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SOME INEQUALITIES FOR q AND (q, k) DEFORMED GAMMA FUNCTIONS

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Abstract − In this short paper, the authors establish some inequalities involving the q and (q, k) deformed Gamma functions by employing some basic analytical techniques.

Keywords − *Gamma function, q-deformation, (q, k)-deformation, q-addition, inequality.*

1 Introduction

Let $\Gamma(x)$ be the classical Gamma function and $\psi(x)$ be the classical Psi or Digamma function defined for $x \in R^+$ as:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt,$$

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

It is common knowledge in literature that the Gamma function satisfies the following properties.

$$\Gamma(n+1) = n!, \quad n \in Z^+,$$
$$\Gamma(x+1) = x\Gamma(x), \quad x \in R^+.$$

Also, let $\Gamma_q(x)$ be the q-deformed Gamma function (also known as the q-Gamma function or the q-analogue of the Gamma function) and $\psi_q(x)$ be the q-deformed Psi function defined for $q \in (0, 1)$ and $x \in R^+$ as (See [6], [7] and the references therein):

$$\Gamma_q(x) = (1-q)^{1-x}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{x+n}} \quad \text{and} \quad \psi_q(x) = \frac{d}{dx}\ln\Gamma_q(x)$$

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with $\Gamma_q(x)$ satisfying the properties:

$$\Gamma_q(n+1) = [n]_q! \quad n \in Z^+, \tag{1}$$
$$\Gamma_q(x+1) = [x]_q \Gamma_q(x) \quad x \in R^+. \tag{2}$$

where $[x]_q = \frac{1-q^x}{1-q}$ and $[x+y]_q = [x]_q + q^x[y]_q$ for $x, y \in R^+$. See [2].

Similarly, let $\Gamma_{(q,k)}(x)$ be the (q,k)-deformed Gamma function and $\psi_{(q,k)}(x)$ be the (q,k)-deformed Psi function defined for $q \in (0,1)$, $k > 0$ and $x \in R^+$ as (See [2], [8], [10] and the references therein):

$$\Gamma_{(q,k)}(x) = \frac{(1-q^k)_{q,k}^{\frac{x}{k}-1}}{(1-q)^{\frac{x}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^x)_{q,k}^{\infty}.(1-q)^{\frac{x}{k}-1}} \quad \text{and} \quad \psi_{(q,k)}(x) = \frac{d}{dx} \ln \Gamma_{(q,k)}(x)$$

where $(x+y)_{q,k}^n = \prod_{j=0}^{n-1}(x + q^{jk}y)$ with $\Gamma_{(q,k)}(x)$ satisfying the following property:

$$\Gamma_{(q,k)}(x+k) = [x]_q \Gamma_{(q,k)}(x), \quad x \in R^+. \tag{3}$$

The q-addition (otherwise known as the q-analogue or q-deformation of the ordinary addition) can be defined in the following two ways:

The Nalli-Ward-Alsalam q-addition, $\oplus_q$ is defined (See [11], [1], [3]) as:

$$(a \oplus_q b)^n := \sum_{k=1}^{n} \binom{n}{k}_q a^k b^{n-k} \quad \text{for} \quad a, b \in R, \, n \in N. \tag{4}$$

where $\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$ is the q-binomial coefficient.

The Jackson-Hahn-Cigler q-addition, $\boxplus_q$ is defined (See [4], [5], [3]) as:

$$(a \boxplus_q b)^n := \sum_{k=1}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{2}} a^{n-k} b^k \quad \text{for} \quad a, b \in R, \, n \in N. \tag{5}$$

Notice that both $\oplus_q$ and $\boxplus_q$ reduce to the ordinary addition, $+$ when $q = 1$.

In a recent paper [9], the inequalities:

$$\frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)} < \frac{(m+n)^{m+n}}{m^m n^n}, \qquad m, n \in Z^+ \tag{6}$$
$$\frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} \leq \frac{(x+y)^{x+y}}{x^x y^y}, \qquad x, y \in R^+ \tag{7}$$

which occur in the study of probability theory were presented together with some other results. In this paper, the objective is to establish related inequalities for the q and (q,k) deformed Gamma functions. The results are presented in the following section.

2 Main Results

Theorem 2.1. Let $q \in (0,1)$ and $m, n \in Z^+$. Then, the inequality:

$$\frac{\Gamma_q(m+n+1)}{\Gamma_q(m+1)\Gamma_q(n+1)} \leq \frac{(m \oplus_q n)^{m+n}}{m^m n^n} \tag{8}$$

holds true.

Proof. By equation (4) we obtain;

$$(m \oplus_q n)^{m+n} \geq \binom{m+n}{m}_q m^m n^n$$

since the binomial expansion of $(m \oplus_q n)^{m+n}$ includes the term $\binom{m+n}{m}_q m^m n^n$ as well as some other terms. That implies,

$$\frac{[m+n]_q!}{[m]_q![n]_q!} \leq \frac{(m \oplus_q n)^{m+n}}{m^m n^n}.$$

Now using relation (1) yields,

$$\frac{\Gamma_q(m+n+1)}{\Gamma_q(m+1)\Gamma_q(n+1)} \leq \frac{(m \oplus_q n)^{m+n}}{m^m n^n}$$

completing the proof.

□

Theorem 2.2. Let $q \in (0,1)$ and $m, n \in Z^+$. Then, the inequality:

$$\frac{\Gamma_q(m+n+1)}{\Gamma_q(m+1)\Gamma_q(n+1)} \leq \frac{(m \boxplus_q n)^{m+n} q^{\frac{n(1-n)}{2}}}{m^m n^n} \tag{9}$$

holds true.

Proof. Similarly, by equation (5) we obtain;

$$(m \boxplus_q n)^{m+n} \geq \binom{m+n}{n}_q q^{\frac{n(n-1)}{2}} m^m n^n.$$

Implying,

$$\frac{[m+n]_q!}{[m]_q![n]_q!} \leq \frac{(m \boxplus_q n)^{m+n} q^{\frac{n(1-n)}{2}}}{m^m n^n}.$$

By relation (1), we obtain;

$$\frac{\Gamma_q(m+n+1)}{\Gamma_q(m+1)\Gamma_q(n+1)} \leq \frac{(m \boxplus_q n)^{m+n} q^{\frac{n(1-n)}{2}}}{m^m n^n}$$

concluding the proof.

□

Lemma 2.3. If $q \in (0,1)$ and $x \in (0,1)$ then,

$$\ln(1-q^x) - \ln(1-q) < 0. \tag{10}$$

Proof. We have $q^x > q$ for all $q \in (0,1)$ and $x \in (0,1)$. That implies, $1 - q^x < 1 - q$. Taking the logarithm of both sides concludes the proof.

□

Theorem 2.4. Let $q \in (0,1)$ fixed, $x \in (0,1)$ and $y \in (0,1)$ be such that $\psi_q(x+1) > 0$. Then, the inequality:

$$\frac{\Gamma_q(x+y+1)}{\Gamma_q(x+1)\Gamma_q(y+1)} \geq \frac{[x+y]_q^{[x+y]_q}}{[x]_q^{[x]_q}[y]_q e^{q^x[y]_q}\Gamma_q(y)} \tag{11}$$

holds true.

Proof. Let Q and T be defined for $q \in (0,1)$ fixed, $x \in (0,1)$ and $y \in (0,1)$ by,

$$Q(x) = \frac{e^{[x]_q}\Gamma_q(x+1)}{[x]_q^{[x]_q}} \quad \text{and} \quad T(x,y) = \frac{Q(x+y)}{Q(x)Q(y)}.$$

Let $\mu(x) = \ln Q(x)$. That is,

$$\begin{aligned} \mu(x) &= [x]_q + \ln\Gamma_q(x+1) - [x]_q \ln[x]_q. \quad \text{Then,} \\ \mu(x)' &= \psi_q(x+1) + (\ln q)\frac{q^x}{1-q}\ln[x]_q \\ &= \psi_q(x+1) + (\ln q)\frac{q^x}{1-q}\left(\ln(1-q^x) - \ln(1-q)\right) > 0 \end{aligned}$$

This is as a result of Lemma 2.3 and the fact that $\ln q < 0$ for $q \in (0,1)$. Hence $Q(x)$ is increasing.

Next, we have,

$$T(x,y) = \frac{Q(x+y)}{Q(x)Q(y)} = \frac{Q(x+y)}{Q(x)}.\frac{1}{Q(y)} \geq \frac{1}{Q(y)} = \frac{[y]_q^{[y]_q}}{e^{[y]_q}[y]_q\Gamma_q(y)}$$

since $Q(x)$ is increasing and $\Gamma_q(y+1) = [y]_q\Gamma(y)$. That implies,

$$\begin{aligned} T(x,y) &= \frac{[x]_q^{[x]_q}[y]_q^{[y]_q}}{[x+y]_q^{[x+y]_q}}.\frac{e^{[x+y]_q}}{e^{[x]_q+[y]_q}}.\frac{\Gamma_q(x+y+1)}{\Gamma_q(x+1)\Gamma_q(y+1)} \\ &= \frac{[x]_q^{[x]_q}[y]_q^{[y]_q}}{[x+y]_q^{[x+y]_q}}.\frac{e^{[x]_q+q^x[y]_q}}{e^{[x]_q+[y]_q}}.\frac{\Gamma_q(x+y+1)}{\Gamma_q(x+1)\Gamma_q(y+1)} \geq \frac{[y]_q^{[y]_q}}{e^{[y]_q}[y]_q\Gamma_q(y)} \end{aligned}$$

yielding the results as in (11). □

Remark 2.5. Let $B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$ be the q-deformation of the classical Beta function. Then, inequality (11) can be rearranged as follows.

$$B_q(x,y) \leq \frac{[x]_q^{[x]_q-1}e^{q^x[y]_q}\Gamma_q(y)}{[x+y]_q^{[x+y]_q-1}}.$$

Theorem 2.6. Let $q \in (0,1)$ fixed, $k > 0$ and $x \in (0,1)$ be such that $\psi_{(q,k)}(x+k) > 0$. Then, the inequality:

$$\frac{\Gamma_{(q,k)}(x+y+k)}{\Gamma_{(q,k)}(x+k)\Gamma_{(q,k)}(y+k)} \geq \frac{[x+y]_q^{[x+y]_q}}{[x]_q^{[x]_q}[y]_q e^{q^x[y]_q}\Gamma_{(q,k)}(y)} \tag{12}$$

is valid.

Proof. Let G and H be defined for $q \in (0,1)$ fixed, $k > 0$, $x \in (0,1)$ and $y \in (0,1)$ by,

$$G(x) = \frac{e^{[x]_q}\Gamma_{(q,k)}(x+k)}{[x]_q^{[x]_q}} \quad \text{and} \quad H(x,y) = \frac{G(x+y)}{G(x)G(y)}.$$

In a similar fashion, let $\lambda(x) = \ln G(x)$. That is,

$$\begin{aligned} \lambda(x) &= [x]_q + \ln\Gamma_{(q,k)}(x+k) - [x]_q\ln[x]_q. \quad \text{Then,} \\ \lambda(x)' &= \psi_{(q,k)}(x+k) + (\ln q)\frac{q^x}{1-q}\left(\ln(1-q^x) - \ln(1-q)\right) > 0. \end{aligned}$$

Hence $G(x)$ is increasing.

Next, observe that,

$$H(x,y) = \frac{G(x+y)}{G(x)G(y)} = \frac{G(x+y)}{G(x)}.\frac{1}{G(y)} \geq \frac{1}{G(y)} = \frac{[y]_q^{[y]_q}}{e^{[y]_q}[y]_q\Gamma_{(q,k)}(y)}$$

since $G(x)$ is increasing and $\Gamma_{(q,k)}(y+k) = [y]_q\Gamma_{(q,k)}(y)$. That implies,

$$H(x,y) = \frac{[x]_q^{[x]_q}[y]_q^{[y]_q}}{[x+y]_q^{[x+y]_q}}.\frac{e^{[x]_q+q^x[y]_q}}{e^{[x]_q+[y]_q}}.\frac{\Gamma_{(q,k)}(x+y+k)}{\Gamma_{(q,k)}(x+k)\Gamma_{(q,k)}(y+k)} \geq \frac{[y]_q^{[y]_q}}{e^{[y]_q}[y]_q\Gamma_{(q,k)}(y)}$$

establishing the results as in (12). □

Remark 2.7. Let $B_{(q,k)}(x,y) = \frac{\Gamma_{(q,k)}(x)\Gamma_{(q,k)}(y)}{\Gamma_{(q,k)}(x+y)}$ be the (q,k)-deformation of the classical Beta function. Then, inequality (12) can be written as follows.

$$B_{(q,k)}(x,y) \leq \frac{[x]_q^{[x]_q-1}e^{q^x[y]_q}\Gamma_{(q,k)}(y)}{[x+y]_q^{[x+y]_q-1}}.$$

3 Concluding Remarks

Some new inequalities related to (6) and (7) have been established for the q and (q,k) deformed Gamma functions. In particular, if we allow $q \to 1$ in either inequality (8) or (9), then, inequality (6) is restored as a special case. Also, by allowing $q \to 1$ in (12), then we obtain the k-analogue of inequality (11).

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