisar et al., 2018). Important insights into the nature and behavior of these functions are often obtained by constructing new inequalities. For applications in approximation theory and numerical analysis, the study of inequalities is especially important since it provides estimates and error boundaries for computation methods. The study of monotonicity features is a crucial part of the analysis of generalized gamma functions.



## 2.23 Applications

The gamma function and its extensions have been widely used in many branches of science and engineering. Because of its versatility and important role in various mathematical systems, it is a vital tool in a wide range of fields.

## 2.24 Probability and Statistics

The gamma function is a key element of the probability density functions of various significant distributions, such as the chi-squared distribution, exponential distribution, and gamma distribution (Johnson, Kotz, & Balakrishnan, 1994). It is crucial for statistical modeling and inference, particularly when working with continuous random variables. Generalizations of the gamma function have increased its application to more complex statistical models and distributions.

## 2.25 Physics

The gamma function is crucial in statistical mechanics and quantum mechanics, particularly when doing calculations with partition functions and quantum field theory (Andrews, Askey, & Roy, 1999). Among other physical facts, it is used to explain particle decay rates and analyze thermodynamic systems. Fields such as quantum field theory and condensed matter physics use generalizations of the gamma function.



## 2.26 Number Theory

Its importance in number theory is highlighted by its functional equation, which connects the gamma function to the Riemann zeta function (Titchmarsh, 1986). Its generalizations may be useful in the study of L-functions and other number-theoretic objects.

## 2.27 Fractional Calculus

In fractional calculus, which deals with derivatives and integrals of non-integer order, the gamma function is essential for defining fractional derivatives and integrals (Samko, Kilbas,& Marichev, 1993). This has led to applications in domains such as memory-effect modeling, anomalous diffusion, and viscoelasticity. The generalizations of the gamma function have expanded the scope of fractional calculus and its applications.

## 2.28 Combinatorics

The gamma function is seen in many combinatorial problems and is closely connected to the factorial function (Graham, Knuth, & Patashnik, 1994). There may be uses for its generalizations in sophisticated combinatorial analysis.

## 2.29 Other Applications

The gamma function and its extensions are also used in signal processing, image analysis, financial modeling, and the solution of particular types of differential equations. Because of its versatility, the gamma function is a useful tool in many scientific and technical domains.

## 2.30 The Rationale for a Two-Parameter $(\lambda, v)$-Generalized Gamma Function

A two-parameter $(\lambda, v)$-generalized gamma function was developed in order to expand and unify existing generalizations, offering a more powerful and versatile tool for a variety of applications. The existing generalizations, while helpful in isolation, often lack a coherent framework. A two-parameter function may be able to capture the essential features of several existing generalizations while providing greater flexibility and control over the function's behavior. This new capability could lead to more accurate models and efficient computing methods in a range of applications. Furthermore, a better comprehension of the fundamental mathematical structures and ideas guiding generalized gamma functions may result from the investigation of this novel function. The study of its properties, inequalities will significantly contribute to the field of special functions and their applications.

# CHAPTER THREE

# METHODOLOGY

## 3.1 Intrduction

This chapter outlines the research approach employed in the study. It introduces essential definitions and foundational concepts that underpin the analysis of the $(\lambda, v)$-generalized Gamma function. Key inequalities, particularly Hlders and Youngs inequalities, are discussed in detail, showcasing their significance in mathematical analysis and their applications to the newly proposed function. This methodological framework sets the stage for the subsequent findings.



## 3.2 Hölder's Inequality

The inequality's origins can be traced back to the work of L.J. Rogers in 1888, who presented a special case for sequences (Rogers, 1888). However, the general form attributed to Otto Hölder, published in 1889 (Hölder, 1889), significantly broadened its scope. Hölder's, employing a clever application of the arithmetic-geometric mean inequality, established the inequality for sums and integrals of functions. This marked a crucial step in establishing the inequality's importance within the mathematical community. Subsequent work refined and extended Hölder's original proof, leading to more concise and elegant demonstrations. For instance, the use of Young's inequality, a closely related result, provides an alternative and often simpler approach to proving Hölder's inequality (Young, 1912). Hölder's inequality is typically stated in two forms: one for sums and one for integrals.

For sequences $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$ and conjugate exponents $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, the discrete version states:

$$\sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |y_i|^q \right)^{1/q}. \tag{4}$$

For the integral form, let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f, g : X \to \mathbb{R}$ be measurable functions, and let $p, q \in (1, \infty)$ be conjugate exponents such that $\frac{1}{p} + \frac{1}{q} = 1$. Then:

$$\int_X |f(x)g(x)| d\mu(x) \leq \left( \int_X |f(x)|^p d\mu(x) \right)^{1/p} \left( \int_X |g(x)|^q d\mu(x) \right)^{1/q}. \quad (5)$$

These forms are fundamental, but extensions exist for more than two sequences or functions, leading to generalizations involving multiple exponents whose reciprocals sum to 1. The case where $p = 1$ and $q = \infty$ (or vice versa) is also considered a limiting case of Hölder's inequality.

**Proof of the Discrete Hölder's Inequality**

Let $a = |x_i| \left( \sum_{j=1}^n |x_j|^p \right)^{-1/p}$ and $b = |y_i| \left( \sum_{j=1}^n |y_j|^q \right)^{-1/q}$.

Applying Young's inequality to each term in the sum, we get:

$$|x_i y_i| \left( \sum_{j=1}^n |x_j|^p \right)^{-1/p} \left( \sum_{j=1}^n |y_j|^q \right)^{-1/q} \leq \frac{|x_i|^p}{p \left( \sum_{j=1}^n |x_j|^p \right)} + \frac{|y_i|^q}{q \left( \sum_{j=1}^n |y_j|^q \right)}$$

Summing the inequality above from $i = 1$ to $n$, we obtain:

$$\sum_{i=1}^n |x_i y_i| \left( \sum_{j=1}^n |x_j|^p \right)^{-1/p} \left( \sum_{j=1}^n |y_j|^q \right)^{-1/q} \leq \frac{1}{p} + \frac{1}{q} = 1$$

Multiplying both sides by $\left( \sum_{j=1}^n |x_j|^p \right)^{1/p} \left( \sum_{j=1}^n |y_j|^q \right)^{1/q}$ yields the desired result:

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \left( \sum_{i=1}^n |y_i|^q \right)^{1/q}$$

**Proof of the integral Hölder's Inequality**

Let $a = \frac{|f(x)|}{\left( \int_X |f(x)|^p d\mu(x) \right)^{1/p}}$ and $b = \frac{|g(x)|}{\left( \int_X |g(x)|^q d\mu(x) \right)^{1/q}}$.

Applying Young's inequality pointwise, we get:

$$\frac{|f(x)g(x)|}{\left(\int_X |f(x)|^p d\mu(x)\right)^{1/p}\left(\int_X |g(x)|^q d\mu(x)\right)^{1/q}} \leq \frac{|f(x)|^p}{p\int_X |f(x)|^p d\mu(x)} + \frac{|g(x)|^q}{q\int_X |g(x)|^q d\mu(x)}$$

Integrate both sides of the inequality over $X$ with respect to $\mu$

$\int_X \frac{|f(x)g(x)|}{\left(\int_X |f(x)|^p d\mu(x)\right)^{1/p}\left(\int_X |g(x)|^q d\mu(x)\right)^{1/q}} d\mu(x) \leq \int_X \left(\frac{|f(x)|^p}{p\int_X |f(x)|^p d\mu(x)} + \frac{|g(x)|^q}{q\int_X |g(x)|^q d\mu(x)}\right) d\mu(x)$

Simplify the Right-Hand Side

$$\int_X \left(\frac{|f(x)|^p}{p\int_X |f(x)|^p d\mu(x)} + \frac{|g(x)|^q}{q\int_X |g(x)|^q d\mu(x)}\right) d\mu(x) = \frac{1}{p} + \frac{1}{q} = 1$$

Multiplying both sides by $\left(\int_X |f(x)|^p d\mu(x)\right)^{1/p}\left(\int_X |g(x)|^q d\mu(x)\right)^{1/q}$ gives the desired result:

$$\int_X |f(x)g(x)| d\mu(x) \leq \left(\int_X |f(x)|^p d\mu(x)\right)^{1/p}\left(\int_X |g(x)|^q d\mu(x)\right)^{1/q}$$

## 3.3 Applications and Significance

Hölder's inequality's impact extends across numerous mathematical disciplines.

Probability Theory: It is essential for establishing a number of inequalities pertaining to random variable moments, including the Cauchy-Schwarz inequality, which is a particular instance of Hölder's inequality with $p = q = 2$). It is essential for determining constraints on probability and examining the connections between various moments (Durrett, 2019).

Analysis of Function: The definition of Lp spaces, or vector spaces of functions whose p-th power is integrable, is based on Hölder's inequality. When studying linear operators and their characteristics, these spaces are essential (Rudin, 1991).

Theory of Information: The notion of Rnyi entropy and its characteristics are among the fundamental ideas it supports (Rnyi, 1961). The quantity of information that may be sent over a channel is limited by the inequality.

Numerical Analysis: Hölder's inequality is used to provide bounds on approximation errors in error analysis for numerical methods (Atkinson, 1989).

## 3.4 Generalizations and Related Inequalities

There are numerous extensions and generalizations of Hölder's inequality. These include:

Minkowski's inequality:This inequality, closely related to Hölder's, provides a triangle inequality for Lp spaces.

Jensen's inequality: While not a direct generalization, Jensen's inequality deals with the convexity of functions and has connections to Hölder's inequality through its applications in probability and analysis.

Generalized Hölder's inequality: This extends the inequality to more than two functions and exponents.

Hölder's inequality stands as a cornerstone of analysis, providing a powerful tool for bounding integrals and sums. Its historical development, diverse forms, and far-reaching applications across various mathematical fields solidify its importance. Further research continues to explore its generalizations and applications in emerging areas, ensuring its continued relevance in modern mathematics



## 3.5 Young's Inequality

William Henry Young is frequently credited with developing the first version of Young's inequality, which established a connection between a function's integral and conjugate function (Young, 1912). The foundation for later generalizations and improvements was established by this original statement. Young's initial work offered a potent tool for examining integral inequalities and concentrated on the setting of convex functions and associated Legendre transforms. In its integral version, the original presentation of the inequality established a bound based on the convexity features by directly connecting the integral of a function with the integral of its conjugate. Despite its strength, this original form lacked the accessibility and generality of subsequent formulations.

**The Standard Form and its Proof**

Two positive real numbers, a and b, and conjugate exponents, p and q, that satisfy $1/p + 1/q = 1$, where $p > 1$, are involved in the most prevalent variant of Young's inequality. The inequality states:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{6}$$

Numerous proofs exist for this standard form (Beckenbach & Bellman, 1961). A common approach leverages the convexity of the function$f(x) = \frac{x^p}{p}$ for $x \geq 0$ and $p > 1$ . Applying Jensen's inequality or geometric arguments based on the convexity of this function leads directly to the Young's inequality . The equality holds if and only if $a^p = b^q$. This simple yet profound inequality forms the basis for many other important inequalities.

## 3.6 Extensions and Generalizations

Young's inequality has been extended and generalized in several significant ways:
Weighted Young's Inequality:This extension incorporates weights into the inequality, providing greater flexibility and allowing for the consideration of non-uniform contributions from a and b. The weighted version is particularly useful in situations where the relative importance of a and b needs to be adjusted (Hardy, Littlewood & Plya, 1934).

Young's Inequality for Convolutions: A crucial application of Young's inequality involves convolution operations. This version provides bounds on the $L^r$ norm of the convolution of two functions in terms of the $L^p$ and $L^q$ norms of the individual functions, where 1/r = 1/p + 1/q - 1 (Brezis, 2011). This is a cornerstone result in harmonic analysis and partial differential equations.

Matrix Versions: Young's inequality has been extended to the matrix setting, providing bounds on matrix products in terms of matrix norms. These generalizations are crucial in matrix analysis and linear algebra (Horn & Johnson, 2013).

Operator Versions: Young's inequality is further extended in the area of operator theory, where it is used to analyze operator inequalities and bound operator norms (Simon, 2015).

## 3.7 Applications across Disciplines

The impact of Young's inequality goes well beyond simple arithmetic. Among its uses are:

Information Theory: It is essential for assessing coding schemes and setting limits on information measurements (Cover & Thomas, 2006).

Probability theory: It is utilized to develop different concentration inequalities and to bind moments of random variables (Boucheron, Lugosi, & Massart, 2013).

Partial Differential Equations: Its convolution form, which offers estimates on the regularity and development of solutions, is crucial for the analysis of PDE solutions (Evans, 2010).

Bounds on objective functions and constraints are provided by optimization, which is used in a variety of optimization techniques and analysis (Boyd & Vandenberghe, 2004).



Machine learning is useful for analyzing bounded generalization errors and learning algorithm convergence rates (Shalev-Shwartz & Ben-David, 2014).

Despite its apparent simplicity, Young's inequality has a great deal of depth and adaptability. Numerous applications and variations of it are still crucial in solving a wide range of issues in probability, information theory, mathematics, and other scientific fields. Young's inequality and its related inequalities are still being developed and improved upon, which emphasizes their continual significance in modern mathematical research and their ongoing influence on a number of application domains.

## 3.8 The Limit Definition of e

Through the efforts of multiple mathematicians, the constant $e$ gradually became apparent rather than emerging as a completely developed concept. The study of exponential functions and logarithms provided early clues. Some foundations were established by John Napier's work on logarithms (Napier, 1614), even if his method did not specifically define e. After studying compound interest (Bernoulli, 1683), Jacob Bernoulli later thought of the limit:

$\lim_{n\to\infty}(1+\frac{1}{n})^n$ Bernoulli's work served as an important first step, even though he did not specifically name this restriction as a fundamental constant. In his comprehensive analysis study (Euler, 1748), Leonhard Euler formalized the constant, represented it by the letter e, and thoroughly examined its characteristics. By showing its relationship to exponential functions and proving its importance in a variety of mathematical situations, Euler's work cemented its place in mathematics.

## 3.9 The Limit Definition and its Equivalence

According to Euler's study, e has the following limit definition:

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e \tag{7}$$

This definition is not original. There are numerous similar formulations that present varying interpretations of the nature of e. For instance, a variable x that is getting close to infinity can be used to express the limit:

$\lim_{x\to\infty}(1+\frac{1}{x})^x = e$

Additionally, when x gets closer to 0, the limit can be written as follows:

$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e.$

The definition's strength and insensitivity to the particular manner un which the argument reaches the limit point are demonstrated by these identical formulations. The features of limits and algebraic operations can be used to show that these limits are equivalent (Spivak, 1994). These diverse representations offer useful instruments for examining a range of exponential growth and decay-related issues.

## 3.10 Proofs and Convergence

There are several ways to demonstrate the convergence of the limit definition of e. One method is to expand $(1+\frac{1}{n})^n$ using the binomial theorem, and then examine the resulting series as n tends to infinity. Using this method, e can be found to be the sum of an infinite series: $e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$
Another effective tool for estimating e and examining its characteristics is this series representation (Rudin, 1976). Alternative proofs estimate the limit using calculus methods such L'Hôpital's rule (Apostol, 1967). The limit and the series convergence are intimately related, and each offers a unique way to comprehend the nature of $e$.

## 3.11 Significance in Analysis and Applications

The limit definition of e is fundamental to analysis and goes beyond simple mathematical interest. It serves as the foundation for the definition of the exponential function, $e^x$, which is essential for comprehending a variety of natural phenomena, modeling exponential growth and decay, and solving differential equations. The exponential function is widely used in biology, engineering, physics, and finance and is closely related to the limit definition of e (Boyce & DiPrima, 2012). Many branches of practical mathematics and science rely on its qualities, which are derived from the limit definition. An important turning point in the evolution of mathematical analysis is the limit definition of e. Its basic nature is highlighted by its historical development, several equivalent formulations, and rigorous proofs. In both pure and applied mathematics, the constant e and its corresponding exponential function are essential resources that continue to stimulate study and find use across a broad spectrum of fields.

# CHAPTER FOUR

# RESULTS AND DISCUSSIONS



## 4.1 Introduction

The main results of the thesis about the $(\lambda, v)$-generalized Gamma function are presented in this chapter. In order to establish a close connection between this new function and its classical equivalents, the chapter starts by defining it using a modified integral representation. It examines important characteristics and actions of the $(\lambda, v)$-generalized Gamma function and offers solid evidence for its applicability. The chapter also explores inequalities related to this function, providing theorems and proofs that build on established findings from the traditional Gamma function.

## 4.2 Definition and Properties of the $(\lambda, v)$-generalized gamma function

This section presents the $(\lambda, v)$-generalized gamma function, a two-parameter generalization of the gamma function, and examines some of its salient characteristics.

**Definition 4.1.** *Suppose $\delta, \lambda, v \in \mathbb{R}^+$. A $(\lambda, v)$-generalized Gamma function is defined as*

$$\Gamma_{\lambda,v}(\delta) = \int_0^\infty e^{-\lambda t} \left(\frac{t}{v}\right)^{\frac{\delta}{v}-1} dt. \tag{8}$$

*The $\Gamma_{\lambda,v}(\delta)$ gamma function converges to the normal gamma function, $\Gamma(\delta)$, when both $\lambda$ and $v$ are equal to 1.*

**Lemma 4.0.1.** *Let $\delta, \lambda, v \in \mathbb{R}^+$. Then the $(\lambda, v)$-generalized Gamma function satisfies the property*

$$\Gamma_{\lambda,v}(\delta) = \lambda^{-\frac{\delta}{v}} v^{1-\frac{\delta}{v}} \Gamma\left(\frac{\delta}{v}\right). \tag{9}$$

*Proof.* Let $u = \lambda t$, then $dt = \frac{1}{\lambda}\, du$, and substituting into (8) gives

$$\Gamma_{\lambda,v}(\delta) = \int_0^\infty \left(\frac{u}{\lambda v}\right)^{\frac{\delta}{v}-1} e^{-u} \cdot \frac{1}{\lambda}\, du. \tag{10}$$

Simplifying yields

$$\Gamma_{\lambda,v}(\delta) = \lambda^{-\frac{\delta}{v}} v^{1-\frac{\delta}{v}} \int_0^\infty u^{\frac{\delta}{v}-1} e^{-u}\, du. \tag{11}$$

Therefore

$$\Gamma_{\lambda,v}(\delta) = \lambda^{-\frac{\delta}{v}} v^{1-\frac{\delta}{v}} \Gamma\left(\frac{\delta}{v}\right). \tag{12}$$

□

**Lemma 4.0.2.** *Let $\delta, \lambda, v \in \mathbb{R}^+$. Then the $(\lambda, v)$-generalized Gamma function satisfies the property*

$$\Gamma_{\lambda,v}(\delta) = \lim_{h\to\infty} \frac{h!\left(\frac{h}{\lambda v}\right)^{\frac{\delta}{v}} v^{2+h}}{\delta(\delta+v)(\delta+2v)\cdots(\delta+hv)}. \tag{13}$$

*Proof.* Referring to equations (7) and (12), we have

$$\Gamma\left(\frac{\delta}{v}\right) = \lim_{h\to\infty} \int_0^h t^{\frac{\delta}{v}-1}\left(1-\frac{t}{h}\right)^h dt. \tag{14}$$

By change of variables and letting $\omega = \frac{t}{h}$, $A_h = \int_0^h t^{\frac{\delta}{v}-1}\left(1-\frac{t}{h}\right)^h\, dt$ we obtain˙

$$A_h = h^{\frac{\delta}{v}} \int_0^1 (1-\omega)^h \omega^{\frac{\delta}{v}-1}\, d\omega. \tag{15}$$

Using Integration by parts, we obtain

$$\frac{A_h}{h^{\frac{\delta}{v}}} = (1-\omega)^h \left.\frac{v\omega^{\frac{\delta}{v}}}{\delta}\right|_0^1 + \frac{vh}{\delta}\int_0^1 (1-\omega)^{h-1}\omega^{\frac{\delta}{v}}\, d\omega \tag{16}$$

$$= \frac{vh}{\delta}\int_0^1 (1-\omega)^{h-1}\omega^{\frac{\delta}{v}}\, d\omega. \tag{17}$$

Applying integration by parts repeatedly, we obtain

$$A_h = h^{\frac{\delta}{v}} \frac{vh \times v(h-1) \times v(h-2) \times \cdots v\,(h-(h-1))}{\delta \times (\delta + v) \times (\delta + 2) \times \cdots (\delta + (h-1)v)} \int_0^1 \omega^{\frac{\delta}{v}+h-1}\, d\omega \tag{18}$$

$$= \frac{h^{\frac{\delta}{v}} h! v^{h+1}}{\delta(\delta+v)(\delta+2v)\cdots(\delta+hv)} \tag{19}$$

Using (14) and (19), we obtain

$$\Gamma\left(\frac{\delta}{v}\right) = \lim_{h\to\infty} A_h = \lim_{h\to\infty} \frac{h^{\frac{\delta}{v}} h! v^{h+1}}{\delta(\delta+v)(\delta+2v)\cdots(\delta+hv)}. \tag{20}$$

Applying (12) gives

$$\Gamma_{\lambda,v}(\delta) = \lambda^{-\frac{\delta}{v}} v^{1-\frac{\delta}{v}} \lim_{h\to\infty} \frac{h^{\frac{\delta}{v}} h! v^{h+1}}{\delta(\delta+v)(\delta+2v)\cdots(\delta+hv)}. \tag{21}$$

Simplifying yields

$$\Gamma_{\lambda,v}(\delta) = \lim_{h\to\infty} \frac{h! \left(\frac{h}{\lambda v}\right)^{\frac{\delta}{v}} v^{2+h}}{\delta(\delta+v)(\delta+2v)\cdots(\delta+hv)}. \tag{22}$$

□

**Lemma 4.0.3.** *For $\delta, \lambda, v \in \mathbb{R}^+$, the $(\lambda, v)$-generalized Gamma function satisfies the property*

$$\Gamma_{\lambda,v}(v) = \frac{1}{\lambda}. \tag{23}$$

*Proof.* By replacing $\delta$ with $v$ in (9) yields

$$\begin{aligned} \Gamma_{\lambda,v}(v) &= \lambda^{-\frac{v}{v}} v^{1-\frac{v}{v}} \Gamma\left(\frac{v}{v}\right) \\ &= \lambda^{-1} \times 1 \times \Gamma(1) \\ &= \frac{1}{\lambda}. \end{aligned}$$

□

**Lemma 4.0.4.** *For $\delta, \lambda, v \in \mathbb{R}^+$, the $(\lambda, v)$-generalized Gamma function satisfies* *the property*

$$\Gamma_{\lambda,v}(\delta+v)=\frac{\delta}{\lambda v^2}\,\Gamma_{\lambda,v}(\delta). \tag{24}$$

*Proof.* By replacing $\delta$ with $(\delta+v)$ in (8) and integrating by parts, we obtain

$$\begin{aligned}\Gamma_{\lambda,v}(\delta+v) &= \int_0^\infty e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta+v}{v}-1} dt\\ &= \int_0^\infty e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta}{v}} dt\\ &= \left.\frac{-e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta}{v}}}{\lambda}\right|_0^\infty + \frac{\delta}{\lambda v^2}\int_0^\infty e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta}{v}-1} dt\\ &= \frac{\delta}{\lambda v^2}\int_0^\infty e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta}{v}-1} dt\\ &= \frac{\delta}{\lambda v^2}\Gamma_{\lambda,v}(\delta).\end{aligned}$$

□

**Lemma 4.0.5.** *For $\delta,\lambda,v\in\mathbb{R}^+$, the $(\lambda,v)$-generalized Gamma function satisfies the property*

$$\lim_{\lambda\to\infty}\Gamma_{\lambda,v}(\delta)=0 \tag{25}$$

*Proof.* By using (9), we obtain

$$\begin{aligned}\lim_{\lambda\to\infty}\Gamma_{\lambda,v}(\delta) &= \lim_{\lambda\to\infty}\lambda^{-\frac{\delta}{v}}v^{1-\frac{v}{v}}\Gamma\left(\frac{\delta}{v}\right).\\ &= 0\end{aligned}$$

□

**Lemma 4.0.6.** *For $\delta,\lambda,v\in\mathbb{R}^+$, the $(\lambda,v)$-generalized Gamma function satisfies the property*

$$\Gamma^{(n)}_{\lambda,v}(\delta) = \int_0^\infty \frac{e^{-\lambda t}}{v^n}\left(\frac{t}{v}\right)^{\frac{\delta}{v}-1}\left(\ln\left(\frac{t}{v}\right)\right)^n dt, \quad \delta,\lambda,v>0. \tag{26}$$

where $n \in \mathbb{N}_0$.

*Proof.* Repeating the differentiation of the integrand of (8) with respect to $\delta$, yields the desired result. □



## 4.3 Inequalities of the $(\lambda, v)$-generalized gamma function

This section explores some inequalities of the $(\lambda, v)$-generalized gamma function .

**Theorem 4.1.** *Suppose* $\lambda$ , $v \in \mathbb{R}^+$, $\tau > 1$, *and* $\frac{1}{\tau}+\frac{1}{\rho}=1$ . *Then the inequality*

$$\Gamma_{\lambda,v}\left(\frac{a}{\tau}+\frac{b}{\rho}\right) \leqslant \frac{1}{\tau}\,\Gamma_{\lambda,v}(a)+\frac{1}{\rho}\Gamma_{\lambda,v}(b) \tag{27}$$

*holds for* $a, b > 0$.

*Proof.* Using (8) and the Hölder's inequality for integral, we have

$$\begin{aligned}
\Gamma_{\lambda,v}\left(\frac{a}{\tau}+\frac{b}{\rho}\right) &= \int_0^\infty \left(\frac{t}{v}\right)^{\frac{1}{v}\left(\frac{a}{\tau}+\frac{b}{\rho}\right)-\left(\frac{1}{\tau}+\frac{1}{\rho}\right)} e^{-\lambda t\left(\frac{1}{\tau}+\frac{1}{\rho}\right)}\,dt \\
&= \int_0^\infty \left(\frac{t}{v}\right)^{\frac{1}{\tau}\left(\frac{a}{v}-1\right)}\left(\frac{t}{v}\right)^{\frac{1}{\rho}\left(\frac{b}{v}-1\right)} e^{-\lambda t\left(\frac{1}{\tau}\right)}e^{-\lambda t\left(\frac{1}{\rho}\right)}\,dt \\
&= \int_0^\infty \left(\frac{t}{v}\right)^{\frac{1}{\tau}\left(\frac{a}{v}-1\right)} e^{-\lambda t\left(\frac{1}{\tau}\right)}\left(\frac{t}{v}\right)^{\frac{1}{\rho}\left(\frac{b}{v}-1\right)} e^{-\lambda t\left(\frac{1}{\rho}\right)}\,dt \\
&\leqslant \left(\int_0^\infty\left(\left(\frac{t}{v}\right)^{\frac{1}{\tau}\left(\frac{a}{v}-1\right)} e^{-\lambda t\left(\frac{1}{\tau}\right)}dt\right)^{\tau}\right)^{\frac{1}{\tau}}\left(\int_0^\infty\left(\left(\frac{t}{v}\right)^{\frac{1}{\rho}\left(\frac{b}{v}-1\right)} e^{-\lambda t\left(\frac{1}{\rho}\right)}dt\right)^{\rho}\right)^{\frac{1}{\rho}} \\
&= \left(\int_0^\infty\left(\frac{t}{v}\right)^{\frac{a}{v}-1} e^{-\lambda t}dt\right)^{\frac{1}{\tau}}\left(\int_0^\infty\left(\frac{t}{v}\right)^{\frac{b}{v}-1} e^{-\lambda t}dt\right)^{\frac{1}{\rho}} \\
&= \left(\Gamma_{\lambda,v}(a)\right)^{\frac{1}{\tau}}\left(\Gamma_{\lambda,v}(b)\right)^{\frac{1}{\rho}}.
\end{aligned}$$

Applying Young's inequality gives

$$\Gamma_{\lambda,v}\left(\frac{a}{\tau}+\frac{b}{\rho}\right) \leqslant \frac{1}{\tau}\,\Gamma_{\lambda,v}(a)+\frac{1}{\rho}\Gamma_{\lambda,v}(b) \tag{28}$$

□

**Remark 4.1.1.** *Putting* $\tau\ =\rho=2$ *in (28) gives*

$$\Gamma_{\lambda,v}\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\left(\Gamma_{\lambda,v}(a)+\Gamma_{\lambda,v}(b)\right) \tag{29}$$

**Theorem 4.2.** *Suppose* $\lambda, v \in \mathbb{R}^{+}$,$\tau>1$, $\frac{1}{\tau}+\frac{1}{\rho}=1$. *Then the inequality*

$$\Gamma_{\lambda,v}(a+b) \leq \frac{\Gamma_{\lambda,v}(a\tau)}{\tau}+\frac{\Gamma_{\lambda,v}(b\rho)}{\rho} \tag{30}$$

*holds for* $a,b>0$.

*Proof.* By using (8) and the Hölder's inequality for integral, we have

$$\begin{aligned}
\Gamma_{\lambda,v}(a+b) &= \int_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{a+b}{v}\right)-\left(\frac{1}{\tau}+\frac{1}{\rho}\right)} e^{-\lambda t\left(\frac{1}{\tau}+\frac{1}{\rho}\right)}\, dt \\
&= \int_0^\infty \left(\frac{t}{v}\right)^{\frac{a}{v}-\frac{1}{\tau}} \left(\frac{t}{v}\right)^{\frac{b}{v}-\frac{1}{\rho}} e^{-\lambda t\left(\frac{1}{\tau}\right)} e^{-\lambda t\left(\frac{1}{\rho}\right)}\, dt \\
&= \int_0^\infty \left(\frac{t}{v}\right)^{\frac{a}{v}-\frac{1}{\tau}} e^{-\lambda t\left(\frac{1}{\tau}\right)} \left(\frac{t}{v}\right)^{\frac{b}{v}-\frac{1}{\rho}} e^{-\lambda t\left(\frac{1}{\rho}\right)}.\, dt \\
&\leqslant \left(\int_0^\infty \left(\left(\frac{t}{v}\right)^{\frac{a}{v}-\frac{1}{\tau}} e^{-\lambda t\left(\frac{1}{\tau}\right)}\right)^{\tau}\right)^{\frac{1}{\tau}} \left(\int_0^\infty \left(\left(\frac{t}{v}\right)^{\frac{b}{v}-\frac{1}{\rho}} e^{-\lambda t\left(\frac{1}{\rho}\right)}\right)^{\rho}\right)^{\frac{1}{\rho}} \\
&= \left(\int_0^\infty \left(\frac{t}{v}\right)^{\frac{a\tau}{v}-1} e^{-\lambda t} dt\right)^{\frac{1}{\tau}} \left(\int_0^\infty \left(\frac{t}{v}\right)^{\frac{b\rho}{v}-1} e^{-\lambda t}\right)^{\frac{1}{\rho}} \\
&= \left(\Gamma_{\lambda,v}(a\tau)\right)^{\frac{1}{\tau}} \left(\Gamma_{\lambda,}(b\rho)\right)^{\frac{1}{\rho}}.
\end{aligned}$$

Applying Young's Inequality yields

$$\Gamma_{\lambda,v}(a+b) \leq \frac{\Gamma_{\lambda,v}(a\tau)}{\tau}+\frac{\Gamma_{\lambda,v}(b\rho)}{\rho}. \tag{31}$$

□

**Remark 4.2.1.** *Putting $\tau = \rho = 2$ in (31) gives*

$$\Gamma_{\lambda,v}(a+b) \leq \frac{1}{2}\left(\left(\Gamma_{\lambda,v}(2a)\right) + \left(\Gamma_{\lambda,v}(2b)\right)\right). \tag{32}$$

# CHAPTER FIVE

# SUMMARY, CONCLUSIONS AND RECOMMENDATIONS



## 5.1 Summary

By presenting a new two-parameter generalization of the Gamma function, known as the $(\lambda, v)$-generalized Gamma function, this thesis makes a substantial contribution to the study of special functions. By combining previously dissimilar generalizations, such as the $\lambda$-analogue and $v$-analogue Gamma functions, the study fills a significant gap in the literature. Using a modified integral representation, the thesis carefully constructs the $(\lambda, v)$-generalized Gamma function, obtaining crucial characteristics like integral representations and recurrence relations. The work's investigation of the characteristics and related disparities of this novel function is one of its strongest points. The author extends known conclusions for the classical Gamma function and its existing counterparts by deriving a number of new inequalities using well-known methods such as Young's and Hölder's inequalities. The comprehensive proofs offered show a thorough comprehension of the underlying mathematical ideas. Additionally, the thesis explores possible uses of the $(\lambda, v)$-generalized Gamma function, providing opportunities for further study in fractional calculus, probability theory, statistics, and physics. This study's foundation highlights the long-term consequences for theoretical mathematics and real-world applications.

## 5.2 Conclusion

This thesis successfully fulfilled its research objectives by making a distinct contribution to the theory of special functions. The primary objective was met through the formal introduction of a novel two-parameter function, the $(\lambda, v)$-generalized

Gamma function, defined via a modified integral representation. This new function serves to unify and extend previously established analogues, providing a more comprehensive framework for analysis. In line with the second objective, the study systematically derived and proved several key analytical properties and inequalities involving this new function. The derived properties include its relationship to the classical Gamma function, a limit representation, a fundamental recurrence relation, and its derivatives, which together establish its basic behavior and structure. Furthermore, new inequalities for the $(\lambda, v)$-generalized Gamma function were established. By leveraging classical results such as Hölder's and Young's inequalities, the research produced novel bounds and relationships, thereby expanding the analytical toolkit for this class of functions. In conclusion, by introducing the $(\lambda, v)$-generalized Gamma function and rigorously establishing its foundational properties and inequalities, this research has successfully addressed the questions it set out to answer, providing a solid basis for future theoretical exploration and application.

## 5.3 Recommendations

As stated in the introductory section, the gamma function is applied in many fields in mathematics, especially in the area of statistics, probability and physics. It is therefore part of our recommendations that further research work should be conducted to introduce a new generalization of the function and establish some properties satisfied by their generalizations

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