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Some Inequalities for the q-Digamma Function

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Abstract

Some inequalities involving the q-digamma function are presented. These results are the q-analogues of some recent results.

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1 Introduction and Preliminaries

The classical Euler's Gamma function $\Gamma(t)$ and the digamma function $\psi(t)$ are commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \quad \text{and} \quad \psi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$

Similarly the q-Gamma and q-digamma functions are defined as (see [1])

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0.$$

and

$$\psi_q(t) = \frac{d}{dt} \ln(\Gamma_q(t)) = \frac{\Gamma_q'(t)}{\Gamma_q(t)}$$

The functions $\psi(t)$ and $\psi_q(t)$ as defined above exhibit the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)}, \quad t > 0.$$

$$\psi_q(t) = -\ln(1-q) + (\ln q)\sum_{n=1}^{\infty} \frac{q^{nt}}{1-q^n}, \quad q \in (0,1), \quad t > 0.$$

where γ is the Euler-Mascheroni's constant.

By taking the m-th derivative of the above functions, we arrive at the following statements for $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+t)^{m+1}}, \quad t > 0.$$

$$\psi_q^{(m)}(t) = (\ln q)^{m+1} \sum_{n=1}^{\infty} \frac{n^m q^{nt}}{1-q^n}, \quad q \in (0,1), \quad t > 0.$$

In 2011, Sulaiman [3] presented the following results.

$$\psi(t+s) \geq \psi(t) + \psi(s) \tag{1}$$

where $t > 0$ and $0 < s < 1$.

$$\psi^{(m)}(t+s) \leq \psi^{(m)}(t) + \psi^{(m)}(s) \tag{2}$$

where m is a positive odd integer and $t, s > 0$.

$$\psi^{(m)}(t+s) \geq \psi^{(m)}(t) + \psi^{(m)}(s) \tag{3}$$

where m is a positive even integer and $t, s > 0$.

In a recent paper, Sroysang [2] presented the following geralizations of the above inequalities.

$$\psi(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi(t) + \sum_{i=1}^{\alpha} \beta_i \psi(s_i) \tag{4}$$

where $t > 0$, $\beta_i > 0$ and $0 < s_i < 1$ for all $i \in N_\alpha$.

$$\psi^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \leq \psi^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi^{(m)}(s_i) \tag{5}$$

where m is a positive odd integer, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$.

$$\psi^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi^{(m)}(s_i) \tag{6}$$

where m is a positive even integer, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$.

The objective of this paper is to establish that the inequalities (4), (5) and (6) still hold true for the q-digamma function.

2 Main Results

We now present our results.

Theorem 2.1. *Let $q \in (0, 1)$, $t > 0$, $\beta_i > 0$ and $0 < s_i < 1$ for all $i \in N_\alpha$. Then the following inequality is valid.*

$$\psi_q(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi_q(t) + \sum_{i=1}^{\alpha} \beta_i \psi_q(s_i) \tag{7}$$

Proof. Let $u(t) = \psi_q(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_q(t) - \sum_{i=1}^{\alpha} \beta_i \psi_q(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned} u'(t) &= \psi_q'(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_q'(t) \\ &= (\ln q)^2 \sum_{n=1}^{\infty} \left[\frac{nq^{n(t+\sum_{i=1}^{\alpha} \beta_i s_i)}}{1-q^n} - \frac{nq^{nt}}{1-q^n} \right] \\ &= (\ln q)^2 \sum_{n=1}^{\infty} \frac{nq^{nt}(q^{n\sum_{i=1}^{\alpha} \beta_i s_i} - 1)}{1-q^n} \leq 0. \end{aligned}$$

That implies u is non-increasing. Moreover,

$$\begin{aligned} \lim_{t\to\infty} u(t) &= \lim_{t\to\infty} \left[\psi_q(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_q(t) - \sum_{i=1}^{\alpha} \beta_i \psi_q(s_i) \right] \\ &= \ln(1-q) \sum_{i=1}^{\alpha} \beta_i \\ &\quad + (\ln q) \lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{q^{n(t+\sum_{i=1}^{\alpha} \beta_i s_i)}}{1-q^n} - \frac{q^{nt}}{1-q^n} - \sum_{i=1}^{\alpha} \frac{\beta_i q^{ns_i}}{1-q^n} \right] \\ &= \ln(1-q) \sum_{i=1}^{\alpha} \beta_i - (\ln q) \sum_{n=1}^{\infty} \sum_{i=1}^{\alpha} \frac{\beta_i q^{ns_i}}{1-q^n} \geq 0. \end{aligned}$$

Therefore $u(t) \geq 0$ concluding the proof.

Theorem 2.2. *Let $q \in (0,1)$, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$. Suppose that m is a positive odd integer, then the following inequality is valid.*

$$\psi_q^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \leq \psi_q^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi_q^{(m)}(s_i) \tag{8}$$

Proof. Let $v(t) = \psi_q^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_q^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_q^{(m)}(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned} v'(t) &= \psi_q^{(m+1)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_q^{(m+1)}(t) \\ &= (\ln q)^{m+2} \sum_{n=1}^{\infty} \left[\frac{n^{m+1} q^{n(t+\sum_{i=1}^{\alpha} \beta_i s_i)}}{1-q^n} - \frac{n^{m+1} q^{nt}}{1-q^n} \right] \\ &= (\ln q)^{m+2} \sum_{n=1}^{\infty} \left[\frac{n^{m+1} q^{nt} (q^{n \sum_{i=1}^{\alpha} \beta_i s_i} - 1)}{1-q^n} \right] \geq 0. \text{ (since } m \text{ is odd)} \end{aligned}$$

That implies v is non-decreasing. Moreover,

$$\begin{aligned} \lim_{t\to\infty} v(t) &= \lim_{t\to\infty} \left[\psi_q^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_q^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_q^{(m)}(s_i) \right] \\ &= (\ln q)^{m+1} \lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{n^m q^{n(t+\sum_{i=1}^{\alpha} \beta_i s_i)}}{1-q^n} - \frac{n^m q^{nt}}{1-q^n} - \sum_{i=1}^{\alpha} \beta_i \frac{n^m q^{n s_i}}{1-q^n} \right] \\ &= -(\ln q)^{m+1} \sum_{n=1}^{\infty} \sum_{i=1}^{\alpha} \beta_i \frac{n^m q^{n s_i}}{1-q^n} \leq 0. \text{ (since } m \text{ is odd)} \end{aligned}$$

Therefore $v(t) \leq 0$ concluding the proof.

Theorem 2.3. *Let $q \in (0,1)$, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$. Suppose that m is a positive even integer, then the following inequality is valid.*

$$\psi_q^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi_q^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi_q^{(m)}(s_i) \tag{9}$$

Proof. Let $w(t) = \psi_q^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_q^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_q^{(m)}(s_i)$. Then fixing

s_i for each i we have,

$$\begin{aligned}
w'(t) &= \psi_q^{(m+1)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_q^{(m+1)}(t) \\
&= (\ln q)^{m+2}\sum_{n=1}^{\infty}\left[\frac{n^{m+1}q^{n(t+\sum_{i=1}^{\alpha}\beta_i s_i)}}{1-q^n} - \frac{n^{m+1}q^{nt}}{1-q^n}\right] \\
&= (\ln q)^{m+2}\sum_{n=1}^{\infty}\left[\frac{n^{m+1}q^{nt}(q^{n\sum_{i=1}^{\alpha}\beta_i s_i}-1)}{1-q^n}\right] \leq 0. \text{ (since } m \text{ is even)}
\end{aligned}$$

That implies w is non-increasing. Moreover,

$$\begin{aligned}
\lim_{t\to\infty} w(t) &= \lim_{t\to\infty}\left[\psi_q^{(m)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_q^{(m)}(t) - \sum_{i=1}^{\alpha}\beta_i\psi_q^{(m)}(s_i)\right] \\
&= (\ln q)^{m+1}\lim_{t\to\infty}\sum_{n=1}^{\infty}\left[\frac{n^m q^{n(t+\sum_{i=1}^{\alpha}\beta_i s_i)}}{1-q^n} - \frac{n^m q^{nt}}{1-q^n} - \sum_{i=1}^{\alpha}\beta_i\frac{n^m q^{ns_i}}{1-q^n}\right] \\
&= -(\ln q)^{m+1}\sum_{n=1}^{\infty}\sum_{i=1}^{\alpha}\beta_i\frac{n^m q^{ns_i}}{1-q^n} \geq 0. \text{ (since } m \text{ is even)}
\end{aligned}$$

Therefore $w(t) \geq 0$ concluding the proof.

Remark 2.4. If we let $q \to 1^-$ in inequalities (7), (8) and (9) then we repectively recover the inequalities (4), (5) and (6).

References

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