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Generalizations of Some Sharp Inequalities for the Ratio of Gamma Functions

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Abstract

In this paper, we present some generalizations of the inequalities presented by the authors in [4].

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1 Introduction

The classical Euler's Gamma function, $\Gamma(t)$ is commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \qquad t > 0.$$

The digamma function $\psi(t)$, also known as the psi function is defined as follows.

$$\psi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}$$

The (p,q)-analogue of the Gamma function, $\Gamma_{(p,q)}(t)$ is defined by Krasniqi and Merovci [2] as

$$\Gamma_{(p,q)}(t) = \frac{[p]_q^t [p]_q!}{[t]_q [t+1]_q \dots [t+p]_q}, \quad t > 0, \quad p \in N, \quad q \in (0,1)$$

where $[p]_q = \frac{1-q^p}{1-q}$.

The (p,q)-analogue of the digamma function, $\psi_{(p,q)}(t)$ is also defined as follows.

$$\psi_{(p,q)}(t) = \frac{d}{dt}\ln(\Gamma_{(p,q)}(t)) = \frac{\Gamma'_{(p,q)}(t)}{\Gamma_{(p,q)}(t)}$$

Also, the (q,k)-analogue of the Gamma function, $\Gamma_{(q,k)}(t)$ is defined as (see [1],[3])

$$\Gamma_{(q,k)}(t) = \frac{(1-q^k)^{\frac{t}{k}-1}_{q,k}}{(1-q)^{\frac{t}{k}-1}} = \frac{(1-q^k)^{\infty}_{q,k}}{(1-q^t)^{\infty}_{q,k}(1-q)^{\frac{t}{k}-1}}, \quad t>0,\ q\in(0,1),\ k>0.$$

Similarly, the (q,k)-analogue of the digamma function, $\psi_{(q,k)}(t)$ is defined as

$$\psi_{(q,k)}(t) = \frac{d}{dt}\ln(\Gamma_{(q,k)}(t)) = \frac{\Gamma'_{(q,k)}(t)}{\Gamma_{(q,k)}(t)}$$

The functions $\psi(t)$, $\psi_{(p,q)}(t)$ and $\psi_{(q,k)}(t)$ as defined above possess the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty}\frac{1}{(1+n)(n+t)} \tag{1}$$

$$\psi_{(p,q)}(t) = \ln[p]_q + (\ln q)\sum_{n=1}^{p}\frac{q^{nt}}{1-q^n} \tag{2}$$

$$\psi_{(q,k)}(t) = \frac{-\ln(1-q)}{k} + (\ln q)\sum_{n=1}^{\infty}\frac{q^{nkt}}{1-q^{nk}} \tag{3}$$

where γ is the Euler-Mascheroni's constant.

Recently, Nantomah and Prempeh [4] established the following results.

$$\frac{e^{-\gamma t}\Gamma(\alpha)}{[p]_q^t\Gamma_{(p,q)}(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_{(p,q)}(\alpha+t)} < \frac{e^{\gamma(1-t)}\Gamma(\alpha+1)}{[p]_q^{t-1}\Gamma_{(p,q)}(\alpha+1)} \tag{4}$$

for $t\in(0,1)$, where $p\in N$, $q\in(0,1)$ and α is a positive real number such that $\alpha+t>1$.

$$\frac{e^{-\gamma t}\Gamma(\alpha)}{(1-q)^{-\frac{t}{k}}\Gamma_{(q,k)}(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_{(q,k)}(\alpha+t)} < \frac{e^{\gamma(1-t)}\Gamma(\alpha+1)}{(1-q)^{\frac{1}{k}(1-t)}\Gamma_{(q,k)}(\alpha+1)} \tag{5}$$

for $t \in (0,1)$, where $q \in (0,1)$, $k > 0$ and α is a positive real number such that $\alpha + t > 1$.

The objective of this paper is to present some generalizations of the above inequalities.

2 Preliminary Results

Lemma 2.1. *Let $a > 0$, $b > 0$ and $t > 1$, Then,*

$$a\gamma + b\ln[p]_q + a\psi(t) - b\psi_{(p,q)}(t) > 0.$$

Proof. Using the series representations in equations (1) and (2) we have,

$$\begin{aligned} &a\gamma + b\ln[p]_q + a\psi(t) - b\psi_{(p,q)}(t) \\ &\qquad = a(t-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+t)} - b(\ln q)\sum_{n=1}^{p}\frac{q^{nt}}{1-q^n} > 0 \end{aligned}$$

Lemma 2.2. *Let $a > 0$, $b > 0$ and $\alpha + \beta t > 1$. Then,*

$$a\gamma + b\ln[p]_q + a\psi(\alpha+\beta t) - b\psi_{(p,q)}(\alpha+\beta t) > 0.$$

Proof. Follows directly from Lemma 2.1.

Lemma 2.3. *Let $a > 0$, $b > 0$ and $t > 1$, Then,*

$$a\gamma - b\frac{\ln(1-q)}{k} + a\psi(t) - b\psi_{(q,k)}(t) > 0.$$

Proof. Using the series representations in equations (1) and (3) we have,

$$\begin{aligned} &a\gamma - b\frac{\ln(1-q)}{k} + a\psi(t) - b\psi_{(q,k)}(t) \\ &\qquad = a(t-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+t)} - b(\ln q)\sum_{n=1}^{\infty}\frac{q^{nkt}}{1-q^{nk}} > 0 \end{aligned}$$

Lemma 2.4. *Let $a > 0$, $b > 0$ and $\alpha + \beta t > 1$. Then,*

$$a\gamma - b\frac{\ln(1-q)}{k} + a\psi(\alpha+\beta t) - b\psi_{(q,k)}(\alpha+\beta t) > 0.$$

Proof. Follows directly from Lemma 2.3.

3 Main Results

Theorem 3.1. *Define a function F by*

$$F(t) = \frac{e^{a\beta\gamma t}\Gamma(\alpha+\beta t)^a}{[p]_q^{-b\beta t}\Gamma_{(p,q)}(\alpha+\beta t)^b}, \quad t \in (0,\infty),\, p \in N,\, q \in (0,1) \tag{6}$$

where a, b, α, β are positive real numbers such that $\alpha + \beta t > 1$. Then F is increasing on $t \in (0,\infty)$ and the inequalities

$$\frac{e^{-a\beta\gamma t}\Gamma(\alpha)^a}{[p]_q^{b\beta t}\Gamma_{(p,q)}(\alpha)^b} < \frac{\Gamma(\alpha+\beta t)^a}{\Gamma_{(p,q)}(\alpha+\beta t)^b} < \frac{e^{a\beta\gamma(1-t)}\Gamma(\alpha+\beta)^a}{[p]_q^{b\beta(t-1)}\Gamma_{(p,q)}(\alpha+\beta)^b} \tag{7}$$

are valid for every $t \in (0,1)$.

Proof. Let $\mu(t) = \ln F(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned}\mu(t) &= \ln \frac{e^{a\beta\gamma t}\Gamma(\alpha+\beta t)^a}{[p]_q^{-b\beta t}\Gamma_{(p,q)}(\alpha+\beta t)^b} \\ &= a\beta\gamma t + b\beta t \ln[p]_q + a\ln\Gamma(\alpha+\beta t) - b\ln\Gamma_{(p,q)}(\alpha+\beta t).\end{aligned}$$

Then,

$$\begin{aligned}\mu'(t) &= a\beta\gamma + b\beta\ln[p]_q + a\beta\psi(\alpha+\beta t) - b\beta\psi_{(p,q)}(\alpha+\beta t) \\ &= \beta\left[a\gamma + b\ln[p]_q + a\psi(\alpha+\beta t) - b\psi_{(p,q)}(\alpha+\beta t)\right] > 0\end{aligned}$$

by Lemma 2.2. That implies μ is increasing on $t \in (0,\infty)$. Hence F is increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$ we have,

$$F(0) < F(t) < F(1)$$

concluding the proof.

Corollary 3.2. *If $t \in (1,\infty)$, then the following inequality is valid.*

$$\frac{\Gamma(\alpha+\beta t)^a}{\Gamma_{(p,q)}(\alpha+\beta t)^b} > \frac{e^{a\beta\gamma(1-t)}\Gamma(\alpha+\beta)^a}{[p]_q^{b\beta(t-1)}\Gamma_{(p,q)}(\alpha+\beta)^b}$$

Proof. If $t \in (1,\infty)$, then we have $F(t) > F(1)$ concluding the proof.

Theorem 3.3. *Define a function G by*

$$G(t) = \frac{e^{a\beta\gamma t}\Gamma(\alpha+\beta t)^a}{(1-q)^{\frac{b\beta t}{k}}\Gamma_{(q,k)}(\alpha+\beta t)^b}, \quad t \in (0,\infty),\, q \in (0,1),\, k > 0 \tag{8}$$

where a, b, α, β are positive real numbers such that $\alpha + \beta t > 1$. Then G is increasing on $t \in (0,\infty)$ and the inequalities

$$\frac{e^{-a\beta\gamma t}\Gamma(\alpha)^a}{(1-q)^{-\frac{b\beta t}{k}}\Gamma_{(q,k)}(\alpha)^b} < \frac{\Gamma(\alpha+\beta t)^a}{\Gamma_{(q,k)}(\alpha+\beta t)^b} < \frac{e^{a\beta\gamma(1-t)}\Gamma(\alpha+\beta)^a}{(1-q)^{\frac{b\beta}{k}(1-t)}\Gamma_{(q,k)}(\alpha+\beta)^b} \tag{9}$$

are valid for every $t \in (0,1)$.

Proof. Let $\lambda(t) = \ln G(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned}\lambda(t) &= \ln \frac{e^{a\beta\gamma t}\Gamma(\alpha+\beta t)^a}{(1-q)^{\frac{b\beta t}{k}}\Gamma_{(q,k)}(\alpha+\beta t)^b} \\ &= a\beta\gamma t - \frac{b\beta t}{k}\ln(1-q) + a\ln\Gamma(\alpha+\beta t) - b\ln\Gamma_{(q,k)}(\alpha+\beta t).\end{aligned}$$

Then,

$$\begin{aligned}\lambda'(t) &= a\beta\gamma - \frac{b\beta}{k}\ln(1-q) + a\beta\psi(\alpha+\beta t) - b\beta\psi_{(q,k)}(\alpha+\beta t) \\ &= \beta\left[a\gamma - \frac{b}{k}\ln(1-q) + a\psi(\alpha+\beta t) - b\psi_{(q,k)}(\alpha+\beta t)\right] > 0\end{aligned}$$

by Lemma 2.4. That implies λ is increasing on $t \in (0,\infty)$. Hence G is increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$ we have,

$$G(0) < G(t) < G(1)$$

yielding the result.

Corollary 3.4. *If $t \in (1,\infty)$, then the following inequality is valid.*

$$\frac{\Gamma(\alpha+\beta t)^a}{\Gamma_{(q,k)}(\alpha+\beta t)^b} > \frac{e^{a\beta\gamma(1-t)}\Gamma(\alpha+\beta)^a}{(1-q)^{\frac{b\beta}{k}(1-t)}\Gamma_{(q,k)}(\alpha+\beta)^b}$$

Proof. If $t \in (1,\infty)$, then we have $G(t) > G(1)$ yielding the result.

4 Concluding Remarks

Remark 4.1. By setting $a = b = \beta = 1$, then the entire results of the paper [4] are obtained as a special case. We have thus generalized our previous results.

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