Further Inequalities Associated with the Classical Gamma Function

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Abstract In this paper, the authors present some double inequalities associated with certain ratios of the Gamma function. The results are further generalizations of several previous results. The approach is based on some monotonicity properties of some functions involving the generalized Gamma functions. At the end, some open problems are posed.

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1. Introduction

Inequalities involving the classical Euler's Gamma function has gained the attention of researchers all over the world. Recent advances in this area include those inequalities involving ratios of the Gamma function. In [1,5,6,10] and [11-17], the authors established some interesting inequalities concerning such ratios, as well as some generalizations. By utilizing similar techniques, this paper seeks to present some new results generalizing the results of [11-17]. At the end, we pose some open problems involving the generalized Psi functions. In the sequel, we recall some basic definitions concerning the Gamma function and its generalizations. These definitions are required in order to establish our results.

The well-known classical Gamma function, $\Gamma(t)$ and the classical Psi or Digamma function $\psi(t)$ are usually defined for $t > 0$ as:

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} dx \text{ and } \psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}.$$

The p-Gamma function, $\Gamma_p(t)$ and the p-Psi function $\psi_p(t)$ are defined for $p \in N$ and $t > 0$ as:

$$\Gamma_p(t) = \frac{p! \, p^t}{t(t+1)\dots(t+p)} \text{ and } \psi_p(t) = \frac{\Gamma_p'(t)}{\Gamma_p(t)}$$

where $\Gamma_p(t) \to \Gamma(t)$ and $\psi_p(t) \to \psi(t)$ as $p \to \infty$. For more information on this function, see [9] and the references therein.

Also, the q-Gamma function, $\Gamma_q(t)$ and the q-Psi function $\psi_q(t)$ are defined for $q \in (0,1)$ and $t > 0$ as:

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^\infty \frac{1-q^n}{1-q^{t+n}} \text{ and } \psi_q(t) = \frac{\Gamma_q'(t)}{\Gamma_q(t)}.$$

where $\Gamma_q(t) \to \Gamma(t)$ and $\psi_q(t) \to \psi(t)$ as $q \to 1^-$.

See also [4,5] and the references therein.

Similarly, the k-Gamma function, $\Gamma_k(t)$ and the k-Psi function $\psi_k(t)$ are defined for $k > 0$ and $t > 0$ as (see [2,7]):

$$\Gamma_k(t) = \int_0^\infty e^{-\frac{x^k}{k}} x^{t-1} dx \text{ and } \psi_k(t) = \frac{\Gamma_k'(t)}{\Gamma_k(t)}$$

where $\Gamma_k(t) \to \Gamma(t)$ and $\psi_k(t) \to \psi(t)$ as $k \to 1$.

Also, the (q,k)-Gamma function $\Gamma_{(q,k)}(t)$ and the (q,k)-Psi function $\psi_{(q,k)}(t)$ are defined for $q \in (0,1)$, $k > 0$ and $t > 0$ as [3]:

$$\Gamma_{(q,k)}(t) = \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}} \text{ and } \psi_{(q,k)}(t) = \frac{\Gamma_{(q,k)}'(t)}{\Gamma_{(q,k)}(t)}$$

where $(t)_{n,k} = \prod_{j=0}^{n-1}(t+jk)$ is the k-generalized Pochhammer symbol and $\Gamma_{(q,k)}(t) \to \Gamma(t)$, $\psi_{(q,k)}(t) \to \psi(t)$ as $q \to 1^-$, $k \to 1$.

Furthermore, the (p,q)-Gamma function $\Gamma_{(p,q)}(t)$ and the (p,q)-Psi function $\psi_{(p,q)}(t)$ are defined for $p \in N$, $q \in (0,1)$ and $t > 0$ as [8]:

$$\Gamma_{(p,q)}(t) = \frac{[p]_q^t [p]_q!}{[t]_q [t+1]_q \cdots [t+p]_q}$$

and

$$\psi_{(p,q)}(t) = \frac{\Gamma'_{(p,q)}(t)}{\Gamma_{(p,q)}(t)}$$

where $[p]_q = \dfrac{1-q^p}{1-q}$, and $\Gamma_{(p,q)}(t) \to \Gamma(t)$, $\psi_{(p,q)}(t) \to \psi(t)$ as $p \to \infty$, $q \to 1^-$.

As defined above, the generalized Psi functions: $\psi_p(t)$, $\psi_q(t)$, $\psi_k(t)$, $\psi_{(q,k)}(t)$ and $\psi_{(p,q)}(t)$ possess the following series forms (see [16,17] and the references therein):

$$\psi_p(t) = \text{In}p - \sum_{n=0}^{p} \frac{1}{n+t} \tag{1}$$

$$\psi_q(t) = -\text{In}(1-q) + (\text{In}q)\sum_{n=1}^{\infty} \frac{q^{nt}}{1-q^n} \tag{2}$$

$$\psi_k(t) = \frac{\text{In}k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} \tag{3}$$

$$\psi_{(p,q)}(t) = \text{In}[p]_q + (\text{In}q)\sum_{n=1}^{p} \frac{q^{nt}}{1-q^n} \tag{4}$$

$$\psi_{(q,k)}(t) = \frac{-\text{In}(1-q)}{k} + (\text{In}q)\sum_{n=1}^{\infty} \frac{q^{nkt}}{1-q^{nk}} \tag{5}$$

with $\gamma = \lim\limits_{n\to\infty} \left(\sum\limits_{k=1}^{n} \frac{1}{k} - \text{In}n \right) = 0.5721566...$ denoting the Euler-Mascheroni's constant.

2. Results

We now present our results. Let us begin with the following Lemmas pertaining to the results.

Lemma 2.1. Assume that $\lambda \ge \mu > 0$, $p \in N$, $q \in (0,1)$ and $g(t) > 0$. Then,

$$\lambda\text{In}(1-q) + \mu\text{In}[p]_q$$
$$+ \lambda\psi_q(g(t)) - \mu\psi_{(p,q)}(g(t)) \le 0.$$

Proof. By using equations (2) and (4) we obtain,

$$\lambda\text{In}(1-q) + \mu\text{In}[p]_q + \lambda\psi_q(g(t)) - \mu\psi_{(p,q)}(g(t))$$
$$= (\text{In}q)\left[\lambda\sum_{n=1}^{\infty} \frac{q^{ng(t)}}{1-q^n} - \mu\sum_{n=1}^{p} \frac{q^{ng(t)}}{1-q^n} \right] \le 0.$$

concluding the proof.

Lemma 2.2. Assume that $\lambda \ge \mu > 0$, $q \in (0,1)$, $k \ge 1$ and $g(t) > 0$. Then,

$$\lambda\text{In}(1-q) - \mu\frac{\text{In}(1-q)}{k}$$
$$+ \lambda\psi_q(g(t)) - \mu\psi_{(q,k)}(g(t)) \le 0.$$

Proof. By using equations (2) and (5) we obtain,

$$\lambda\text{In}(1-q) - \mu\frac{\text{In}(1-q)}{k}$$
$$+ \lambda\psi_q(g(t)) - \mu\psi_{(q,k)}(g(t))$$
$$= (\text{In}q)\sum_{n=1}^{\infty}\left[\lambda\frac{q^{ng(t)}}{1-q^n} - \mu\frac{q^{nkg(t)}}{1-q^{nk}} \right] \le 0$$

concluding the proof.

Lemma 2.3. Assume that $\lambda > 0$, $\mu > 0$, $k > 0$, $p \in N$, $q \in (0,1)$ and $g(t) > 0$. Then,

$$\mu\text{In}[p]_q - \frac{\lambda\text{In}k}{k} + \frac{\lambda\gamma}{k} + \frac{\lambda}{g(t)}$$
$$+ \lambda\psi_k(g(t)) - \mu\psi_{(p,q)}(g(t)) > 0.$$

Proof. By using equations (3) and (4) we obtain,

$$\mu\text{In}[p]_q - \frac{\lambda\text{In}k}{k} + \frac{\lambda\gamma}{k} + \frac{\lambda}{g(t)}$$
$$+ \lambda\psi_k(g(t)) - \mu\psi_{(p,q)}(g(t))$$
$$= \lambda\sum_{n=1}^{\infty} \frac{g(t)}{nk(nk+g(t))} - \mu(\text{In}q)\sum_{n=1}^{p} \frac{q^{ng(t)}}{1-q^n} > 0$$

concluding the proof.

Lemma 2.4. Assume that $\lambda > 0$, $\mu > 0$, $q \in (0,1)$, $k > 0$ and $g(t) > 0$. Then,

$$\frac{\lambda\gamma}{k} + \frac{\lambda}{g(t)} - \frac{\text{In}(k^{\lambda}(1-q)^{\mu})}{k}$$
$$+ \lambda\psi_k(g(t)) - \mu\psi_{(q,k)}(g(t)) > 0.$$

Proof. By using equations (3) and (5) we obtain,

$$\frac{\lambda\gamma}{k} + \frac{\lambda}{g(t)} - \frac{\text{In}(k^{\lambda}(1-q)^{\mu})}{k}$$
$$+ \lambda\psi_k(g(t)) - \mu\psi_{(q,k)}(g(t))$$
$$= \lambda\sum_{n=1}^{\infty} \frac{g(t)}{nk(nk+g(t))} - \mu(\text{In}q)\sum_{n=1}^{\infty} \frac{q^{nkg(t)}}{1-q^{nk}} > 0$$

concluding the proof.

Theorem 2.5. Let $g(t)$ be a positive, increasing and differentiable function, $p \in N$ and $q \in (0,1)$. Then for positive real numbers λ and μ such that $\lambda \ge \mu$, the inequalities:

$$\frac{(1-q)^{\lambda(g(0)-g(x))}\Gamma_q(g(0))^{\lambda}}{[p]_q^{-\mu(g(0)-g(x))}\Gamma_{(p,q)}(g(0))^{\mu}}$$
$$\ge \frac{\Gamma_q(g(x))^{\lambda}}{\Gamma_{(p,q)}(g(x))^{\mu}} \tag{6}$$
$$\ge \frac{(1-q)^{\lambda(g(y)-g(x))}\Gamma_q(g(y))^{\lambda}}{[p]_q^{-\mu(g(y)-g(x))}\Gamma_{(p,q)}(g(y))^{\mu}}$$

hold true for $0 < x < y$.

Proof. Define a function G for $p \in N$ and $q \in (0,1)$ by

$$G(t) = \frac{(1-q)^{\lambda g(t)} \Gamma_q(g(t))^\lambda}{[p]_q^{-\mu g(t)} \Gamma_{(p,q)}(g(t))^\mu}, \quad t \in (0,\infty).$$

Let $u(t) = \mathrm{In}G(t)$. Then,

$$u(t) = \mathrm{In} \frac{(1-q)^{\lambda g(t)} \Gamma_q(g(t))^\lambda}{[p]_q^{-\mu g(t)} \Gamma_{(p,q)}(g(t))^\mu}$$

$$= \lambda g(t)\mathrm{In}(1-q) + \mu g(t)\mathrm{In}[p]_q$$
$$+ \lambda \mathrm{In}\Gamma_q(g(t)) - \mu \mathrm{In}\Gamma_{(p,q)}(g(t)).$$

Then,

$$u'(t) = \lambda g'(t)\mathrm{In}(1-q) + \mu g'(t)\mathrm{In}[p]_q$$
$$+ \lambda g'(t)\psi_q(g(t)) - \mu g'(t)\psi_{(p,q)}(g(t))$$
$$= g'(t)\Big[\lambda \mathrm{In}(1-q) + \mu \mathrm{In}[p]_q$$
$$+ \lambda \psi_q(g(t)) - \mu \psi_{(p,q)}(g(t)) \Big] \le 0$$

as a consequence of Lemma 2.1. That implies u is non-increasing on $t \in (0,\infty)$. Hence $G = e^{u(t)}$ is non-increasing and for $0 < x < y$ we have,

$$G(0) \ge G(x) \ge G(y)$$

establishing the inequalities in (6).

Theorem 2.6. Let $g(t)$ be a positive, increasing and differentiable function, $q \in (0,1)$ and $k \ge 1$. Then for positive real numbers λ and μ such that $\lambda \ge \mu$, the inequalities:

$$\frac{(1-q)^{\lambda(g(0)-g(x))} \Gamma_q(g(0))^\lambda}{(1-q)^{\frac{\mu}{k}(g(0)-g(x))} \Gamma_{(q,k)}(g(0))^\mu}$$

$$\ge \frac{\Gamma_q(g(x))^\lambda}{\Gamma_{(q,k)}(g(x))^\mu} \tag{7}$$

$$\ge \frac{(1-q)^{\lambda(g(y)-g(x))} \Gamma_q(g(y))^\lambda}{(1-q)^{\frac{\mu}{k}(g(y)-g(x))} \Gamma_{(q,k)}(g(y))^\mu}$$

hold true for $0 < x < y$.

Proof. Define a function H for $q \in (0,1)$ and $k \ge 1$ by

$$H(t) = \frac{(1-q)^{\lambda g(t)} \Gamma_q(g(t))^\lambda}{(1-q)^{\frac{\mu g(t)}{k}} \Gamma_{(q,k)}(g(t))^\mu}, \quad t \in (0,\infty).$$

Let $v(t) = \mathrm{In}H(t)$. Then,

$$v(t) = \mathrm{In} \frac{(1-q)^{\lambda g(t)} \Gamma_q(g(t))^\lambda}{(1-q)^{\frac{\mu g(t)}{k}} \Gamma_{(q,k)}(g(t))^\mu}$$

$$= \lambda g(t)\mathrm{In}(1-q) - \frac{\mu g(t)}{k}\mathrm{In}(1\text{-}q)$$
$$+ \lambda \mathrm{In}\Gamma_q(g(t)) - \mu \mathrm{In}\Gamma_{(q,k)}(g(t)).$$

Then,

$$v'(t) = \lambda g'(t)\mathrm{In}(1-q) - \frac{\mu g'(t)}{k}\mathrm{In}(1-q)$$
$$+ \lambda g'(t)\psi_q(g(t)) - \mu g'(t)\psi_{(q,k)}(g(t))$$
$$= g'(t)\Big[\lambda \mathrm{In}(1-q) + \mu \frac{\mathrm{In}(1-q)}{k}$$
$$+ \lambda \psi_q(g(t)) - \mu \psi_{(q,k)}(g(t)) \Big] \le 0$$

as a consequence of Lemma 2.2. That implies v is non-increasing on $t \in (0,\infty)$. Hence $H = e^{v(t)}$ is non-increasing and for $0 < x < y$ we have,

$$H(0) \ge H(x) \ge H(y)$$

establishing the inequalities in (7).

Theorem 2.7. Let $g(t)$ be a positive, increasing and differentiable function, $k > 0$, $p \in N$ and $q \in (0,1)$. Then for positive real numbers λ and μ, the inequalities:

$$\frac{(g(0))^\lambda k^{-\frac{\lambda}{k}(g(0)-g(x))} e^{\frac{\lambda\gamma}{k}(g(0)-g(x))} \Gamma_k(g(0))^\lambda}{(g(x))^\lambda [p]_q^{-\mu(g(0)-g(x))} \Gamma_{(p,q)}(g(0))^\mu}$$

$$< \frac{\Gamma_k(g(x))^\lambda}{\Gamma_{(p,q)}(g(x))^\mu} < \tag{8}$$

$$\frac{(g(y))^\lambda k^{-\frac{\lambda}{k}(g(y)-g(x))} e^{\frac{\lambda\gamma}{k}(g(y)-g(x))} \Gamma_k(g(y))^\lambda}{(g(x))^\lambda [p]_q^{-\mu(g(y)-g(x))} \Gamma_{(p,q)}(g(y))^\mu}$$

hold true for $0 < x < y$.

Proof. Define a function S for $k > 0$, $p \in N$ and $q \in (0,1)$ by

$$S(t) = \frac{(g(t))^\lambda k^{-\frac{\lambda g(t)}{k}} e^{\frac{\lambda\gamma g(t)}{k}} \Gamma_k(g(t))^\lambda}{[p]_q^{-\mu g(t)} \Gamma_{(p,q)}(g(t))^\mu}, \quad t \in (0,\infty).$$

Let $w(t) = \mathrm{In}S(t)$. Then,

$$w(t) = \mathrm{In} \frac{(g(t))^\lambda k^{-\frac{\lambda g(t)}{k}} e^{\frac{\lambda\gamma g(t)}{k}} \Gamma_k(g(t))^\lambda}{[p]_q^{-\mu g(t)} \Gamma_{(p,q)}(g(t))^\mu}$$

$$= \mu g(t)\mathrm{In}[p]_q - \frac{\lambda g(t)}{k}\mathrm{In}k + \frac{\lambda\gamma g(t)}{k} + \lambda \mathrm{In}(g(t))$$
$$+ \lambda \mathrm{In}\Gamma_k(g(t)) - \mu \mathrm{In}\Gamma_{(p,q)}(g(t)).$$

Then,

$$w'(t) = \mu g'(t)\mathrm{In}[p]_q - \frac{\lambda g'(t)\mathrm{In}k}{k} + \frac{\lambda\gamma g'(t)}{k}$$
$$+ \lambda \frac{g'(t)}{g(t)} + \lambda g'(t)\psi_k(g(t)) - \mu g'(t)\psi_{(p,q)}(g(t))$$
$$= g'(t)\Big[\mu \mathrm{In}[p]_q - \frac{\lambda \mathrm{In}k}{k} + \frac{\lambda\gamma}{k} + \frac{\lambda}{g(t)}$$
$$+ \lambda \psi_k(g(t)) - \mu \psi_{(p,q)}(g(t)) \Big] > 0$$

as a result of Lemma 2.3. That implies w is increasing on $t \in (0,\infty)$. Hence $S = e^{w(t)}$ is increasing and for $0 < x < y$ we have,

$$S(0) < S(x) < S(y)$$

establishing the inequalities in (8).

Theorem 2.8. Let $g(t)$ be a positive, increasing and differentiable function, $k > 0$ and $q \in (0,1)$. Then for positive real numbers λ and μ, the inequalities:

$$\frac{(g(x))^{-\lambda}(g(0))^{\lambda} e^{\frac{\lambda\gamma}{k}(g(0)-g(x))} \Gamma_k(g(0))^{\lambda}}{k^{\frac{\lambda}{k}(g(0)-g(x))}(1-q)^{\frac{\mu}{k}(g(0)-g(x))} \Gamma_{(q,k)}(g(0))^{\mu}}$$

$$< \frac{\Gamma_k(g(x))^{\lambda}}{\Gamma_{(q,k)}(g(x))^{\mu}} < \quad (9)$$

$$\frac{(g(x))^{-\lambda}(g(y))^{\lambda} e^{\frac{\lambda\gamma}{k}(g(y)-g(x))} \Gamma_k(g(y))^{\lambda}}{k^{\frac{\lambda}{k}(g(y)-g(x))}(1-q)^{\frac{\mu}{k}(g(y)-g(x))} \Gamma_{(q,k)}(g(y))^{\mu}}$$

hold true for $0 < x < y$.

Proof. Define a function T for $k > 0$ and $q \in (0,1)$ by

$$T(t) = \frac{(g(t))^{\lambda} e^{\frac{\lambda\gamma g(t)}{k}} \Gamma_k(g(t))^{\lambda}}{k^{\frac{\lambda g(t)}{k}}(1-q)^{\frac{\mu g(t)}{k}} \Gamma_{(q,k)}(g(t))^{\mu}}, \quad t \in (0,\infty).$$

Let $\delta(t) = \ln T(t)$. Then,

$$\delta(t) = \ln \frac{(g(t))^{\lambda} e^{\frac{\lambda\gamma g(t)}{k}} \Gamma_k(g(t))^{\lambda}}{k^{\frac{\lambda g(t)}{k}}(1-q)^{\frac{\mu g(t)}{k}} \Gamma_{(q,k)}(g(t))^{\mu}}$$

$$= \lambda \ln g(t) + \frac{\lambda\gamma g(t)}{k} - \frac{\lambda g(t)}{k}\ln k - \frac{\mu g(t)}{k}\ln(1-q)$$

$$+ \lambda \ln\Gamma_k(g(t)) - \mu \ln\Gamma_{(q,k)}(g(t)).$$

Then,

$$\delta'(t) = \frac{\lambda\gamma g'(t)}{k} + \lambda\frac{g'(t)}{g(t)} - \frac{g'(t)\ln(k^{\lambda}(1-q)^{\mu})}{k}$$

$$+ \lambda g'(t)\psi_k(g(t)) - \mu g'(t)\psi_{(q,k)}(g(t))$$

$$= g'(t)\left[\frac{\lambda\gamma}{k} + \frac{\lambda}{g(t)} - \frac{\ln(k^{\lambda}(1-q)^{\mu})}{k}\right.$$

$$\left. + \lambda\psi_k(g(t)) - \mu\psi_{(q,k)}(g(t))\right] > 0$$

as a result of Lemma 2.4. That implies δ is -increasing on $t \in (0,\infty)$. Hence $T = e^{\delta(t)}$ is increasing and for $0 < x < y$ we have,

$$T(0) < T(x) < T(y)$$

establishing the inequalities in (9).

3. Concluding Remarks

In particular, if we let $g(t) = \alpha + \beta t$ for $\alpha > 0$ and $\beta > 0$ on the interval $0 < t < 1$, then we recover the entire results of [17]. Also, by setting $g(t) = \alpha + t$ and $\lambda = \mu = 1$ on the interval $0 < t < 1$, we obtain the results of [16]. The results [11] – [17] are therefore special cases of the results of this paper. For example, let $g(t) = \alpha + \beta t$ for $\alpha, \beta > 0$ on the interval

$0 < t < 1$. Then;

(i) by allowing $q \to 1$ in Theorem 2.5, we recover Theorem 3.7 of [13].

(ii) by allowing $k \to 1$ in Theorem 2.8, we recover Theorem 3.8 of [13].

(iii) by allowing $q \to 1$ in Theorem 2.6, we recover Theorem 3.9 of [13].

(iv) by allowing $k \to 1$ in Theorem 2.7, we recover Theorem 3.1 of [15].

This paper is a slightly modified version of preprint [18].

4. Open Problems

For $k > 0$, $p \in N$ and $q \in (0,1)$, let $\psi_p(t)$, $\psi_q(t)$, $\psi_{(p,q)}(t)$ and $\psi_{(q,k)}(t)$ be the generalized Psi functions as defined in equations (1) – (5).

Problem 1: Under what conditions will the statements:

$$\ln p + \ln(1-q) + \psi_q(t) - \psi_p(t)$$

$$= \sum_{n=0}^{p}\frac{1}{n+t} + (\ln q)\sum_{n=1}^{\infty}\frac{q^{nt}}{1-q^n} \le (\ge)0$$

be valid?

Problem 2: Under what conditions will the statements:

$$-\ln[p]_q - \frac{\ln(1-q)}{k} + \psi_{(p,q)}(t) - \psi_{(q,k)}(t)$$

$$= (\ln q)\left[\sum_{n=1}^{p}\frac{q^{nt}}{1-q^n} - \sum_{n=1}^{\infty}\frac{q^{nkt}}{1-q^{nk}}\right] \le (\ge)0$$

be valid?

Competing Interests

The authors declare that there is no competing interest.

Acknowlegement

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