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Extensions of Some Inequalities for the Gamma Function

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Abstract

In this paper, we improve the results of Shabani [7] concerning some inequalities for the Gamma function. Our approach makes use of the logarithmic derivative of products of the Gamma function. We also present some p-analogues.

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1 Introduction

We begin by recalling some definitions related to the Gamma function.

The classical Euler's Gamma function, $\Gamma(t)$ is defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \qquad t > 0. \tag{1}$$

The logarithmic derivative of the Gamma function is defined as

$$\phi(t) = \frac{d}{dt}\ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \qquad t > 0. \tag{2}$$

The p-analogue of the Gamma Function, $\Gamma_p(t)$ is defined as

$$\Gamma_p(t) = \frac{p!p^t}{t(t+1)\dots(t+p)} = \frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})}, \quad p \in \aleph, \quad t > 0. \tag{3}$$

(see also [3] and [4])

The equivalent definition of $\phi(t)$ in terms of the p-analogue is given as follows.

$$\phi_p(t) = \frac{d}{dt}\ln(\Gamma_p(t)) = \frac{\Gamma_p'(t)}{\Gamma_p(t)}, \quad p \in \aleph, \quad t > 0. \tag{4}$$

and

$$\lim_{p\to\infty} \Gamma_p(t) = \Gamma(t), \qquad \lim_{p\to\infty} \phi_p(t) = \phi(t) \tag{5}$$

Our aim in this paper is to establish and prove an extension of the generalized result of A. S. Shabani:

$$\frac{\Gamma(a+b)^c}{\Gamma(\alpha+\beta)^f} \leq \frac{\Gamma(a+bt)^c}{\Gamma(\alpha+\beta t)^f} \leq \frac{\Gamma(a)^c}{\Gamma(\alpha)^f}, \qquad t \in [0,1] \tag{6}$$

where $a, b, c, \alpha, \beta, f$ are positive real numbers such that $a+bt > 0$, $\alpha+\beta t > 0$, $a+bt \leq \alpha+\beta t$, $0 < bc \leq \beta f$ and $\phi(a+bt) > 0$ or $\phi(\alpha+\beta t) > 0$.
The result (6) is a generalisation of some earlier results by Alsina and Tomas [1], Bougoffa [2], Sandor [5] and Shabani [6].

2 Preliminaries

We present the following auxiliary results.

Lemma 2.1. *Let $t > 0$. Then $\phi(t)$ has the following series representation*

$$\phi(t) = -\gamma + (t-1)\sum_{k=0}^{\infty} \frac{1}{(1+k)(t+k)} \tag{7}$$

where γ is the Euler-Mascheroni's constant.

Proof. See [8].

Lemma 2.2. *Let $s > 0$, $t > 0$ with $s \leq t$, then*

$$\phi(s) \leq \phi(t). \tag{8}$$

Proof. From (7), we have the following.

$$\begin{aligned}
\phi(s) - \phi(t) &= (s-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+s)} - (t-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+t)} \\
&= \sum_{k=0}^{\infty}\frac{1}{(k+1)}\left(\frac{s-1}{k+s} - \frac{t-1}{k+t}\right) \\
&= \sum_{k=0}^{\infty}\frac{(s-t)}{(k+s)(k+t)} \leq 0
\end{aligned}$$

and the proof is complete.

Lemma 2.3. *Let a, b, α, β be real numbers such that $a + bt > 0$, $\alpha + \beta t > 0$. Then $a + bt \leq \alpha + \beta t$ implies $\phi(a + bt) \leq \phi(\alpha + \beta t)$.*

Proof. The proof follows directly from Lemma 2.2. (See also[6] and the references therein.)

We also have the following lemma from the paper [7].

Lemma 2.4. *Let $a, b, \alpha, \beta, r, q,$ be real numbers such that*
$a + bt > 0$, $\alpha + \beta t > 0$, $a + bt \leq \alpha + \beta t$ and $q\beta \geq rb$.
If (i) $\phi(a + bt) > 0$ or
(ii) $\phi(\alpha + \beta t) > 0$,
then $rb\phi(a + bt) - q\beta\phi(\alpha + \beta t) \leq 0.$

Proof. (i) If $\phi(a+bt) > 0$, then by Lemma 2.3, we have $\phi(a+bt) \leq \phi(\alpha+\beta t)$.
Multiplying both sides of $q\beta \geq rb$ by $\phi(\alpha + \beta t)$ yields;
$q\beta\phi(\alpha + \beta t) \geq rb\phi(\alpha + \beta t) \geq rb\phi(a + bt)$ which implies;
$rb\phi(a + bt) - q\beta\phi(\alpha + \beta t) \leq 0.$

(ii) From Lemma 2.3, we have $\phi(a + bt) \leq \phi(\alpha + \beta t)$.
If $\phi(\alpha + \beta t) > 0$, then there are two possible values of $\phi(a + bt)$. That is,
Case 1: $\phi(a + bt) \leq 0$ or
Case 2: $\phi(a + bt) > 0$.
For Case 1, we have $rb\phi(a + bt) \leq 0$ and $q\beta\phi(\alpha + \beta t) > 0$.
Hence $rb\phi(a + bt) - q\beta\phi(\alpha + \beta t) \leq 0$.
Case 2 is shown in (i).

Lemma 2.5. *The function $\phi_p(t)$ as defined in (4) has the following series representation.*

$$\phi_p(t) = \ln(p) - \sum_{k=0}^{p} \frac{1}{t+k}, \quad p \in \aleph, \quad t > 0. \tag{9}$$

Proof. From inequality (3), we have

$$\ln \Gamma_p(t) = t \ln p - \left(\ln t + \ln(1+t) + \ln(1+\frac{t}{2}) + \cdots + \ln(1+\frac{t}{p}) \right)$$

Thus

$$\begin{aligned} \phi_p(t) = \frac{d}{dt} \ln(\Gamma(t)) &= \ln p - \left(\frac{1}{t} + \frac{1}{1+t} + \cdots + \frac{1}{p+t} \right) \\ &= \ln p - \sum_{k=0}^{p} \frac{1}{k+t}. \end{aligned}$$

See also [4].

Lemma 2.6. *Let $s > 0$, $t > 0$ with $s \leq t$, then*

$$\phi_p(s) \leq \phi_p(t). \tag{10}$$

Proof. From (9), we have the following.

$$\begin{aligned} \phi_p(s) - \phi_p(t) &= \left(\ln(p) - \sum_{k=0}^{p} \frac{1}{s+k} \right) - \left(\ln(p) - \sum_{k=0}^{p} \frac{1}{t+k} \right) \\ &= \sum_{k=0}^{p} \left(\frac{1}{t+k} - \frac{1}{s+k} \right) \\ &= \sum_{k=0}^{p} \frac{(s-t)}{(t+k)(s+k)} \leq 0 \end{aligned}$$

and that ends the proof.

The following Lemmas (See [4]) are the p-analogues of Lemmas 2.3 and 2.4 with similar proofs.

Lemma 2.7. *Let a, b, α, β be real numbers such that $a + bt > 0$, $\alpha + \beta t > 0$. Then $a + bt \leq \alpha + \beta t$ implies $\phi_p(a + bt) \leq \phi_p(\alpha + \beta t)$.*

Lemma 2.8. *Let $a, b, \alpha, \beta, r, q$, be real numbers such that*
$a + bt > 0$, $\alpha + \beta t > 0$, $a + bt \leq \alpha + \beta t$ and $q\beta \geq rb$.
If (i) $\phi_p(a + bt) > 0$ or
(ii) $\phi_p(\alpha + \beta t) > 0$,
then $\quad rb\phi_p(a + bt) - q\beta\phi_p(\alpha + \beta t) \leq 0.$

3 Main Results

We state and prove the results of this paper here.

Theorem 3.1. *Define a function* Λ *by*

$$\Lambda(t) = \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}}, \qquad t \in [0, \infty) \tag{11}$$

where $a_i, b_i, \alpha_i, \beta_i, r_i, q_i$, $i = 1, 2, ..., n$ *are real numbers such that* $a_i + b_i t > 0$, $\alpha_i + \beta_i t > 0$, $a_i + b_i t \leq \alpha_i + \beta_i t$ *and* $q_i \beta_i \geq r_i b_i$. *If* $\phi(a_i + b_i t) > 0$ *or* $\phi(\alpha_i + \beta_i t) > 0$ *then* Λ *is decreasing and for every* $t \in [0, 1]$*, the following inequality holds.*

$$\frac{\prod_{i=1}^{n} \Gamma(a_i + b_i)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma(a_i)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i)^{q_i}} \tag{12}$$

Proof. Let $g(t) = \ln \Lambda(t)$ for every $t \in [0, \infty)$. Then,

$$\begin{aligned} g(t) &= \ln\left(\frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}}\right) \\ &= \ln\left(\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}\right) - \ln\left(\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}\right) \end{aligned}$$

Then,

$$\begin{aligned} g'(t) &= \sum_{i=1}^{n} r_i b_i \frac{\Gamma'(a_i + b_i t)}{\Gamma(a_i + b_i t)} - \sum_{i=1}^{n} q_i \beta_i \frac{\Gamma'(\alpha_i + \beta_i t)}{\Gamma(\alpha_i + \beta_i t)} \\ &= \sum_{i=1}^{n} r_i b_i \phi(a_i + b_i t) - \sum_{i=1}^{n} q_i \beta_i \phi(\alpha_i + \beta_i t) \\ &= \sum_{i=1}^{n} \left[r_i b_i \phi(a_i + b_i t) - q_i \beta_i \phi(\alpha_i + \beta_i t)\right] \leq 0. \quad \text{(by Lemma 2.4).} \end{aligned}$$

That implies g is decreasing on $t \in [0, \infty)$. Hence, Λ is decreasing for every $t \in [0, \infty)$. Then for every $t \in [0, 1]$ we have,

$$\Lambda(1) \leq \Lambda(t) \leq \Lambda(0) \qquad \text{yielding,}$$

$$\frac{\prod_{i=1}^{n} \Gamma(a_i + b_i)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma(a_i)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i)^{q_i}}.$$

Corollary 3.2. *If $t \in (1, \infty)$, then the following inequality holds.*

$$\frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i)^{q_i}}$$

Proof. If $t \in (1, \infty)$, then we have $\Lambda(t) \leq \Lambda(1)$ yielding the result.

In the following, we present the p-analogues of Theorem 3.1 and Corollory 3.2.

Theorem 3.3. *Define a function Ω by*

$$\Omega(t) = \frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i t)^{q_i}}, \qquad t \in [0, \infty), \quad p \in \aleph \tag{13}$$

where $a_i, b_i, \alpha_i, \beta_i, r_i, q_i$, $i = 1, 2, ..., n$ are real numbers such that $a_i + b_i t > 0$, $\alpha_i + \beta_i t > 0$, $a_i + b_i t \leq \alpha_i + \beta_i t$ and $q_i \beta_i \geq r_i b_i$. If $\phi_p(a_i + b_i t) > 0$ or $\phi_p(\alpha_i + \beta_i t) > 0$ then Ω is decreasing and for every $t \in [0, 1]$, the following inequality holds.

$$\frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma_p(a_i)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i)^{q_i}} \tag{14}$$

Proof. Let $h(t) = \ln \Omega(t)$ for every $t \in [0, \infty)$. Then by a similar argument as in in the proof of Theorem 3.1 we arrive at,

$$h'(t) = \sum_{i=1}^{n} [r_i b_i \phi_p(a_i + b_i t) - q_i \beta_i \phi_p(\alpha_i + \beta_i t)] \leq 0. \quad \text{(by Lemma 2.8)}.$$

That implies h is decreasing on $t \in [0, \infty)$. Hence, Ω is decreasing for every $t \in [0, \infty)$. Then for every $t \in [0, 1]$ we have,

$$\Omega(1) \leq \Omega(t) \leq \Omega(0) \qquad \text{yielding,}$$

$$\frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma_p(a_i)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i)^{q_i}}.$$

Corollary 3.4. *If $t \in (1, \infty)$, then the following inequality holds.*

$$\frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i)^{q_i}}, \quad p \in \aleph$$

Proof. If $t \in (1, \infty)$, then we have $\Omega(t) \leq \Omega(1)$ giving the result.

4 Concluding Remarks

We dedicate this section to some remarks concerning inequalities (12) and (14).

Remark 4.1. In inequality (12), put $i = 1$, $a_1 = b_1 = \alpha_1 = q_1 = 1$ and $\beta_1 = r_1 = n$, then we obtain

$$\frac{1}{n!} \leq \frac{\Gamma(1+t)^n}{\Gamma(1+nt)} \leq 1, \qquad t \in [0,1], \quad n \in \aleph$$

as in [1].

Remark 4.2. In inequality (12), put $i = 1$, $a_1 = \alpha_1 = q_1 = 1$ and $\beta_1 = r_1 = a$, then we obtain

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+t)^a}{\Gamma(1+at)} \leq 1, \qquad t \in [0,1], \quad a \geq 1$$

as in [5]

Remark 4.3. In inequality (12), put $i = 1$, $a_1 = a$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $q_1 = q$ and $r_1 = r$, then

$$\frac{\Gamma(a)^r}{\Gamma(\alpha)^q} \leq \frac{\Gamma(a+bt)^r}{\Gamma(\alpha+\beta t)^q} \leq \frac{\Gamma(a+b)^r}{\Gamma(\alpha+\beta)^q}, \qquad t \in [0,1]$$

where $a \geq b > 0$, r, q are positive real numbers such that $rb \geq q\beta > 0$ and $\phi(a+\beta t) > 0$.

Remark 4.4. Using (5) together with Theorem 3.3 and Corollary 3.4, the entire results of Theorem 3.1 and Corollary 3.2 are respectively recovered.

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