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Some Inequalities Involving the Ratio of Gamma Functions

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Abstract

In this paper, we present and prove some inequalities involving the ratios $\frac{\Gamma_k(t)}{\Gamma_p(t)}$ and $\frac{\Gamma_k(t)}{\Gamma_q(t)}$. Our approach makes use of the series representations of the functions $\psi_p(t)$, $\psi_q(t)$ and $\psi_k(t)$.

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1. Introduction

We begin by recalling some basic definitions related to the Gamma function.

The classical Euler's Gamma function $\Gamma(t)$ is defined by,

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \qquad t > 0. \tag{1}$$

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The p-Gamma function $\Gamma_p(t)$, also known as the p-analogue of the Gamma function is defined as (see [3], [2])

$$\Gamma_p(t) = \frac{p!p^t}{t(t+1)\dots(t+p)} = \frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})}, \quad p \in N, \quad t > 0. \tag{2}$$

The p-psi function $\psi_p(t)$ is defined as the logarithmic derivative of the p-Gamma function. That is,

$$\psi_p(t) = \frac{d}{dt}\ln(\Gamma_p(t)) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \qquad t > 0. \tag{3}$$

The q-Gamma function, $\Gamma_q(t)$ is defined as (see [5])

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0. \tag{4}$$

The q-psi function, $\psi_q(t)$ is also defined as,

$$\psi_q(t) = \frac{d}{dt}\ln(\Gamma_q(t)) = \frac{\Gamma'_q(t)}{\Gamma_q(t)}, \qquad t > 0. \tag{5}$$

The k-Gamma function, $\Gamma_k(t)$ is defined as (see [1], [6])

$$\Gamma_k(t) = \int_0^{\infty} e^{-\frac{x^k}{k}} x^{t-1}\, dx, \quad k > 0, \quad t > 0. \tag{6}$$

The k-psi function, $\psi_k(t)$ is similarly defined as follows.

$$\psi_k(t) = \frac{d}{dt}\ln(\Gamma_k(t)) = \frac{\Gamma'_k(t)}{\Gamma_k(t)}, \quad t > 0. \tag{7}$$

In a recent paper [4], Krasniqi and Shabani proved the following result.

$$\frac{p^{-t}e^{-\gamma t}\Gamma(\alpha)}{\Gamma_p(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_p(\alpha+t)} < \frac{p^{1-t}e^{\gamma(1-t)}\Gamma(\alpha+1)}{\Gamma_p(\alpha+1)} \tag{8}$$

for $t \in (0,1)$, where α is a positive real number such that $\alpha + t > 1$.

Also in [2], Krasniqi, Mansour and Shabani proved the following result.

$$\frac{(1-q)^t e^{-\gamma t}\Gamma(\alpha)}{\Gamma_q(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{(1-q)^{t-1}e^{\gamma(1-t)}\Gamma(\alpha+1)}{\Gamma_q(\alpha+1)} \tag{9}$$

for $t \in (0,1)$, where α is a positive real number such that $\alpha + t > 1$ and $q \in (0,1)$.

Our objective is to establish and prove some results similar to (8) and (9).

2. Preliminaries

We present the following auxiliary results.

Lemma 2.1. *The function $\psi_p(t)$ as defined in (3) has the following series representation.*

$$\psi_p(t) = \ln p - \sum_{n=0}^{p} \frac{1}{n+t} \tag{10}$$

Proof. See [4].

Lemma 2.2. *The function $\psi_q(t)$ as defined in (5) has the following series representation.*

$$\psi_q(t) = -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{t+n}}{1-q^{t+n}} \tag{11}$$

Proof. See [2].

Lemma 2.3. *The function $\psi_k(t)$ as defined in (7) also has the following series representation.*

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} \tag{12}$$

where γ is the Euler-Mascheroni's constant.

Proof. See [6]

Lemma 2.4. *Let $t > 0$. Then,*

$$-\frac{\ln k - \gamma}{k} + \ln p + \frac{1}{t} + \psi_k(t) - \psi_p(t) > 0$$

Proof. Using the series representations in equations (10) and (12) we have,

$$-\frac{\ln k - \gamma}{k} + \ln p + \frac{1}{t} + \psi_k(t) - \psi_p(t) = \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} + \sum_{n=0}^{p} \frac{1}{(n+t)} > 0$$

Lemma 2.5. *Let α be a positive real number such that $\alpha + t > 0$. Then,*

$$-\frac{\ln k - \gamma}{k} + \ln p + \frac{1}{t} + \psi_k(\alpha + t) - \psi_p(\alpha + t) > 0$$

Proof. Follows directly from Lemma 2.4

Lemma 2.6. *Let $t > 0$. Then,*

$$-\frac{\ln k - \gamma}{k} - \ln(1-q) + \frac{1}{t} + \psi_k(t) - \psi_q(t) > 0$$

Proof. Using the series representations in equations (11) and (12) we have,

$$-\frac{\ln k-\gamma}{k}-\ln(1-q)+\frac{1}{t}+\psi_k(t)-\psi_q(t)=\sum_{n=1}^{\infty}\frac{t}{nk(nk+t)}-\ln q\sum_{n=0}^{\infty}\frac{q^{x+n}}{1-q^{x+n}}>0$$

Lemma 2.7. *Let α be a positive real number such that $\alpha+t>0$. Then,*

$$-\frac{\ln k-\gamma}{k}-\ln(1-q)+\frac{1}{t}+\psi_k(\alpha+t)-\psi_q(\alpha+t)>0$$

Proof. Follows directly from Lemma 2.6

3. Main Results

We now state and prove the results of this paper.

Theorem 3.1. *Define a function Ω by*

$$\Omega(t)=\frac{te^{-t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+t)}{p^{-t}\Gamma_p(\alpha+t)},\quad t\in(0,\infty),\ k>0,\ p\in N. \tag{13}$$

where α is a positive real number. Then Ω is increasing on $t\in(0,\infty)$ and the inequality

$$0<\frac{\Gamma_k(\alpha+t)}{\Gamma_p(\alpha+t)}<\frac{e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{tp^{t-1}\Gamma_p(\alpha+1)} \tag{14}$$

holds for every $t\in(0,1)$.

Proof. Let $u(t)=\ln\Omega(t)$ for every $t\in(0,\infty)$. Then,

$$\begin{aligned}u(t)&=\ln\frac{te^{-t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+t)}{p^{-t}\Gamma_p(\alpha+t)}\\&=\ln t+t\ln p-t(\frac{\ln k-\gamma}{k})+\ln\Gamma_k(\alpha+t)-\ln\Gamma_p(\alpha+t)\end{aligned}$$

Then,

$$u'(t)=-\frac{\ln k-\gamma}{k}+\ln p+\frac{1}{t}+\psi_k(\alpha+t)-\psi_p(\alpha+t)>0.\quad\text{(by Lemma 2.5)}$$

That implies u is increasing on $t\in(0,\infty)$. Hence Ω is increasing on $t\in(0,\infty)$ and for every $t\in(0,1)$ we have,

$$\Omega(0)<\Omega(t)<\Omega(1)\qquad\text{yielding,}$$

$$0<\frac{\Gamma_k(\alpha+t)}{\Gamma_p(\alpha+t)}<\frac{e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{tp^{t-1}\Gamma_p(\alpha+1)}.$$

Corollary 3.2. *If $t \in [1, \infty)$, then the following inequality holds.*

$$\frac{e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{tp^{t-1}\Gamma_p(\alpha+1)} \leq \frac{\Gamma_k(\alpha+t)}{\Gamma_p(\alpha+t)}$$

Proof. If $t \in [1, \infty)$, then we have $\Omega(1) \leq \Omega(t)$ yielding the result.

Theorem 3.3. *Define a function ϕ by*

$$\phi(t) = \frac{te^{-t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+t)}{(1-q)^t\Gamma_q(\alpha+t)}, \quad t \in (0, \infty),\ k > 0,\ q \in (0, 1). \tag{15}$$

where α is a positive real number. Then ϕ is increasing on $t \in (0, \infty)$ and the inequality

$$0 < \frac{\Gamma_k(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{t(1-q)^{1-t}\Gamma_q(\alpha+1)} \tag{16}$$

holds for every $t \in (0, 1)$.

Proof. Let $v(t) = \ln \phi(t)$ for every $t \in (0, \infty)$. Then,

$$\begin{aligned} v(t) &= \ln \frac{te^{-t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+t)}{(1-q)^t\Gamma_q(\alpha+t)} \\ &= \ln t - t\ln(1-q) - t(\frac{\ln k-\gamma}{k}) + \ln\Gamma_k(\alpha+t) - \ln\Gamma_q(\alpha+t) \end{aligned}$$

Then,

$$v'(t) = -\frac{\ln k-\gamma}{k} - \ln(1-q) + \frac{1}{t} + \psi_k(\alpha+t) - \psi_q(\alpha+t) > 0. \quad \text{(by Lemma 2.7)}$$

That implies v is increasing on $t \in (0, \infty)$. Hence ϕ is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have,

$$\phi(0) < \phi(t) < \phi(1) \qquad \text{yielding,}$$

$$0 < \frac{\Gamma_k(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{t(1-q)^{1-t}\Gamma_q(\alpha+1)}.$$

Corollary 3.4. *If $t \in [1, \infty)$, then the following inequality holds.*

$$\frac{e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{t(1-q)^{1-t}\Gamma_q(\alpha+1)} \leq \frac{\Gamma_k(\alpha+t)}{\Gamma_q(\alpha+t)}$$

Proof. If $t \in [1, \infty)$, then we have $\phi(1) \leq \phi(t)$ yielding the result.

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