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Some Sharp Inequalities for the Ratio of Gamma Functions

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Abstract

In this paper, we present some sharp inequalities involving ratios of the functions Γ, $\Gamma_{(p,q)}$ and $\Gamma_{(q,k)}$.

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1 Introduction

The classical Euler's Gamma function, $\Gamma(t)$ is commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}\,dx, \qquad t > 0.$$

The p-analogue of the Gamma function, $\Gamma_p(t)$ is defined as (see [7])

$$\Gamma_p(t) = \frac{p!p^t}{t(t+1)\dots(t+p)} = \frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})}, \quad p \in N, \quad t > 0.$$

The q-analogue of the Gamma function, $\Gamma_q(t)$ is defined as (see [2])

$$\Gamma_q(t) = (1-q)^{1-t}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0.$$

The k-analogue of the Gamma function $\Gamma_k(t)$ is also defined as (see [1],[3])

$$\Gamma_k(t) = \int_0^\infty e^{-\frac{x^k}{k}} x^{t-1}\, dx, \quad k > 0 \text{ a real number}, \quad t > 0.$$

In 2012, Krasniqi and Merovci [4] defined the (p,q)-analogue of the Gamma function, $\Gamma_{(p,q)}(t)$ as

$$\Gamma_{(p,q)}(t) = \frac{[p]_q^t [p]_q!}{[t]_q[t+1]_q \ldots [t+p]_q}, \quad t > 0, \quad p \in N, \quad q \in (0,1)$$

where $[p]_q = \frac{1-q^p}{1-q}$.

Also, the (q,k)-analogue of the Gamma function, $\Gamma_{(q,k)}(t)$ is defined as (see [5],[6])

$$\Gamma_{(q,k)}(t) = \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}(1-q)^{\frac{t}{k}-1}}, \quad t > 0,\ q \in (0,1),\ k > 0.$$

The digamma function, (p,q)-analogue of the digamma function and (q,k)-analogue of the digamma function are respectively defined as follows.

$$\psi(t) = \frac{d}{dt} \ln \Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$

$$\psi_{(p,q)}(t) = \frac{d}{dt} \ln \Gamma_{(p,q)}(t) = \frac{\Gamma'_{(p,q)}(t)}{\Gamma_{(p,q)}(t)}, \quad t > 0,\ p \in N,\ q \in (0,1).$$

$$\psi_{(q,k)}(t) = \frac{d}{dt} \ln \Gamma_{(q,k)}(t) = \frac{\Gamma'_{(q,k)}(t)}{\Gamma_{(q,k)}(t)}, \quad t > 0,\ q \in (0,1),\ k > 0.$$

These functions exhibit the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)} \tag{1}$$

$$\psi_{(p,q)}(t) = \ln[p]_q + (\ln q)\sum_{n=1}^{p} \frac{q^{nt}}{1-q^n} \tag{2}$$

$$\psi_{(q,k)}(t) = \frac{-\ln(1-q)}{k} + (\ln q)\sum_{n=1}^{\infty} \frac{q^{nkt}}{1-q^{nk}} \tag{3}$$

where γ is the Euler-Mascheroni's constant.

In [8], Krasniqi and Shabani presented the following results.

$$\frac{p^{-t}e^{-\gamma t}\Gamma(\alpha)}{\Gamma_p(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_p(\alpha+t)} < \frac{p^{1-t}e^{\gamma(1-t)}\Gamma(\alpha+1)}{\Gamma_p(\alpha+1)} \tag{4}$$

for $t \in (0,1)$, where α is a positive real number such that $\alpha + t > 1$.

Also in [7], Krasniqi, Mansour and Shabani presented the following.

$$\frac{(1-q)^t e^{-\gamma t}\Gamma(\alpha)}{\Gamma_q(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{(1-q)^{t-1}e^{\gamma(1-t)}\Gamma(\alpha+1)}{\Gamma_q(\alpha+1)} \tag{5}$$

for $t \in (0,1)$, where α is a positive real number such that $\alpha + t > 1$ and $q \in (0,1)$.

In a recent paper, Nantomah [9] also presented the following.

$$\frac{k^{-\frac{t}{k}}e^{-t\left(\frac{k\gamma-\gamma}{k}\right)}\Gamma(\alpha)}{\Gamma_k(\alpha)} \leq \frac{\Gamma(\alpha+t)}{\Gamma_k(\alpha+t)} \leq \frac{k^{\frac{1-t}{k}}e^{(1-t)\left(\frac{k\gamma-\gamma}{k}\right)}\Gamma(\alpha+1)}{\Gamma_k(\alpha+1)} \tag{6}$$

for $t \in (0,1)$, where α is a positive real number.

Results of this nature and some generalizations can also be found in the papers [10], [11] and [12].

The objective of this paper is to establish similar results involving ratios of the functions Γ, $\Gamma_{(p,q)}$ and $\Gamma_{(q,k)}$.

2 Preliminary Results

Lemma 2.1. *Let $t > 1$, Then,*

$$\gamma + \ln[p]_q + \psi(t) - \psi_{(p,q)}(t) > 0.$$

Proof. Using the series representations in equations (1) and (2) we have,

$$\gamma + \ln[p]_q + \psi(t) - \psi_{(p,q)}(t) = (t-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+t)} - (\ln q)\sum_{n=1}^{p}\frac{q^{nt}}{1-q^n} > 0$$

Lemma 2.2. *Let α be a positive real number such that $\alpha + t > 1$. Then,*

$$\gamma + \ln[p]_q + \psi(\alpha+t) - \psi_{(p,q)}(\alpha+t) > 0.$$

Proof. Follows directly from Lemma 2.1.

Lemma 2.3. *Let $t > 1$, Then,*

$$\gamma - \frac{\ln(1-q)}{k} + \psi(t) - \psi_{(q,k)}(t) > 0.$$

Proof. Using the series representations in equations (1) and (3) we have,

$$\gamma - \frac{\ln(1-q)}{k} + \psi(t) - \psi_{(q,k)}(t) = (t-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+t)} - (\ln q)\sum_{n=1}^{\infty}\frac{q^{nkt}}{1-q^{nk}} > 0$$

Lemma 2.4. *Let α be a positive real number such that $\alpha + t > 1$. Then,*

$$\gamma - \frac{\ln(1-q)}{k} + \psi(\alpha+t) - \psi_{(q,k)}(\alpha+t) > 0.$$

Proof. Follows directly from Lemma 2.3.

3 Main Results

Theorem 3.1. *Define a function U by*

$$U(t) = \frac{e^{\gamma t}\Gamma(\alpha+t)}{[p]_q^{-t}\Gamma_{(p,q)}(\alpha+t)}, \quad t \in (0,\infty), p \in N, q \in (0,1) \tag{7}$$

where α is a positive real number such that $\alpha + t > 1$. Then U is increasing on $t \in (0,\infty)$ and the inequalities

$$\frac{e^{-\gamma t}\Gamma(\alpha)}{[p]_q^t\Gamma_{(p,q)}(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_{(p,q)}(\alpha+t)} < \frac{e^{\gamma(1-t)}\Gamma(\alpha+1)}{[p]_q^{t-1}\Gamma_{(p,q)}(\alpha+1)} \tag{8}$$

are valid for every $t \in (0,1)$.

Proof. Let $\mu(t) = \ln U(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned}\mu(t) &= \ln\frac{e^{\gamma t}\Gamma(\alpha+t)}{[p]_q^{-t}\Gamma_{(p,q)}(\alpha+t)}\\ &= \gamma t + t\ln[p]_q + \ln\Gamma(\alpha+t) - \ln\Gamma_{(p,q)}(\alpha+t).\end{aligned}$$

Then,

$$\mu'(t) = \gamma + \ln[p]_q + \psi(\alpha+t) - \psi_{(p,q)}(\alpha+t) > 0$$

by Lemma 2.2. That implies μ is increasing on $t \in (0,\infty)$. Hence U is increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$ we have,

$$U(0) < U(t) < U(1)$$

yielding the result.

Corollary 3.2. *If $t \in (1, \infty)$, then the following inequality is valid.*

$$\frac{\Gamma(\alpha+t)}{\Gamma_{(p,q)}(\alpha+t)} > \frac{e^{\gamma(1-t)}\Gamma(\alpha+1)}{[p]_q^{t-1}\Gamma_{(p,q)}(\alpha+1)}$$

Proof. If $t \in (1, \infty)$, then we have $U(t) > U(1)$ yielding the result.

Theorem 3.3. *Define a function V by*

$$V(t) = \frac{e^{\gamma t}\Gamma(\alpha+t)}{(1-q)^{\frac{t}{k}}\Gamma_{(q,k)}(\alpha+t)}, \quad t \in (0,\infty),\ q \in (0,1),\ k > 0 \tag{9}$$

where α is a positive real number such that $\alpha + t > 1$. Then V is increasing on $t \in (0, \infty)$ and the inequalities

$$\frac{e^{-\gamma t}\Gamma(\alpha)}{(1-q)^{-\frac{t}{k}}\Gamma_{(q,k)}(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_{(q,k)}(\alpha+t)} < \frac{e^{\gamma(1-t)}\Gamma(\alpha+1)}{(1-q)^{\frac{1}{k}(1-t)}\Gamma_{(q,k)}(\alpha+1)} \tag{10}$$

are valid for every $t \in (0, 1)$.

Proof. Let $\lambda(t) = \ln V(t)$ for every $t \in (0, \infty)$. Then,

$$\begin{aligned}\lambda(t) &= \ln \frac{e^{\gamma t}\Gamma(\alpha+t)}{(1-q)^{\frac{t}{k}}\Gamma_{(q,k)}(\alpha+t)} \\ &= \gamma t - \frac{t}{k}\ln(1-q) + \ln\Gamma(\alpha+t) - \ln\Gamma_{(q,k)}(\alpha+t).\end{aligned}$$

Then,

$$\lambda'(t) = \gamma - \frac{\ln(1-q)}{k} + \psi(\alpha+t) - \psi_{(q,k)}(\alpha+t) > 0$$

by Lemma 2.4. That implies λ is increasing on $t \in (0, \infty)$. Hence V is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have,

$$V(0) < V(t) < V(1)$$

establishing the result.

Corollary 3.4. *If $t \in (1, \infty)$, then the following inequality is valid.*

$$\frac{\Gamma(\alpha+t)}{\Gamma_{(q,k)}(\alpha+t)} > \frac{e^{\gamma(1-t)}\Gamma(\alpha+1)}{(1-q)^{\frac{1}{k}(1-t)}\Gamma_{(q,k)}(\alpha+1)}$$

Proof. If $t \in (1, \infty)$, then we have $V(t) > V(1)$ yielding the result.

4 Concluding Remarks

Remark 4.1. If in (8) we allow $q \to 1$, then the inequality (4) is recovered.

Remark 4.2. If in (8) we allow $p \to \infty$ or if in (10) we allow $k \to 1$ then the inequality (5) is recovered.

Remark 4.3. If in (10) we allow $q \to 1$ then the inequality (6) is recovered.

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