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Generalization of Some Inequalities for the Ratio of Gamma Functions

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Abstract

We present some monotonic functions and some generalized inequalities involving the ratios of analogues of the Gamma function.

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1 Introduction

The classical Euler's Gamma function $\Gamma(t)$ is commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \qquad t > 0. \tag{1}$$

The p-digamma function $\psi_p(t)$, q-digamma function $\psi_q(t)$ and k-digamma function $\psi_k(t)$ are respectively defined as follows.

$$\psi_p(t) = \frac{d}{dt}\ln(\Gamma_p(t)) = \frac{\Gamma_p'(t)}{\Gamma_p(t)}, \qquad t > 0. \tag{2}$$

where $\Gamma_p(t)$ is the p-analogue of the Gamma function defined by (see [2], [3])

$$\Gamma_p(t) = \frac{p!p^t}{t(t+1)\dots(t+p)} = \frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})}, \quad p \in N, \quad t > 0, \tag{3}$$

$$\psi_q(t) = \frac{d}{dt}\ln(\Gamma_q(t)) = \frac{\Gamma_q'(t)}{\Gamma_q(t)}, \qquad t > 0 \tag{4}$$

where $\Gamma_q(t)$ is the q-analogue of the Gamma function defined by (see [4])

$$\Gamma_q(t) = (1-q)^{1-t}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0, \tag{5}$$

and

$$\psi_k(t) = \frac{d}{dt}\ln(\Gamma_k(t)) = \frac{\Gamma_k'(t)}{\Gamma_k(t)}, \quad t > 0 \tag{6}$$

where $\Gamma_k(t)$ is the k-analogue of the Gamma function defined by (see [1], [5])

$$\Gamma_k(t) = \int_0^{\infty} e^{-\frac{x^k}{k}} x^{t-1}\,dx, \quad k > 0, \quad t > 0. \tag{7}$$

In a recent paper [6], Nantomah and Iddrisu proved that the following double inequalities hold:

$$0 < \frac{\Gamma_k(\alpha+t)}{\Gamma_p(\alpha+t)} < \frac{e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{tp^{t-1}\Gamma_p(\alpha+1)}, \quad k > 0,\, p \in N \tag{8}$$

and

$$0 < \frac{\Gamma_k(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{e^{(-t)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{t(1-q)^{1-t}\Gamma_q(\alpha+1)}, \quad k > 0,\, q \in (0,1) \tag{9}$$

for $t \in (0,1)$ and for a positive real number α.

Our objective in this paper is to establish some generalizations of the inequalities (8) and (9).

2 Preliminary Results

The following auxiliary results are crucial to the main results of the paper.

Lemma 2.1. *The functions $\psi_p(t)$, $\psi_q(t)$ and $\psi_k(t)$ as defined above have the following series representations.*

$$\psi_p(t) = \ln p - \sum_{n=0}^{p} \frac{1}{n+t}, \quad p \in N, \quad t > 0 \tag{10}$$

$$\psi_q(t) = -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{t+n}}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0 \tag{11}$$

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)}, \quad k > 0, \quad t > 0. \tag{12}$$

where γ is the Euler-Mascheroni's constant.

Proof. See [3], [5] and [6] and the references therein.

Lemma 2.2. *Let $a > 0$, $b > 0$ and $t > 0$. Then,*

$$-a(\frac{\ln k - \gamma}{k}) + b\ln p + \frac{a}{t} + a\psi_k(t) - b\psi_p(t) > 0.$$

Proof. Using the series representations in equations (10) and (12) we have,

$$\begin{aligned} & -a(\frac{\ln k - \gamma}{k}) + b\ln p + \frac{a}{t} + a\psi_k(t) - b\psi_p(t) \\ & \qquad = a\sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} + b\sum_{n=0}^{p} \frac{1}{(n+t)} > 0. \end{aligned}$$

Lemma 2.3. *Let $a > 0$, $b > 0$ and $\alpha + \beta t > 0$. Then*

$$-a(\frac{\ln k - \gamma}{k}) + b\ln p + \frac{a}{t} + a\psi_k(\alpha + \beta t) - b\psi_p(\alpha + \beta t) > 0$$

Proof. This follows directly from Lemma 2.2.

Lemma 2.4. *Let $a > 0$, $b > 0$ and $t > 0$. Then,*

$$-a(\frac{\ln k - \gamma}{k}) + b\ln(1-q) + \frac{a}{t} + a\psi_k(t) - b\psi_q(t) > 0$$

Proof. Using the series representations in equations (11) and (12) we have,

$$
\begin{aligned}
&-a(\frac{\ln k-\gamma}{k})+b\ln(1-q)+\frac{a}{t}+a\psi_k(t)-b\psi_q(t)\\
&\qquad\qquad = a\sum_{n=1}^{\infty}\frac{t}{nk(nk+t)}-b\ln q\sum_{n=0}^{\infty}\frac{q^{t+n}}{1-q^{t+n}}>0
\end{aligned}
$$

Lemma 2.5. *Let $a>0$, $b>0$ and $\alpha+\beta t>0$. Then,*

$$-a(\frac{\ln k-\gamma}{k})+b\ln(1-q)+\frac{a}{t}+a\psi_k(\alpha+\beta t)-b\psi_p(\alpha+\beta t)>0$$

Proof. This follows directly from Lemma 2.4.

3 Main Results

We now state and prove the results of this paper.

Theorem 3.1. *Define a function Λ by*

$$\Lambda(t)=\frac{t^{a\beta}e^{-a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta t)^a}{p^{-b\beta t}\Gamma_p(\alpha+\beta t)^b},\quad t\in(0,\infty),\ k>0,\ p\in N. \tag{13}$$

where a, b, α, β are positive real numbers. Then Λ is increasing on $t\in(0,\infty)$ and the inequality

$$0<\frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_p(\alpha+\beta t)^b}<\frac{e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{t^{a\beta}p^{b\beta(t-1)}\Gamma_p(\alpha+\beta)^b} \tag{14}$$

holds for every $t\in(0,1)$.

Proof. Let $g(t)=\ln\Lambda(t)$ for every $t\in(0,\infty)$. Then,

$$
\begin{aligned}
g(t)&=\ln\frac{t^{a\beta}e^{-a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta t)^a}{p^{-b\beta t}\Gamma_p(\alpha+\beta t)^b}\\
&=-a\beta t(\frac{\ln k-\gamma}{k})+b\beta t\ln p+a\beta\ln t+a\ln\Gamma_k(\alpha+\beta t)-b\ln\Gamma_p(\alpha+\beta t)
\end{aligned}
$$

Then,

$$
\begin{aligned}
g'(t)&=-a\beta(\frac{\ln k-\gamma}{k})+b\beta\ln p+\frac{a\beta}{t}+a\beta\psi_k(\alpha+\beta t)-b\beta\psi_p(\alpha+\beta t)\\
&=\beta\left[-a(\frac{\ln k-\gamma}{k})+b\ln p+\frac{a}{t}+a\psi_k(\alpha+\beta t)-b\psi_p(\alpha+\beta t)\right]>0
\end{aligned}
$$

as a result of Lemma 2.3. This proves that g is increasing on $t \in (0,\infty)$. Hence Λ is increasing on $t \in (0,\infty)$. Thus, for every $t \in (0,1)$ we have

$$\Lambda(0) < \Lambda(t) < \Lambda(1),$$

yielding

$$0 < \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_p(\alpha+\beta t)^b} < \frac{e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{t^{a\beta}p^{b\beta(t-1)}\Gamma_p(\alpha+\beta)^b}.$$

Corollary 3.2. *If $t \in [1,\infty)$, then the following inequality holds.*

$$\frac{e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{t^{a\beta}p^{b\beta(t-1)}\Gamma_p(\alpha+\beta)^b} \le \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_p(\alpha+\beta t)^b}$$

Proof. If $t \in [1,\infty)$, then we have $\Lambda(1) \le \Lambda(t)$ yielding the result.

Theorem 3.3. *Define a function Υ by*

$$\Upsilon(t) = \frac{t^{a\beta}e^{-a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta t)^a}{(1-q)^{b\beta t}\Gamma_q(\alpha+\beta t)^b}, \quad t \in (0,\infty),\ k>0,\ q\in(0,1). \tag{15}$$

where a, b, α, β are positive real numbers. Then Υ is increasing on $t \in (0,\infty)$ and the inequality

$$0 < \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_q(\alpha+\beta t)^b} < \frac{e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{t^{a\beta}(1-q)^{b\beta(1-t)}\Gamma_q(\alpha+\beta)^b} \tag{16}$$

holds for every $t \in (0,1)$.

Proof. Let $h(t) = \ln\Upsilon(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned} h(t) &= \ln\frac{t^{a\beta}e^{-a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta t)^a}{(1-q)^{b\beta t}\Gamma_q(\alpha+\beta t)^b} \\ &= -a\beta t(\frac{\ln k-\gamma}{k}) - b\beta t\ln(1-q) + a\beta\ln t + a\ln\Gamma_k(\alpha+\beta t) - b\ln\Gamma_q(\alpha+\beta t) \end{aligned}$$

Then,

$$\begin{aligned} h'(t) &= -a\beta(\frac{\ln k-\gamma}{k}) - b\beta\ln(1-q) + \frac{a\beta}{t} + a\beta\psi_k(\alpha+\beta t) - b\beta\psi_q(\alpha+\beta t) \\ &= \beta\left[-a(\frac{\ln k-\gamma}{k}) - b\ln(1-q) + \frac{a}{t} + a\psi_k(\alpha+\beta t) - b\psi_p(\alpha+\beta t)\right] > 0 \end{aligned}$$

as a result of Lemma 2.5. This proves that h is increasing on $t \in (0, \infty)$. Hence Υ is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have

$$\Upsilon(0) < \Upsilon(t) < \Upsilon(1)$$

yielding

$$0 < \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_q(\alpha+\beta t)^b} < \frac{e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{t^{a\beta}(1-q)^{b\beta(1-t)}\Gamma_q(\alpha+\beta)^b}.$$

Corollary 3.4. *If $t \in [1, \infty)$, then the following inequality holds.*

$$\frac{e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{t^{a\beta}(1-q)^{b\beta(1-t)}\Gamma_q(\alpha+\beta)^b} \leq \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_q(\alpha+\beta t)^b}$$

Proof. If $t \in [1, \infty)$, then we have $\Upsilon(1) \leq \Upsilon(t)$ yielding the result.

4 Concluding Remarks

Remark 4.1. By putting $a = b = \beta = 1$ into inequalities (14) and (16), we thus obtain respectively, inequalities (8) and (9) as in [6].

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