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SOME INEQUALITIES FOR THE RATIOS OF GENERALIZED DIGAMMA FUNCTIONS

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Abstract. In this paper, some inequalities for the ratios of generalized digamma functions are presented. The approache makes use of the series representations of the (q,k)-digamma and (p,q)-digamma functions.

Keywords: digamma function; (q,k)-digamma function; (p,q)-digamma function; inequality.

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1. Introduction and preliminaries

The classical Euler's Gamma function $\Gamma(t)$ and the digamma function $\psi(t)$ are commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}\,dx, \qquad \psi(t) = \frac{d}{dt}\ln\Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t>0.$$

In 2005, Díaz and Teruel [1] defined the (q,k)-Gamma function, $\Gamma_{q,k}(t)$ as

$$\Gamma_{q,k}(t) = \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}(1-q)^{\frac{t}{k}-1}}, \quad t>0, k>0, q\in(0,1)$$

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with the (q,k)-digamma function, $\psi_{q,k}(t)$ is defined as

$$\psi_{q,k}(t)=\frac{d}{dt}\ln\Gamma_{q,k}(t)=\frac{\Gamma'_{q,k}(t)}{\Gamma_{q,k}(t)},\quad t>0, k>0, q\in(0,1).$$

Also in 2012, Krasniqi and Merovci [2] gave the (p,q)-Gamma function, $\Gamma_{p,q}(t)$ as

$$\Gamma_{p,q}(t)=\frac{[p]_q^t[p]_q!}{[t]_q[t+1]_q\ldots[t+p]_q},\quad t>0, p\in N, q\in(0,1),$$

where $[p]_q=\frac{1-q^p}{1-q}$.

Similarly, the (p,q)-digamma function, $\psi_{p,q}(t)$ is defined as

$$\psi_{p,q}(t)=\frac{d}{dt}\ln\Gamma_{p,q}(t)=\frac{\Gamma'_{p,q}(t)}{\Gamma_{p,q}(t)},\quad t>0, p\in N, q\in(0,1).$$

The functions $\psi_{q,k}(t)$ and $\psi_{p,q}(t)$ as defined above exhibit the following series representations.

$$\psi_{q,k}(t)=\frac{-\ln(1-q)}{k}+(\ln q)\sum_{n=1}^{\infty}\frac{q^{nkt}}{1-q^{nk}},\qquad t>0, \tag{1}$$

$$\psi_{p,q}(t)=\ln[p]_q+(\ln q)\sum_{n=1}^{p}\frac{q^{nt}}{1-q^{n}},\qquad t>0. \tag{2}$$

By taking derivatives of these functions, it can easily be established that

$$\psi'_{q,k}(t)=(\ln q)^2\sum_{n=1}^{\infty}\frac{nk.q^{nkt}}{1-q^{nk}},\quad t>0, \tag{3}$$

$$\psi'_{p,q}(t)=(\ln q)^2\sum_{n=1}^{p}\frac{n.q^{nt}}{1-q^{n}},\quad t>0. \tag{4}$$

In [3], Nantomah presented the following results for the digamma function.

$$\frac{[\psi(a)]^{\alpha}}{[\psi(c)]^{\beta}}\leq\frac{[\psi(a+bt)]^{\alpha}}{[\psi(c+dt)]^{\beta}}\leq\frac{[\psi(a+b)]^{\alpha}}{[\psi(c+d)]^{\beta}}, \tag{5}$$

where a, b, c, d, α, β are positive real numbers such that $\beta d\leq\alpha b$, $a+bt\leq c+dt$, $\psi(a+bt)>0$ and $\psi(c+dt)>0$. The k-analogue of these inequalities can be found in [4].

The purpose of this paper is to extend inequalities (5) to the (q,k) and (p,q)-digamma functions.

2. Results and discussion

We now present the results of this paper.

Lemma 2.1. *Let $0 < s \leq t$, then the following statement is valid.*

$$\psi_{q,k}(s) \leq \psi_{q,k}(t).$$

Proof. From (1), we have

$$\psi_{q,k}(s) - \psi_{q,k}(t) = (\ln q) \sum_{n=1}^{\infty} \left[\frac{q^{nks} - q^{nkt}}{1 - q^{nk}} \right] \leq 0.$$

Lemma 2.2. *Let $0 < s \leq t$, then the following statement is valid.*

$$\psi'_{q,k}(s) \geq \psi'_{q,k}(t).$$

Proof. From (3) we have,

$$\psi'_{q,k}(s) - \psi'_{q,k}(t) = (\ln q)^2 \sum_{n=1}^{\infty} \left[\frac{nk(q^{nks} - q^{nkt})}{1 - q^{nk}} \right] \geq 0.$$

Lemma 2.3. *Let a, b, c, d, α, β be positive real numbers such that $a+bt \leq c+dt$, $\beta d \leq \alpha b$, $\psi_{q,k}(a+bt) > 0$ and $\psi_{q,k}(c+dt) > 0$. Then*

$$\alpha b \psi_{q,k}(c+dt)\psi'_{q,k}(a+bt) - \beta d \psi_{q,k}(a+bt)\psi'_{q,k}(c+dt) \geq 0.$$

Proof. Since $0 < a+bt \leq c+dt$, then by Lemmas 2.1 and 2.2 we have

$$0 < \psi_{q,k}(a+bt) \leq \psi_{q,k}(c+dt)$$

and

$$\psi'_{q,k}(a+bt) \geq \psi'_{q,k}(c+dt) > 0.$$

This implies

$$\begin{aligned} \psi_{q,k}(c+dt)\psi'_{q,k}(a+bt) &\geq \psi_{q,k}(c+dt)\psi'_{q,k}(c+dt) \\ &\geq \psi_{q,k}(a+bt)\psi'_{q,k}(c+dt). \end{aligned}$$

Further, $\alpha b \geq \beta d$ implies

$$\begin{aligned} \alpha b \psi_{q,k}(c+dt)\psi'_{q,k}(a+bt) &\geq \alpha b \psi_{q,k}(a+bt)\psi'_{q,k}(c+dt) \\ &\geq \beta d \psi_{q,k}(a+bt)\psi'_{q,k}(c+dt). \end{aligned}$$

Hence, we have

$$\alpha b\psi_{q,k}(c+dt)\psi'_{q,k}(a+bt)-\beta d\psi_{q,k}(a+bt)\psi'_{q,k}(c+dt)\geq 0.$$

Theorem 2.4. *Define a function G by*

$$G(t)=\frac{\left[\psi_{q,k}(a+bt)\right]^{\alpha}}{\left[\psi_{q,k}(c+dt)\right]^{\beta}},\quad t\in[0,\infty), \tag{6}$$

where a, b, c, d, α, β are positive real numbers such that $a+bt\leq c+dt$, $\beta d\leq\alpha b$, $\psi_{q,k}(a+bt)>0$ and $\psi_{q,k}(c+dt)>0$. Then G is nondecreasing on $t\in[0,\infty)$ and the inequalities

$$\frac{\left[\psi_{q,k}(a)\right]^{\alpha}}{\left[\psi_{q,k}(c)\right]^{\beta}}\leq\frac{\left[\psi_{q,k}(a+bt)\right]^{\alpha}}{\left[\psi_{q,k}(c+dt)\right]^{\beta}}\leq\frac{\left[\psi_{q,k}(a+b)\right]^{\alpha}}{\left[\psi_{q,k}(c+d)\right]^{\beta}} \tag{7}$$

are valid for every $t\in[0,1]$.

Proof. Let $g(t)=\ln G(t)$ for every $t\in[0,\infty)$. Then,

$$g=\ln\frac{\left[\psi_{q,k}(a+bt)\right]^{\alpha}}{\left[\psi_{q,k}(c+dt)\right]^{\beta}}=\alpha\ln\psi_{q,k}(a+bt)-\beta\ln\psi_{q,k}(c+dt)$$

and

$$\begin{aligned}g'(t)&=\alpha b\frac{\psi'_{q,k}(a+bt)}{\psi_{q,k}(a+bt)}-\beta d\frac{\psi'_{q,k}(c+dt)}{\psi_{q,k}(c+dt)}\\&=\frac{\alpha b\psi'_{q,k}(a+bt)\psi_{q,k}(c+dt)-\beta d\psi'_{q,k}(c+dt)\psi_{q,k}(a+bt)}{\psi_{q,k}(a+bt)\psi_{q,k}(c+dt)}\geq 0\end{aligned}$$

as a result of Lemma 2.3. That implies g as well as G are nondecreasing on $t\in[0,\infty)$ and for every $t\in[0,1]$ we have

$$G(0)\leq G(t)\leq G(1)$$

concluding the proof.

Corollary 2.5. *If $t\in(1,\infty)$, then the following inequality is valid.*

$$\frac{\left[\psi_{q,k}(a+bt)\right]^{\alpha}}{\left[\psi_{q,k}(c+dt)\right]^{\beta}}\geq\frac{\left[\psi_{q,k}(a+b)\right]^{\alpha}}{\left[\psi_{q,k}(c+d)\right]^{\beta}} \tag{8}$$

Proof. For each $t\in(1,\infty)$, we have $G(t)\geq G(1)$ yielding the result.

Lemma 2.6. *Let $0<s\leq t$, then the following statement is valid.*

$$\psi_{p,q}(s)\leq\psi_{p,q}(t).$$

Proof. From (2) we have

$$\psi_{p,q}(s)-\psi_{p,q}(t)=(\ln q)\sum_{n=1}^{p}\left[\frac{q^{ns}-q^{nt}}{1-q^{n}}\right]\leq 0.$$

Lemma 2.7. *Let $0<s\leq t$, then the following statement is valid.*

$$\psi'_{p,q}(s)\geq \psi'_{p,q}(t).$$

Proof. From (4) we have

$$\psi'_{p,q}(s)-\psi'_{p,q}(t)=(\ln q)^2\sum_{n=1}^{p}\left[\frac{n(q^{ns}-q^{nt})}{1-q^{n}}\right]\geq 0.$$

Lemma 2.8. *Let a, b, c, d, α, β be positive real numbers such that $a+bt\leq c+dt$, $\beta d\leq \alpha b$, $\psi_{p,q}(a+bt)>0$ and $\psi_{p,q}(c+dt)>0$. Then*

$$\alpha b\psi_{p,q}(c+dt)\psi'_{p,q}(a+bt)-\beta d\psi_{p,q}(a+bt)\psi'_{p,q}(c+dt)\geq 0.$$

Proof. Follows the same argument as in the proof of Lemma 2.3.

Theorem 2.9. *Define a function H by*

$$H(t)=\frac{\left[\psi_{p,q}(a+bt)\right]^{\alpha}}{\left[\psi_{p,q}(c+dt)\right]^{\beta}},\quad t\in[0,\infty), \tag{9}$$

where a, b, c, d, α, β are positive real numbers such that $a+bt\leq c+dt$, $\beta d\leq \alpha b$, $\psi_{p,q}(a+bt)>0$ and $\psi_{p,q}(c+dt)>0$. Then H is nondecreasing on $t\in[0,\infty)$ and the inequalities

$$\frac{\left[\psi_{p,q}(a)\right]^{\alpha}}{\left[\psi_{p,q}(c)\right]^{\beta}}\leq\frac{\left[\psi_{p,q}(a+bt)\right]^{\alpha}}{\left[\psi_{p,q}(c+dt)\right]^{\beta}}\leq\frac{\left[\psi_{p,q}(a+b)\right]^{\alpha}}{\left[\psi_{p,q}(c+d)\right]^{\beta}} \tag{10}$$

are valid for every $t\in[0,1]$.

Proof. Follows the same procedure as in Theorem 2.4. Using Lemma 2.3, we conclude that H is nondecreasing on $t\in[0,\infty)$ and for every $t\in[0,1]$ we have, $H(0)\leq H(t)\leq H(1)$ ending the proof.

Corollary 2.10. *If $t\in(1,\infty)$, then the following inequality is valid.*

$$\frac{\left[\psi_{p,q}(a+bt)\right]^{\alpha}}{\left[\psi_{p,q}(c+dt)\right]^{\beta}}\geq\frac{\left[\psi_{p,q}(a+b)\right]^{\alpha}}{\left[\psi_{p,q}(c+d)\right]^{\beta}} \tag{11}$$

Proof. For each $t\in(1,\infty)$, we have $H(t)\geq H(1)$ yielding the result.

3. Concluding remarks

This section is dedicated to some remarks concerning our results.

Remark 3.1. If in (7) we allow $k \to 1$, then we obtain the q-analogue of (5).

Remark 3.2. If in (7) we allow $q \to 1^-$, then we obtain the k-analogue of (5) as presented in Theorem 3.7 of the paper [4].

Remark 3.3. If in (7) we allow $q \to 1^-$ as $k \to 1$, then we obtain (5).

Remark 3.4. If in (10) we allow $q \to 1^-$, then we obtain the p-analogue of (5).

Remark 3.5. If in (10) we allow $p \to \infty$, then we obtain the q-analogue of (5).

Remark 3.6. If in (10) we allow $p \to \infty$ as $q \to 1^-$, then we obtain (5).

Conflict of Interests.

The author declares that there is no conflict of interests.

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