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SOME INEQUALITIES FOR THE Q-EXTENSION OF THE GAMMA FUNCTION

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ABSTRACT

In this paper, the authors establish some inequalities involving the *q*-extension of the classical Gamma function. These inequalities provide bounds for certain ratios of the *q*-extended Gamma function. The procedure makes use of geometric convexity and monotonicity properties of certain functions associated with the *q*-extended Gamma function.

Keywords: Gamma function; *q*-extension; geometrically convex function; inequality.

Mathematics subject classification: 33B15, 33D05.

1 Introduction and Preliminaries

In recent years, the theory of inequalities has developed from a collection of isolated formulas into a vibrant independent area of research. This is manifested by the emergence of several new journals devoted to this area of research. Particularly, inequalities involving special functions have been studied intensively by researchers across the globe. In this study, we establish some new inequalities involving the *q*-extension of the Gamma function. Before we present our results, let us recall the following definitions pertaining to the results.

The classical Euler's Gamma function, $\Gamma(x)$ is usually defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = \lim_{n\to\infty}\left[\frac{n!n^x}{x(x+1)\cdots(x+n)}\right].$$

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It is well-known in literature that the Gamma function satisfies the following basic properties.

$$\Gamma(x+1) = x\Gamma(x), \quad x \in R^+ \tag{1}$$

$$\Gamma(n+1) = n!, \quad n \in Z^+ \tag{2}$$

Let $\psi(x)$ be the digamma or psi function defined for $x > 0$ as the logarithmic derivative of the Gamma function. That is,

$$\psi(x) = \frac{d}{dx} \text{In } \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} .$$

The following series representations hold true for $\psi(x)$, $x > 0$ [1].

$$\psi(x) = -\gamma + (x-1)\sum_{n=0}^{\infty} \frac{1}{(1+n)(n+x)} = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)} \tag{3}$$

where γ is the Euler-Mascheroni's constant given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \text{In } n \right) = -\psi(1) = 0.577215664\cdots \tag{4}$$

Let $\Gamma_q(x)$ be the *q*-extension (also known as, *q*-analogue, *q*-deformation or *q*-generalization) of the Gamma function defined for $x > 0$ and for fixed $q \in (0,1)$ by (see [2,3] and the references therein).

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}} = (1-q)^{1-x} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{n+x}} .$$

Similarly, $\Gamma_q(x)$ satisfies the following properties [4].

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad x \in R^+ \tag{5}$$

$$\Gamma_q(n+1) = [n]_q!, \quad n \in Z^+ \tag{6}$$

where $[x]_q = \frac{1-q^x}{1-q}$. Note that equations (5) and (6) are respectively the *q*-extensions of equations (1) and (2).

Likewise, the *q*-extension of the digamma function is defined for $x > 0$ and $q \in (0,1)$ as the logarithmic derivative of the function $\Gamma_q(x)$. That is,

$$\psi_q(x) = \frac{d}{dx} \text{In } \Gamma_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} .$$

It also exhibits the following series representations (see [5,6] and the related references therein).

$$\begin{aligned}\psi_q(x) &= -\text{In}(1-q) + (\text{In}q)\sum_{n=0}^{\infty}\frac{q^{n+x}}{1-q^{n+x}} \\ &= -\text{In}(1-q) + (\text{In}q)\sum_{n=1}^{\infty}\frac{q^{nx}}{1-q^{nx}}\end{aligned} \tag{7}$$

The function $\psi_q(x)$ is increasing for $x > 0$ [7, Lemma 2.2]. Also, for $q > 0$ and $x > 0$, $\psi_q(x)$ has a uniquely determined positive root [8, Lemma 4.5].

Further, let γ_q be the *q*-extension of the Euler-Mascheroni's constant (see [9-11]). Then,

$$\gamma_q = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{[k]_q} - \text{In}[n]_q\right) = -\psi_q(1)$$

The following limit relations are valid (see [10-12).

$$\lim_{n\to\infty}\Gamma_q(x) = \Gamma(x), \quad \lim_{n\to\infty}\psi_q(x) = \psi(x) \quad \text{and} \quad \lim_{n\to\infty}\gamma_q = \gamma.$$

Remark 1.1. Unlike the value of γ which is fixed, the value γ_q varies according to the value of q. Tables of some approximate values of γ_q can be found in [9,10].

By taking the *m*-th derivative of (7), it can easily be shown that

$$\psi_q^{(m)}(x) = (\text{In}q)^{m+1}\sum_{n=1}^{\infty}\frac{n^m q^{nx}}{1-q^{nx}}$$

for $m \geq 1$. The functions $\psi_q^{(m)}(x)$ are called the *q*-extension of the polygamma functions.

Definition 1.2. ([13-15]). Let $f : I \subseteq (0,\infty) \to (0,\infty)$ be a continuous function. Then, f is said to be *geometrically (or multiplicatively) convex* on I if there exist $n \geq 2$ such that one of the following two inequalities holds:

$$f\left(\sqrt{x_1 x_2}\right) \leq \sqrt{f(x_1)f(x_2)}, \tag{8}$$

$$f\left(\prod_{i=1}^{n} x_i^{\lambda_i}\right) \leq \prod_{i=1}^{n}[f(x_i)]^{\lambda_i} \tag{9}$$

where $x_1, x_2, \cdots, x_n \in I$ and $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$ with $\sum_{i=1}^{n}\lambda_i = 1$. If inequalities (8) and (9) are reversed, then f is said to be *geometrically (or multiplicatively) concave* on I.

In 1971, Kečkić and Vasić [16, Theorem 1] established the double inequality

$$\frac{x^{x-1}e^{y}}{y^{y-1}e^{x}} \leq \frac{\Gamma(x)}{\Gamma(y)} \leq \frac{x^{x-\frac{1}{2}}e^{y}}{y^{y-\frac{1}{2}}e^{x}} \tag{10}$$

for $x \geq y > 1$, by employing the monotonicity properties of certain functions involving the Gamma function.

Also, in 2007, Zhang, Xu and Situ [15, Theorem 1.2] established the double inequality

$$\frac{x^{x}}{y^{y}}\left(\frac{x}{y}\right)^{y[\psi(y)-\text{In}\, y]} e^{y-x} \leq \frac{\Gamma(x)}{\Gamma(y)} \leq \frac{x^{x}}{y^{y}}\left(\frac{x}{y}\right)^{x[\psi(x)-\text{In}\, x]} e^{y-x} \tag{11}$$

for $x > 0$ and $y > 0$, by using the geometric convexity of a certain function related to the Gamma function, and as a byproduct, inequality (10) was recovered.

Furthermore, in 2010, Krasniqi and Shabani [13, Theorem 3.5] also established the following related inequality for the *p*-Gamma function.

$$\left(\frac{x}{y}\right)^{y[1+\psi(y)]} e^{y-x} \leq \frac{\Gamma_{p}(x)}{\Gamma_{p}(y)} \leq \left(\frac{x}{y}\right)^{x[1+\psi(x)]} e^{y-x} \tag{12}$$

for $x > 0$ and $y > 0$.

For more information on inequalities of this nature, one could access the review article by Qi [17].

Lemma 1.3. *Let* $f : I \subseteq (0,\infty) \rightarrow (0,\infty)$ *be a differentiable function. Then* f *is said to be geometrically convex if and only if the function* $\frac{xf'(x)}{f(x)}$ *is non-decreasing.*

Lemma 1.4. *Let* $f : I \subseteq (0,\infty) \rightarrow (0,\infty)$ *be a differentiable function. Then* f *is said to be geometrically convex if and only if the function* $\frac{f(x)}{f(y)} \geq \left(\frac{x}{y}\right)^{\frac{yf'(y)}{f(y)}}$ *holds for any* $x, y \in I$ *.*

For proofs of Lemmas 1.3 and 1.4, see [14].

The purpose of this paper is to establish some related inequalities for the *q*-extension of the Gamma function, by using geometric convexity and monotonicity features of certain functions associated with the *q*-extended Gamma function. We present our results in the following section.

2 Main Results

Theorem 2.1. *Let* $x \geq 1$, $y \geq 1$ *and* $q \in (0,1)$. *Then the double inequality*

$$\left(\frac{x}{y}\right)^{y\left(-\frac{\text{In}\, q}{1-q}q^y + \psi_q(y)\right)} e^{\left(\frac{q^x - q^y}{1-q}\right)} \leq \frac{\Gamma_q(x)}{\Gamma_q(y)} \leq \left(\frac{x}{y}\right)^{x\left(-\frac{\text{In}\, q}{1-q}q^x + \psi_q(x)\right)} e^{\left(\frac{q^x - q^y}{1-q}\right)} \tag{13}$$

holds true.

Proof. Define a function f for $x \geq 1$ and $q \in (0,1)$ by $f(x) = e^{[x]_q}\Gamma_q(x)$. Then, $\text{In}\, f(x) = [x]_q + \text{In}\Gamma_q(x)$ which implies $\frac{f'(x)}{f(x)} = [x]_q' + \psi_q(x) = -\frac{(\text{In}\, q)q^x}{1-q} + \psi_q(x)$.

That further implies,

$$\left(\frac{xf'(x)}{f(x)}\right) = -x\frac{(\text{In}\, q)q^x}{1-q} + x\psi_q(x) \text{ yielding,}$$

$$\begin{aligned}
\left(\frac{xf'(x)}{f(x)}\right)' &= -\left(\frac{(\text{In}\, q)q^x}{1-q} + x\frac{(\text{In}\, q)^2 q^x}{1-q}\right) + \psi_q(x) + x\psi_q'(x) \\
&= -\frac{(\text{In}\, q)q^x}{1-q} - x\frac{(\text{In}\, q)^2 q^x}{1-q} - \text{In}(1-q) + (\text{In}\, q)\sum_{n=1}^{\infty}\frac{q^{nx}}{1-q^n} + x(\text{In}\, q)^2\sum_{n=1}^{\infty}\frac{nq^{nx}}{1-q^n} \\
&= -\text{In}(1-q) + (\text{In}\, q)\sum_{n=2}^{\infty}\frac{q^{nx}}{1-q^n} + x(\text{In}\, q)^2\sum_{n=2}^{\infty}\frac{nq^{nx}}{1-q^n} \\
&= -\text{In}(1-q) + \sum_{n=2}^{\infty}\left[\frac{(\text{In}\, q)q^{nx} + nx(\text{In}\, q)^2 q^{nx}}{1-q^n}\right] \geq 0
\end{aligned}$$

Thus, $\frac{xf'(x)}{f(x)}$ is non-decreasing. Therefore, by Lemmas 1.3 and 1.4, f is geometrically convex, and consequently $\frac{f(x)}{f(y)} \geq \left(\frac{x}{y}\right)^{\frac{yf'(y)}{f(y)}}$ resulting to

$$\frac{e^{[x]_q}\Gamma_q(x)}{e^{[y]_q}\Gamma_q(y)} \geq \left(\frac{x}{y}\right)^{y\left([y]_q' + \psi_q(y)\right)} \tag{14}$$

and

$$\frac{e^{[y]_q}\Gamma_q(y)}{e^{[x]_q}\Gamma_q(x)} \geq \left(\frac{y}{x}\right)^{x\left([x]_q' + \psi_q(x)\right)} \tag{15}$$

Combining (14) and (15) concludes the proof of Theorem 2.1. Observe that $[y]_q - [x]_q = \frac{q^x - q^y}{1-q}$.

Corollary 2.2. *For* $x>0$ *and* $q \in (0,1)$, *the inequalities*

$$\left(\frac{x+1}{x+\frac{1}{2}}\right)^{\left(x+\frac{1}{2}\right)\left(-\frac{\ln q}{1-q}q^{\left(x+\frac{1}{2}\right)}+\psi_q\left(x+\frac{1}{2}\right)\right)} e^{q^x\left(\frac{q-\sqrt{q}}{1-q}\right)} \le \frac{\Gamma_q(x+1)}{\Gamma_q\left(x+\frac{1}{2}\right)} \le \left(\frac{x+1}{x+\frac{1}{2}}\right)^{(x+1)\left(-\frac{\ln q}{1-q}q^{(x+1)}+\psi_q(x+1)\right)} e^{q^x\left(\frac{q-\sqrt{q}}{1-q}\right)} \tag{16}$$

are valid.

Proof. This follows directly from Theorem 2.1 by substituting x by $x+1$, and y by $x+\frac{1}{2}$.

Theorem 2.3. *Let* $x>0$, $y>0$ *and* $\alpha \ge x^*$, *where* x^* *is the unique positive root of* $\psi_q(x)$. *Then for fixed* $q \in (0,1)$, *the double inequality*

$$e^{y-x}\frac{(x+\alpha)}{(y+\alpha)}\left(\frac{x}{y}\right)^{y\left(\frac{y+\alpha-1}{y+\alpha}+\psi_q(y+\alpha)\right)} \le \frac{\Gamma_q(x+\alpha)}{\Gamma_q(y+\alpha)} \le e^{y-x}\frac{(x+\alpha)}{(y+\alpha)}\left(\frac{x}{y}\right)^{x\left(\frac{x+\alpha-1}{x+\alpha}+\psi_q(x+\alpha)\right)} \tag{17}$$

holds true.

Proof. Define a function g for $x>0$ and $q \in (0,1)$ by $g(x) = \frac{e^x \Gamma_q(x+\alpha)}{x+\alpha}$. Then,

$\ln g(x) = x + \ln\Gamma_q(x+\alpha) - \ln(x+\alpha)$. This implies $\frac{g'(x)}{g(x)} = 1 + \psi_q(x+\alpha) - \frac{1}{x+\alpha}$.

Then,

$$\frac{xg'(x)}{g(x)} = x + x\psi_q(x+\alpha) - \frac{x}{x+\alpha}$$

from which we obtain,

$$\left(\frac{xg'(x)}{g(x)}\right)' = 1 + \psi_q(x+\alpha) + x\psi_q'(x+\alpha) - \frac{1}{(x+\alpha)^2} > 0$$

Thus, $\frac{xg'(x)}{g(x)}$ is nondecreasing. Therefore, by Lemmas 1.3 and 1.4, g is geometrically convex, and thus $\frac{g(x)}{g(y)} \ge \left(\frac{x}{y}\right)^{\frac{yg'(y)}{g(y)}}$. Consequently, we obtain

$$\frac{(y+\alpha)e^x\Gamma_q(x+\alpha)}{(x+\alpha)e^y\Gamma_q(y+\alpha)} \ge \left(\frac{x}{y}\right)^{y\left(\frac{y+\alpha-1}{y+\alpha}+\psi_q(y+\alpha)\right)}$$

and

$$\frac{(x+\alpha)e^{y}\Gamma_q(y+\alpha)}{(y+\alpha)e^{x}\Gamma_q(x+\alpha)} \geq \left(\frac{y}{x}\right)^{x\left(\frac{x+\alpha-1}{x+\alpha}+\psi_q(x+\alpha)\right)}$$

concluding the proof of Theorem 2.3.

Theorem 2.4. *Let* $x > y > 0$. *Then for fixed* $q \in (0,1)$, *the double inequality*

$$e^{(x-y)\psi_q(y)} < \frac{\Gamma_q(x)}{\Gamma_q(y)} < e^{(x-y)\psi_q(x)} \tag{18}$$

holds true.

Proof. Define a function h_q for $t > 0$ and $q \in (0,1)$ by $h_q(t) = \text{In}\Gamma_q(t)$. Let (y, x) be fixed.

Then, by the well-known mean value theorem, there exists a $c \in (y, x)$ such that

$$h_q'(c) = \frac{\text{In}\Gamma_q(x) - \text{In}\Gamma_q(y)}{x-y}$$

implying,

$$\psi_q(c) = \frac{1}{x-y}\text{In}\frac{\Gamma_q(x)}{\Gamma_q(y)}.$$

Since $\psi_q(t)$ is increasing for $t > 0$, then for $c \in (y, x)$ we obtain

$$\psi_q(y) < \frac{1}{x-y}\text{In}\frac{\Gamma_q(x)}{\Gamma_q(y)} < \psi_q(x).$$

That is,

$$(x-y)\psi_q(y) < \text{In}\frac{\Gamma_q(x)}{\Gamma_q(y)} < (x-y)\psi_q(x).$$

Exponentiating yields the desired results.

Remark 2.5. The double inequality (18) provides the *q*-extension of [15, Corollary 1.5] and [18, Corollary 2].

Corollary 2.6. *For* $x > 0$, $\mu > \lambda > 0$ *and* $q \in (0,1)$, *the inequalities*

$$e^{(\mu-\lambda)\psi_q(x+\lambda)} < \frac{\Gamma_q(x+\mu)}{\Gamma_q(x+\lambda)} < e^{(\mu-\lambda)\psi_q(x+\mu)} \tag{19}$$

hold true.

Proof. This follows directly from Theorem 2.4 by substituting x by $x+\mu$, and y by $x+\lambda$.

Remark 2.7. If we set $\mu=1$ in Corollary 2.6, then we obtain the *q*-extension of the result of Laforgia and Natalini [19, Theorem 3.1].

Corollary 2.8. *For* $x>0$ *and* $q\in(0,1)$, *the inequalities*

$$e^{\frac{1}{2}\psi_q(x+\frac{1}{2})} < \frac{\Gamma_q(x+1)}{\Gamma_q(x+\frac{1}{2})} < e^{\frac{1}{2}\psi_q(x+1)} \tag{20}$$

hold true.

Proof. Follows from Theorem 2.4 by substituting x by $x+1$, and y by $x+\frac{1}{2}$.

Remark 2.9. By virtue of relation (5), inequalities (20) can be rearranged as

$$\left(\frac{1-q}{1-q^x}\right)e^{\frac{1}{2}\psi_q(x+\frac{1}{2})} < \frac{\Gamma_q(x)}{\Gamma_q(x+\frac{1}{2})} < \left(\frac{1-q}{1-q^x}\right)e^{\frac{1}{2}\psi_q(x+1)}. \tag{21}$$

Remark 2.10. Results similar to inequalities (16) and (20) can also be found in [20-22].

3 Conclusion

In the paper, the authors have established some inequalities for the *q*-extension of the classical Gamma function. The results provide generalizations for several previous results. The findings of this research could provide useful information for researchers interested in *q*-analysis in particular, and the theory of inequalities in general. In addition, a further research could be conducted to see if similar results could be obtained for other special functions like the *q*-Beta and *q*-Psi functions. This could further expand the potential applications of our results.

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Competing Interests

Authors have declared that no competing interests exist.

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