UNIVERSITY FOR DEVELOPMENT STUDIES

T-INVERSE EXPONENTIAL {Y} FAMILY OF DISTRIBUTIONS

ABUDULAI SANDOW

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T-INVERSE EXPONENTIAL {Y} FAMILY OF DISTRIBUTIONS

BY

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SEPTEMBER, 2021

DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original work and that no part of it has been presented for another degree in this University or elsewhere.

Signature: …………………………. Date……………………….…..

Abudulai Sandow

(UDS/DAS/0002/18)

Supervisors' Declaration

We hereby declare that the preparation and presentation of this thesis was supervised in accordance with the guidelines on supervision of thesis laid down by the University for Development Studies.

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(Co-Supervisor)

ABSTRACT

Recently, statisticians have been exploring the generalization and extension of classes of distributions to make them more flexible for data analysis. There are a lot of data sets that are either extremely skewed, bimodal, bathtub or have heavy tails. However, inverse exponential distribution cannot model data sets that exhibit these features properly. In this study, a new family of inverse exponential distribution called T-inverse exponential {Y} family based on quantile function approach has been developed to fill some of the gaps identified in the inverse exponential distribution. Statistical properties such as quantile function, mode, entropy and moments of the family have been derived. Three sub-families namely, T-IE{Weibull} T-IE{Logistic} and T-IE{Lomax} families were defined. Three (3) special cases of these family of distributions namely, Log-logistic-IE{Weibull}, Weibull-IE {Logistic} and Gumbel-IE{Lomax} distributions were developed. Monte Carlo simulations were done to investigate the properties of the maximum likelihood estimation, ordinary least square estimators, weighted least square estimators, Cramér-von Mises minimum distance estimators and Percentile based estimators for estimating the parameters of the special distributions. It was revealed that, maximum likelihood estimators for the parameters of the special distributions were consistent. Empirical applications of the special distributions to real life data sets were done and they showed greater flexibility for different kinds of lifetime data sets than other competing distributions. It is recommended that, parametric regression models for the special distributions can be developed to examine the relationship between the dependent and the independent variables of the distributions.

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DEDICATION

To my mother, Madam Mariama Sagre.

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LIST OF ABBREVIATIONS

ABias	Average Bias
AIC	Akaike Information Criteria
AICc	corrected Akaike Information Criteria
BFGS	Broyden-Fletcher-Goldfarb-Shannon
BIC	Bayesian Information Criteria
BXII	Burr XII
CDF	Cumulative Distribution Function
CME	Cramér-von Mises Estimators
CVM	Cramér-von Mises Test
EEFr	Exponentiated-exponential Fréchet
EIRD	Exponentiated Inverse Raleigh Distribution
GIE	Generalized Inverse Exponential
GIEL	Gumbel Inverse Exponential Logistic
IE	Inverse Exponential Distribution
IRD	Inverse Raleigh Distribution
IW	Inverse Weibull
KIE	Kumaraswamy Inverse Exponential
K-S	Kolmogorove-Smirnov
LLIEW	Log-logistic Inverse Exponential Weibull
LRT	Likelihood Ratio Test
LWLL	Lomax Weibull {Log-logistic}
MBIII	Modified Burr III
MLE	Maximum Likelihood Estimators
MSE	Mean Square Error
OLS	Ordinary Least Square
PDF	Probability Density Function
PCE	Percentile Based estimators
RD	Raleigh Distribution
TTT	Total Test on Time

WBXII	Weibull Burr XII
WD	Weibull Distribution
WIEL	Weibull Inverse Exponential Lomax
WLS	Weighted Least Square
WLx	Weibull Lomax

CHAPTER ONE

INTRODUCTION

1.1 Background

Data scientists usually model data using either parametric or non-parametric statistical models. Parametric statistical modeling demand the knowledge of appropriate distributional assumptions of data sets, while non-parametric statistical modeling does not (Sheskin, 2000).

Parametric statistical distributions have widely been used in the field of social and applied sciences to model data sets and make statistical inferences. For instance, modeling the survival times of heart attack patients in medical sciences; modeling life expectancy in demography; modeling urban air pollution in environmental science among others. In these area of applied research however, the data generating process may have varied degrees of features in terms of the skewness and kurtosis. The data may also exhibit non-monotonic failure rates. As a result, modeling data with the existing classical distributions may provide an approximation of a parametric fit rather than a reality and the conclusions might be incorrect if the data set deviates from the distributional assumptions (Nasiru, 2018; Hoskin, 2010).

On the other hand, non-parametric statistics which does not rely on certain assumptions could be used for modeling to reduce the challenges of parametric statistics. It is however realized that, non-parametric procedures also has some serious weakness which include, lower power if the underlying distributional assumptions of the data set are known, low precision of measurements, loss of information, computational difficulties and explanation of non-parametric methods can also be more difficult as compared to parametric procedures (Nasiru, 2018; Hoskin, 2010 ; Allison, 1995).

As a result of these challenges, most recent literatures are focused on classical statistical distributions with the aim of improving the existing distributions to make them more flexible and also, develop new statistical distributions for modeling data sets from various fields of study. This is because, it is desirable to use a probability distribution that provide a better fit to a data set than to transform it as this may affect the originality of the data set and the results (Oguntunde, 2017).

There is a growing demand for modification, generalization and extension of the existing standard probability distributions to make them flexible and fit for the various situations. It is a fact that, an existing distribution may be tractable, that makes it easy for simulation but not flexible. With this, many statisticians have extended the existing standard probability distributions using several approaches to make them flexible. One of such is by using generators to formulate the generalization of the existing probability distribution. With generalization, an extra shape parameter(s) from the family of distributions is added to the distribution. The extra shape parameter(s) is to vary the tail weight of the resulting compound probability distribution, in so doing inducing it with skewness (Oguntunde, 2017). In addition, the flexibility of a probability distribution can be gotten by combining two or more standard distributions.

Among the distributions generalized in literature, the inverse exponential (IE) distribution in recent times has received much attention by researchers. IE distribution is used to model data sets with inverted bathtub failure rate. If X follows IE distribution with parameter λ, then the probability density function (PDF) is given by;

$$g(x) = (\tfrac{\lambda}{x^2})\exp(-\tfrac{\lambda}{x}), x > 0, \lambda > 0. \quad (1.1)$$

In order to make the IE more flexible, a number of researches have been done which include, inverse generalized exponential models (Khan et al., 2009); exponential IE distribution with applications to lifetime data (Oguntunde et al., 2017); the beta generalized IE distribution with real data applications (Bakoban et al., 2017) and exponentiated generalized IE distribution (Oguntunde et al., 2017). Despite these attempts, there still exist some drawbacks. For instance some of the distributions are unable to give better parametric fit to data sets that exhibit non-monotonic failure rates, some fail to properly model symmetric data (Oguntunde, 2017) and no extra shape parameter for improving the flexibility of modified distributions (Alzaatreh et al., 2013). Hence, this research is aimed at developing and studying the statistical properties of the T-inverse exponential{Y} family of distributions using $T-X\{Y\}$ framework pioneered by Aljarrah et al. (2014) that will be more flexible for life time data analysis.

1.2 Problem Statement

For lifetime data analysis, the selection of appropriate distribution is very vital for modeling a given data set. Data from different fields of study may exhibit different traits such as heavy tails, skewness, kurtosis and non-monotonic failure rates. However, the existing statistical distributions usually do not offer realistic fit to certain data sets. Hence, researchers started developing generalized classes of distributions to overcome these challenges. IE distribution is used to model data sets with inverted bathtub failure rates but cannot model data sets that are extremely skewed or that have heavy tails properly (Abouammoh and Alshingiti, 2009). Also, the IE is not suitable for modeling data that exhibit bimodality and bathtub failure rate.

Dey and Pradhan (2014) indicated that there is little literature on the generalization of the IE distribution compare to other distributions for flexible modeling of data. It is necessary therefore to rigorously explore the IE distribution so that data sets with these features can be modeled easily.

Although the literature is filled with barrage of generalized classes of distributions, there is no specific generalized class of distributions that is appropriate for all kinds of data. Thus, developing new generalized classes of distributions in addition to the existing ones is necessary. This will help provide several choices of distributions from which researchers can compare and select the best for a given data set. This study therefore proposes another new class of distributions called the T-inverse exponential {Y} family of distributions to fill some of the gaps identified in the existing distributions.

1.3 Objectives of the study

1.3.1 General Objective

The main objective is to develop and study the statistical properties of T-inverse exponential {Y} family of distributions.

1.3.2 Specific Objectives

The specific objectives are to;

1. Develop the T-inverse exponential {Y} family of distributions.
2. Derive the statistical properties of the new family of distributions.
3. Develop estimators for estimating the parameters of T-inverse exponential{Y} family.
4. Perform simulations to compare the performance of the estimators
5. Demonstrate the applications of the distributions using real data set.

1.5 Significance of the Study

The usefulness of probability distributions in predicting and describing real life phenomenon cannot be over emphasized. Although a lot of distributions have been developed to model and analyze lifetime data, there are always rooms for developing new distributions. This study proposes another new class of distributions called the T-inverse exponential {Y} family of distributions to model lifetime data sets that are bimodal, extremely skewed, heavy tailed, monotonic and non-monotonic failure rates.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Developing new distributions and extending the existing ones has been ongoing for some time now. This chapter captures the review of literature that are related to the development of the new statistical distributions. To enhance clarity, this chapter is divided into two sections. First section reviews literature on modification and generalization of IE distributions and the second section reviews literature that used the $T-X\{Y\}$ family of distributions.

2.2 Generalization and Modification of the Inverse Exponential Distribution

In order to induce skewness to the IE distribution given in equation (1.1), to make it more flexible for modeling lifetime data, a lot of generalizations and modifications has been done on the IE distribution. Examples include; generalized IE distribution by Abouammoh and Alshingiti (2009) and beta generalized IE distribution by Bakoban and Abu-zinadah (2017).

Beta-generated family of distributions was introduced by Eugene et al. (2002). This family of distributions is the generalization of the distribution of ordered statistics. Despite these enormous literature on this class of distributions, Cordeiro et al. (2009) concluded in their study that, the beta class of distributions were not fairly tractable. This among others led other researchers to suggest alternative bounded distributions for which Kumaraswamy distribution named after Pondi Kumaraswamy is an example Kumaraswamy (1980). This distribution has been identified as a good alternative to beta distribution because they both have the same basic shape properties and also Kumaraswamy distribution is deemed mathematically tractable because of its mild algebraic properties.

Abouammoh and Alshingiti (2009) generalized the IE distribution by adding another parameter to the IE distribution and called it the generalized IE distribution. The application of the generalized IE distribution was demonstrated using life time data and it proves to perform better than the IE distribution. Though better than IE distribution, this distribution cannot model data sets that exhibit bimodality or bathtub failure rate properly.

Elbatal and Muhammed (2014) proposed four parametric-distribution known as exponentiated generalized inverse Weibull distribution. It can be used to model various failures including bathtub and cost effectiveness in reliability test.

Oguntunde and Adejumo (2015) introduced the transmuted IE distribution as another generalization of the IE. The hazard function of the distribution has inverted bathtub shape. It should be noted that this distribution reduces back to the IE distribution if the transmutation parameter is equal to zero.

Bakoban and Abu-zinadah (2017) employed the beta generated family to develop the beta generalized IE distribution as a generalization of the IE distribution proposed by Keller and Kamath (1982). This compound distribution is a combination of the beta function and generalized IE function. The beta generalized IE distribution is unimodal, positively skewed and has a long right tail.

Yahaya and Mohammed (2017) proposed transmuted Kumaraswamy-IE distribution with four parameters. This distribution is an extension of Kumaraswamy-IE distribution. It was observed from their study that, the distribution is positively skewed with non-existent moments.

Oguntunde et al. (2017) proposed exponentiated generalized IE distribution using exponentiated family of distributions. They claimed that, this distribution better fit lifetime data sets than competing distributions such as Kumaraswamy IE, generalized IE and IE distribution except when the variance of a data set is larger than the mean.

Oguntunde (2017) proposed the generalization of the IE with special cases as, the Kumaraswamy IE, transmuted IE, exponentiated generalized IE and Weibull IE distributions. He however, indicated that the derived distributions cannot properly model symmetric and over-dispersed data sets.

2.4 Generating *T-X{Y}* Family of Distributions using Quantile Function

Modifying an existing distributions has gained a lot of attention recently. The result of the modifications is that one or more extra parameter is added to the existing distribution making it more flexible for data modelling than the baseline distribution. A number of procedures have been adopted over the years in modifying baseline distributions. For instance, Azzalini (1985) modified the normal distribution by adding an extra parameter to the normal distribution and called the new distribution, skewed normal distribution. This new distribution performed better than the normal distribution since it has skewness parameter that makes it more flexible. This procedure could be applied to modify other symmetric distributions. When the skewed parameter equals zero, the proposed distribution reduces back to normal distribution.

Elal-olivero (2010) introduced the alpha-skew-normal distribution. This distribution is an extension of the normal distribution which can be used to model skewed data sets.

Acitas et al. (2015) introduced alpha-skew generalized t distribution. This is an extension of the distribution formulated by Elal-olivero (2010). This distribution is more flexible than the alpha-skew-normal distribution.

Hutson et al. (2020) proposed and studied Log-epsilon-skew normal distribution as a generalization of the log-normal distribution. This new distribution is related to the epsilon-skew normal distribution and has log-normal distribution as a special case.

Mudholkar and Srivastava (1993) introduced a new method and used it to proposed and study exponentiated-Weibull distribution. This approach added an extra shape parameter to the two parameter Weibull distribution. This new distribution performed better than the existing distribution due to its flexibility. When the value of the extra shape parameter equals one, then the exponentiated-Weibull distribution returns to the two-parameter Weibull distribution.

Gupta et al. (1998) proposed the general class of exponentiated distributions. This approach has widely been accepted and used by many researchers. For example, Elbatal and Muhammed (2014) introduced exponentiated generalized inverse Weibull distribution, Rao and Mbwambo (2019) used the procedure to introduce exponentiated inverse Rayleigh distribution, Elgarhy et al. (2017) introduced exponentiated extended-G family of distributions and Mansoor et al. (2016) introduced exponentiated exponential Fréchet distribution.

Eugene et al. (2002) introduced a new way of generating distributions called the Beta-G family of distributions. This approach adds two extra shape parameters to the baseline distribution. This procedure has widely been used by scientists to generate distributions with two extra shape parameters.

Nadarajah and Kotz (2004) introduced beta Gumbel distribution. Gumbel distribution is one the distributions that is widely been used in the field of engineering. The beta Gumbel distribution is an extension of the Gumbel distribution which was developed by using the beta distribution as the generator.

Famoye et al. (2005) introduced the beta-Weibull distribution. The hazard rate function for beta-Weibull distribution could be increasing, decreasing and bathtub shape. Lee et al. (2007) studied and discussed the properties of this distribution and applied it to censored data set.

Akinsete et al. (2008) proposed and studied the beta-Pareto distribution. The family of Pareto distributions are widely used modelling data set that are heavy-tailed. The beta-Pareto distribution is an extension of the Pareto distribution. This new distribution has a decreasing and inverted bathtub hazard rate function. It is used to model unimodal data set.

Eugene et al. (2012) introduced beta-normal family of distributions. Beta-normal distribution has been applied to a variety of data set including data set that exhibit bimodality. The difference between beta-normal distribution and the skewed normal distribution is that, the beta-normal distribution developed by adding more parameters to the baseline distribution using beta distribution as the generator. The two extra shape parameters introduced control the tail weight of the distribution. Skewed normal on the other hand introduces skewness into normal distribution. The skewness can be determined using the skewness parameter.

Percontini et al. (2013) introduced the beta Weibull Poisson distribution. They proposed beta Weibull Poisson distribution by compounding Weibull Poisson and beta distributions. This distribution is an extension of the Weibull Poisson distribution.

Handique et al. (2017) proposed and studied the beta generated Kumaraswamy-G family of distributions. This is a generalization and improvement of the Kumaraswamy-G family of distributions. The density could be unimodal and reverse-J shape with varied combinations of parameter values.

Bakoban and Abu-zinadah (2017) proposed and studied the beta generalized inverse exponential distribution. The beta generalized inverse exponential distribution is a generalization of the generalized inverse distribution. The PDF of this distribution could be positively skewed and unimodal.

Tablada and Cordeiro (2019) introduced the beta Marshall-Olkin Lomax family of distributions. This distribution was proved to be efficient in modelling lifetime data using uncensored data set.

The beta-generated procedure was extended by Cordeiro and Castro (2009). They use Kumaraswamy distribution as a generator instead of the beta distribution. This approach was seen as an improvement over the beta-generated distributions.

The beta-generated family of distributions was generalized by Alzaatreh et al. (2013). They introduced a new method of parameter induction by defining the transformed-transformer (T-X) family of distributions. This procedure was introduced by replacing the beta PDF with a PDF of any continuous random variable and applying a link function $W(0)$ that satisfied certain specific situations. This family of distributions has no extra shape parameter. This makes it not flexible to model lifetime data sets that have heavy tails and non-monotonic failure rates.

This method of generating distribution was extended by Alzaghal et al. (2013). They introduced the exponentiated T-X family of distributions with some applications. One special case of this family is exponentiated Weibull exponential distribution. The density of the distribution could be right skewed or left skewed. The limitation of adding only one shape parameter is that, the distribution is not flexible enough to model data sets that have heavy tails with varying degrees of skewness and kurtosis.

Aljarrah et al. (2014) used the quantile function of random variable Y as the link function $W(.)$ to generate the T-X{Y} family. Special cases under their study could be applied to data that exhibit either unimodal or bimodal shapes. This approach has wildly been applied by many researchers to extend and develop new families of distributions.

Alzaatreh et al. (2014) introduced T-normal{Y} family of distributions with four special cases. This family of distributions can be applied to data set that are skewed to both right and left. Though, the family can be applied to data set that exhibit bimodality, what is unknown is if the distributions can fit well with data set that have multimodal, modified bathtub and modified inverted bathtub shapes.

Alzaatreh et al. (2016) introduced T-gamma {Y} family of distributions as generalization of gamma distribution. This was to add extra flexibility to the existing distribution. It was observed that the shapes of the densities of the new family could be unimodal, bimodal, right and left skewed. This makes the distributions more flexible than the parent distribution in modeling real life data.

Alzaatreh et al. (2016) introduced the T-Cauchy{Y} family of distributions. This is an improvement on the Cauchy distribution. The generalization of the Cauchy distribution using the quantile function approach induces skewness on to the baseline distribution thereby making it more flexible than the baseline distribution. This new family of distributions is unimodal and might not be able to properly model data sets that exhibit multimodality.

Nasir et al. (2017) proposed and studied a new distribution called T-Burr family of distributions. This family of distributions is an improvement of the Burr type XII distribution. The shape of the density is unimodal with monotone hazard rate function. This implies that, the distribution may not be appropriate for modelling data set that are multimodal and non-monotonic hazard rate functions.

Mansoor et al. (2017) developed the Lomax-Weibull family of distributions and applied it to both censored and uncensored data. They developed a special model as a generalization of Weibull distribution to model lifetime data sets which are both monotonic and non-monotonic. A special case of the new family of distributions has a density function with approximately symmetric, bimodal, negatively skewed, positively skewed and reversed-J shapes.

Aldeni et al. (2017) introduced the $T-R$ {generalized lambda} family of distributions using the quantile function of the generalized lambda distribution. The shapes of the densities could be symmetric, positively skewed, negatively skewed and bimodal.

Zubair et al. (2018) introduced T-exponential$\{Y\}$family of distributions using quantile functions of well-known distributions. Some of the special classes of this distributions are the Weibull-E {Log-logistic}, the gamma-E{log-logistic} and the normal-E{logistic}. It is however observed that, none of these new models could modeled data that exhibits bimodality.

Ghosh and Nadarajah (2018) introduced the Weibull-R family of distributions using Weibull as a baseline distribution. This was one attempt to minimize if not eliminate some of the drawbacks identified by Bain (1978) associated with Weibull distribution. The Weibull-R family of distributions is broader than gamma-generalized distributions and can be used to model censored data.

Hamed et al. (2018) introduced the T-Pareto{Y} family of distributions. These extensions and generalizations are necessary to cure the weaknesses of standard Pareto distribution which is used to fit right-skewed data set. They used quantile functions of six different standard distributions to develop six generalized families of distributions. It can be observed that, the various models introduced can model data sets with various shapes except those which are symmetric in nature and those with increasing failure rate.

Jamal et al. (2018) proposed the T-Burr III{Y} family of distributions. They introduced three Burr-III sub-families and applied them to both censored and uncensored data to confirm their flexibility and robustness in fitting data. In all, the distributions performed better.

Nasir et al. (2017) used the quantile function approach to develop the new Weibull Burr-XII distribution. They generalized the Weibull distribution by inducting two additional parameters that makes it more flexible compare to its competitors. It was observed that, data that exhibit bimodality and bathtub failure rate might not properly be modeled by this family.

Nasiru et al. (2020) used the quantile function method to develop the T-NH{Y} family of distributions. These family of compound distributions proved to perform better in modelling data sets with non-monotonic failure rates than their competing models.

CHAPTER THREE

METHODOLOGY

3.0 Introduction

This chapter presents the methodology that was applied in achieving the objectives of the study. The topics discussed are data and source, the IE distribution, *T-X{Y}* framework, parameter estimation, goodness of fit test, information criteria and total time on test (TTT).

3.1 Inverse Exponential Distribution

The IE distribution was proposed by Keller and Kamath (1982) and has been studied and extensively used for modeling. If a random variable Y has an exponential distribution, then $X = \frac{1}{Y}$ will have IE distribution. If $X \sim IE(\lambda)$, then the probability density function (PDF) is given by;

$$g(x) = \tfrac{\lambda}{x^2}\exp(-\tfrac{\lambda}{x}), x > 0, \lambda > 0 \tag{3.1}$$

and the cumulative distribution function (CDF) is

$$G(x) = \exp(-\tfrac{\lambda}{x}), \ x > 0, \lambda > 0. \tag{3.2}$$

The corresponding survival function and hazard rate function are respectively given by;

$$S(x) = 1 - \exp(-\tfrac{\lambda}{x}), \ x>0, \lambda>0, \tag{3.3}$$

and

$$h(x) = \frac{g(x)}{S(x)} = \frac{\tfrac{\lambda}{x^2}\exp(-\tfrac{\lambda}{x})}{1-\exp(-\tfrac{\lambda}{x})}, x > 0, \lambda>0. \tag{3.4}$$

3.2 The *T-X{Y}* Framework

Aljarrah et al. (2014) used quantile function of continuous random variable Y to generate the $T - X\{Y\}$ family of distributions. Let the CDFs of the random variables T, R and Y be $G_T(x)$, $G_R(x)$ and $G_Y(x)$. Also let their PDFs be $g_T(x)$, $g_R(x)$ and $g_Y(x)$ respectively. If the quantile function of the random variable Z is $Q_Z(u) = \inf\{z : G_Z(z) \geq u\}, 0 < u < 1$. Then, the corresponding quantile functions for the random variables T, R and Y are $Q_T(u)$, $Q_R(u)$ and $Q_Y(u)$ respectively. The CDF, PDF and the hazard rate function are defined respectively as;

$$G_X(x) = \int_a^{Q_Y(G_R(x))} g_T(t)dt = G_T(Q_Y(G_R(x))), \tag{3.5}$$

$$g_X(x) = g_R(x) \times \frac{g_t(Q_Y(G_R(x)))}{g_Y(Q_Y(G_R(x)))}, \tag{3.6}$$

and

$$h_X(x) = h_R(x) \times \frac{h_T(Q_Y(G_R(x)))}{h_Y(Q_Y(G_R(x)))}. \tag{3.7}$$

3.3 Estimation of Parameters

This section presents five (5) parameter estimation procedures for estimating the parameters of the T-IE{Y} family of distributions. These estimators are: Maximum Likelihood estimators, Ordinary Least Square estimators, Weighted Least Square estimators, Percentile based estimators and Cramér-von Mises minimum distance estimators.

3.3.1 Maximum Likelihood Estimation

In estimating the parameters of the T-IE{Y} family of distributions, the maximum likelihood estimation (MLE) for both complete and incomplete samples were used. The maximum likelihood estimation is the most widely used classical approach for approximating parameters due to its desirable properties over others and is based on a likelihood function. At a specific value of the parameters, the likelihood function attains its maximum.

Suppose that X is distributed randomly with size k with PDF $g(x;\theta)$ where $\theta = (\theta_1, \theta_2, ..., \theta_n)'$, $n < k$, is the vector of parameters that governs the PDF. The joint PDF can be expressed as

$$g(x/\theta) = \prod_{i=1}^{k} g(x_i;\theta). \tag{3.8}$$

The joint PDF becomes a function of θ and called likelihood function when the random sample is collected. The likelihood functions for complete and incomplete samples are defined respectively as

$$L(\theta / x) = \prod_{i=1}^{k} g(x_i;\theta), \tag{3.9}$$

$$L(\theta / x) = \prod_{i=1}^{n} [g(x_i,\theta)]^{r_i} [1 - G(x_i,\theta)]^{1-r_i}. \tag{3.10}$$

The estimator $\hat{\theta}$ is the value of θ that maximize the likelihood function through the following procedure.

1. Obtain the likelihood function
2. Find the log-likelihood function
3. Find partial derivatives of the log-likelihood function with respect to the parameters
4. Equate the derived equations to zero and solve simultaneously.

3.3.2 Ordinary Least Square Estimators and Weighted Least Square Estimators

The ordinary least squares (OLS) estimators was proposed by Swain et al. (1988) to estimate the parameters of beta distribution. Given the ordered statistics $x_{(1)}, x_{(2)}, ..., x_{(n)}$ of the random sample of size n from $G(x;\Theta)$, where Θ is a vector of parameters.The OLS estimates Θ for the parameters can be obtained by minimizing the function

$$L(\Theta)=\sum_{i=1}^{n}\left[G(x_{(i)}\,|\,\Theta)-\frac{i}{n+i}\right]^{2}. \tag{3.11}$$

with respect to Θ or equivalently solving the following non-linear function

$$\sum_{i=1}^{n}\left[G(x_{(i)}\,|\,\Theta)-\frac{i}{n+i}\right]\Psi_{s}(x_{(i)}\,|\,\Theta), \tag{3.12}$$

where $s=1,2,\ldots$ and $\Psi_{s}(x_{(i)}\,|\,\Theta)=\frac{\partial}{\partial\Theta}G(x_{(i)};\Theta).$

The Weighted Least Square (WLS) estimates are obtained by minimizing the following function with respect to Θ :

$$L(x_{(i)}\,|\,\Theta)=\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left[G(x_{(i)}\,|\,\Theta)-\frac{i}{n+1}\right]^{2}. \tag{3.13}$$

3.3.3 Estimators Based on Percentile

The percentile based estimators (PCE) was introduced by Kao (1958). The main advantage of this procedure is that, the estimates can be obtained in explicit forms. The PCE are mainly obtained by minimizing the Euclidean distance between the sample percentile and population percentile points. Let $x_{(1)},\ldots,x_{(n)}$ be a sample ordered statistics and $u_i=\frac{i}{(n+1)}$ be unbiased estimator of $G(x_{(i)}\,|\,\Theta)$. Then, the PCE of the distribution are obtained by minimizing the function with respect to Θ :

$$p(x_{(i)}\,|\,\Theta)=\sum_{i=1}^{n}\left(x_{(i)}-G^{-1}(u_i)\right)^{2}. \tag{3.14}$$

3.3.4 The Cramér-von Mises Minimum Distance Estimators

The Cramér- von Mises estimators (CME) is a method that is based on the difference between the estimate of the CDF and the empirical distribution function (Louzada et al., 2016). It is also called maximum goodness-of-fit estimators. Among the minimum distance estimators, CME is less biased. The CME can be obtained by minimizing the following function

$$c(\Theta) = \frac{1}{12n} + \sum_{i=1}^{n} [G(x_{(i)} \mid \Theta) - \tfrac{2i-1}{2n}]^2. \qquad (3.15)$$

3.4 Broyden-Fletcher-Goldfarb-Shannon Algorithm (BFGS)

When the maximum likelihood estimators have no closed form, then the systems of equations are solved using numerical techniques. BFGS was employed in this study to solve such system of equations. The BFGS is an iterative technique for solving unconstrained nonlinear optimization problem and was independently developed by the four researchers as cited by (Nasiru, 2018). It is an optimization technique that is used to maximize likelihood. To optimize a given likelihood function, the ensuing steps are reiterated as θ_i converges to the solution with a preliminary guess θ_0 and an estimated Hessian matrix H_0.

1. First obtain a direction b_i by solving

   $$H_i b_i + \nabla \ell(\theta_i) = 0.$$

2. A one dimensional optimization is then performed to look for an acceptable step size γ_i in the direction found in the first step.
3. Set $a_i = \gamma_i b_i$ and update $\theta_{i+1} = \theta_i + a_i$.
4. Let $y_i = \nabla \ell(\theta_{i+1}) - \nabla \ell(\theta_i)$.

5. $H_{i+1} = H_0 + \frac{y_i y_i^{'}}{y_i^{'} a_i} - \frac{H_i a_i a_i^{'} H_i}{a_i^{'} H_i a_i}$.

Convergence can be checked by noting the norm of the gradient $|\ell(\theta_i)|$. In practice, H_0 can be initialized with $H_0 = \mathrm{I}$, to make the first step equivalent to a gradient descent, but additional steps are refined by the approximation of the Hessian, H_i. Step one of the algorithm is carried out using the inverse of the matrix H_i. This can be achieved proficiently by introducing the Sherman-Morrison formulae to the fifth step of the algorithm. Hence,

$$H_{i+1}^{-1} = (I - \tfrac{a_i y_i^{'}}{y_i^{'} a_i}) H_i^{-1} (I - \tfrac{y_i a_i^{'}}{y_i^{'} a_i}) + \tfrac{a_i a_i^{'}}{y_i^{'} a_i}. \qquad (3.16)$$

Recognizing that H_{i+1}^{-1} is symmetric and $y_i^{'} H_i^{-1} y_i$ and $a_i^{'} y_i$ are scalars, equation (3.10) can be computed efficiently using the following expansion

$$H_{i+1}^{-1} = H_i^{-1} + \tfrac{(a_i^{'} y_i + y_i^{'} H_i^{-1} y_i)(a_i a_i^{'})}{(a_i^{'} y_i)^2} - \tfrac{H_i^{-1} y_i a_i^{'} + a_i y_i^{'} H_i^{-1}}{a_i^{'} y_i}. \qquad (3.17)$$

3.5 Goodness of Fit Test

If $X_1, X_2, X_3, ..., X_k$ is a randomly selected sample from a given distribution, then the goodness-of-fit tests is a method that measures whether the random sample came from a specified theoretical probability distribution. In this study, three tests were employed; likelihood ratio test (LRT), Kolmogorov-Smirnov (K-S) and Cramér-von Misses test (CVM).

3.5.1 Likelihood Ratio Test

This method is usually used to compare the goodness of fit of two models that are nested. Assuming we let X be a random sample with PDF $g(x;\alpha)$. The hypotheses test is of the form

$H_0 : \alpha \in \alpha_0$ verses $H_1 : \alpha \in \alpha_1$, where α_0 and α_1 are the parameter spaces for the reduce and full model respectively. The test statistic is given as

$$\phi = -2\ln(\tfrac{L_0(\hat{\alpha})}{L_1(\hat{\alpha})}), \qquad (3.18)$$

where L_0 and L_1 are the likelihood functions of the two models. The best model is the one that maximizes the likelihood function. When the null hypothesis is not accepted, it implies that the full model gives a better fit than the reduce model.

3.5.2 Kolmogorov-Smirnov (K-S) Test

This is a Hypotheses test procedure for testing whether a given random sample belongs to a population with specific distribution. The K-S test is defined by;

H_0: The sample follows a specified distribution,

against

H_1: The sample does not follow the specified distribution.

If $G(x_i)$ is the value of the CDF of the candidate distribution at x_i and $\hat{G}(x_i)$ is the value of the empirical distribution at x_i, then the test statistic for the K-S test is defined as

$$K - S = \max\{|G(x_i) - \hat{G}(x_i)|, |G(x_i) - \hat{G}(x_{i-1})|\}, i = 1, 2, ..., n. \qquad (3.19)$$

The computed value of test statistic is then compared with a K-S table value at a given significance level to make a decision of rejection or not of the null hypothesis. If multiple distributions are to be compared, the distribution with the smaller K-S value is the most appropriate.

3.5.3 Cramér-von Misses

Suppose $G(x_i, \alpha)$ is the CDF such that the form of G is known and the n-dimensional parameter vector α is unknown. The test statistic Z^*, is obtained as follows:

1. In ascending order, arrange the x_i's and estimate $G(x_i;\hat{\alpha}) = \beta_i$.
2. Estimate $w_i = \Phi^{-1}(\beta_i)$.
3. Compute $Z^2 = \sum_{i=1}^{n}(w_i - \frac{2i-1}{2n})^2 + \frac{1}{12n}$.
4. Transform Z^2 into $Z^* = Z^2(1+\frac{0.5}{n})$ to obtain the test statistic. The model with the smallest test statistic Z^* is the best.

3.6 Information Criteria

The effect of increasing the quantity of parameters is that it improves the fit of a given model and causes the likelihood to increase irrespective of whether the new parameter is important or not. The information criteria enable us do comparison of models when they are not nested. The information criteria discussed in this study are; the Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICc) and Bayesian Information Criterion (BIC).

3.6.1 Akaike Information Criterion

The AIC is a model selection tool derived from the information theory. It is designed to select the model that reduces the Kullback-Leibler distance between the model and the truth (Akaike, 1974). The test statistic is given by

$$AIC = -2\ln L + 2n, \qquad (3.20)$$

where L is the value of the likelihood and n is the number of estimated parameters of the model. AIC introduces good model selection for large samples and is able to penalize models with many parameters. In order to overcome an issue of biasness associated with this method, a Corrected AIC was introduced. The test statistic for AICc is given by

$$AICc = AIC + \frac{2n(n+1)}{k-n-1}. \tag{3.21}$$

3.6.2 Bayesian Information Criterion

BIC is basically a model selection technique that measures the trade-off between model fit and complexity of the model. It assumes that the data is independently and identically distributed (Schwarz, 1978). The test statistic is given as

$$BIC = \ln(k)n - 2\ln\hat{L}, \tag{3.22}$$

where k is the sample size. Just like the AIC, the appropriate model is the one with the minimum BIC value.

3.7 Total Time on Test

There has been a generalization of the TTT in literature. Among these generalizations are TTT-transform and TTT-plot. This study employed the TTT-transform technique. This technique is used to check whether a random sample is from a class of lifetime distributions with hazard rate function displaying bathtub shape. If G is the CDF of a distribution, then TTT-transform is define as

$$G^{-1} = \int_0^{F^{-1}(p)} S(u), p \in (0,1), \tag{3.23}$$

where $S(u)$ is the survival function. The scaled TTT-transform is computed using

$$\varphi F(p) = \frac{G^{-1}(p)}{G^{-1}(1)}. \tag{3.24}$$

The curve of $\varphi F(p)$ verses $0 \le p \le 1$ is scaled TTT-transform curve. The shape of the hazard rate function can be classified as one of the following according to Barlow and Doksum (1972).

1. The hrf is said to be increasing if the scaled TTT-transform curve is concave above the 45^0 line.
2. The hrf is decreasing if the scaled TTT-transform curve is convex beneath the 45^0 line.
3. The hrf exhibits a bathtub shape if the scaled TTT-transform curve is first convex below the 45^0 line and then concave above the line.
4. The hrf is upside down bathtub or unimodal if the scaled TTT- transform curve is first concave above 45^0 line and then convex below the 45^0 line.

Suppose an ordered sample $Y_{1:k}, Y_{2:k}, \ldots, Y_{k:k}$, from a lifetime distribution G, then the TTT test statistics to the i^{th} failure is given as

$$TTT_i = \sum_{j=1}^{i} (k - j + 1)\left(y_{j:k} - y_{j-k:1}\right), i = 1, 2, \ldots, k. \tag{3.19}$$

The empirical scaled TTT-transform is given by

$$TTT_i^* = \frac{TTT_i}{TTT_k},\ \ 0 \le \mathrm{TTT}_k \le 1. \tag{3.20}$$

This curve is obtained by plotting $\frac{i}{k}$ against TTT_i^*.

CHAPTER FOUR

THEORETICAL RESULTS

4.0 Introduction

In this chapter a new family of distributions called T-IE$\{Y\}$ family of distributions is introduced by using the quantile functions of well-known statistical distributions. This idea of parameter induction was pioneered by Aljarrah et al.(2014), this they did by taking a special case of the work of Alzaatreh et al. (2013). The results obtained include, derivations of the T-IE$\{Y\}$ family of distributions and their statistical properties.

4.1 T-IE $\{Y\}$ Family of Distributions

In this section, the CDF, PDF, survival function and hazard rate function of the proposed T-IE $\{Y\}$ family of distributions has been developed.

Suppose R follows the IE random variable with CDF $G_R(x) = \exp(-\lambda/x)$, $x > 0$ and PDF $g_R(x) = \frac{\lambda}{x^2}\exp(-\lambda/x)$ where $x > 0, \lambda > 0$ is scale parameter. Using the technique by Aljarrah et al.(2014) as in equation (3.5), the CDF of T-IE$\{Y\}$ family of distributions is given by

$$G_X(x) = \int_a^{Q_Y(\exp(-\lambda/x))} g(t)dt = G_T(Q_Y[\exp(-\lambda/x)]). \qquad (4.1)$$

Differentiating equation (4.1) with respect to x gives the corresponding PDF as:

$$g_X(x) = \frac{\lambda}{x^2}\exp(-\tfrac{\lambda}{x})Q_Y'(\exp(-\tfrac{\lambda}{x}))g_T\{Q_Y(\exp(-\tfrac{\lambda}{x}))\}. \qquad (4.2)$$

The associated survival function is given as:

$$S_X(x) = 1 - G_T(Q_Y[\exp(-\lambda/x)]). \qquad (4.3)$$

The hazard rate of a system usually depends on time. It is the frequency at which a system or component fails. Unlike the survival function, hazard rate function focuses on the event failing (that is, the event of interest occurring). The hazard rate function of the new family, T-IE$\{Y\}$ family of distribution is given by

$$h_X(x) = \frac{\frac{\lambda}{x^2}\exp(-\lambda/x)Q_Y'(\exp(-\lambda/x))g_T(Q_Y(\exp(-\lambda/x)))}{[1-G_T(Q_Y(\exp(-\lambda/x)))]}. \qquad (4.4)$$

Lemma 4.1. If X follows the T-IE$\{Y\}$ family of distributions given by equation (4.1), then the quantile function is

$$Q_X(u) = -\frac{\lambda}{\log[G_Y(Q_T(u))]}, u \in (0,1).$$

Proof. Equating the CDF of T-IE$\{Y\}$ family of distribution $G_X(x) = G_T\left(Q_Y\left(\exp\left(-\frac{\lambda}{x_u}\right)\right)\right)$ to u and solving for x_u completes the proof.

The quantile function of the T-IE$\{Y\}$ family can be obtained by taking the inverse of equation (4.1). The quantile function $Q_X(u)$ can be used to generate random samples from the T-IE$\{Y\}$ family of distributions. In practice, the first quartile, median and third quartile can be obtained by substituting $u = 0.25$ $u = 0.5$ and $u = 0.75$ respectively into Lemma 4.1.

Lemma 4.2. Suppose X follows the T-IE$\{Y\}$ family of distributions given by equation (4.1), then

$$X = -\frac{\lambda}{(LogG_Y(T))}.$$

Proof. From equation (4.1), equating T to $Q_Y\left(\exp\left(-\frac{\lambda}{X}\right)\right)$ and solving for X completes the proof.

Lemma 4.3. If the distribution of Y follows IE and X follows the T-IE$\{Y\}$ family of distributions given by equation (4.1), then $X \overset{d}{=} T$.

Proof. Substituting $G_Y(T) = \exp\left(-\frac{\lambda}{T}\right)$ into Lemma 4.2 gives $X = -\frac{\lambda}{(LogG_Y(T))} = T$. This completes the proof.

4.2 Sub-families of T-IE{Y} Family of Distributions

The T-IE{Y} family in equation (4.2) can produce many different sub-families for life time data. Three sub-families of T-IE{Y} class using quantile functions of Weibull, Logistic and Lomax distributions have been developed. Quantile functions are used to generate random samples from statistical distributions during simulations. The three sub-families are T-IE{Weibull}, T-IE{Logistic} and T-IE{Lomax} families of distributions.

4.2.1 T-IE{Weibull} Family

Let Y ~Weibull distribution with the CDF $G_Y(x) = 1 - \exp(-\eta x^{\gamma})$ and the PDF $g_Y(x) = \gamma\eta x^{\gamma-1}\exp(-\eta x^{\gamma})$, where η is the scale parameter and γ is the shape parameter. Taking the inverse of the CDF (Weibull distribution) will result in $Q_Y(u) = (-\frac{1}{\eta}\log(1-u))^{\frac{1}{\gamma}}$. Also, from equation (4.1), the CDF of T-IE{Y} class is $G_X(x) = G_T[(Q_Y(\exp(-\lambda/x)))]$.

Let $\eta = 1$ and by substitution the CDF of the IE into the quantile function of the Weibull distribution, the CDF for T-IE{Weibull} family is given by

$$G_X(x) = G_T[\{-\log(1-\exp(-\lambda/x))\}^{\frac{1}{\gamma}}], \tag{4.5}$$

where $T \in (0,\infty)$, $\lambda > 0$ is the scale parameter and $\gamma > 0$ is the shape parameter.

The corresponding PDF of T-IE{Weibull} class of distributions is given by

$$g_X(x) = \frac{\lambda \exp(-\lambda/x)}{\gamma x^2 (1-\exp(-\lambda/x))} \{-\log(1-\exp(-\lambda/x))\}^{\frac{1}{\gamma}-1} g_T[\{-\log(1-\exp(-\lambda/x))\}^{\frac{1}{\gamma}}]. \tag{4.6}$$

It is worth noting that, some distributions are sub-models of T-IE{Weibull} family of distributions. For example if $\gamma = 1$, we obtain T-IE{exponential} family and If $\gamma = 2$, then we obtain the T-IE{Rayleigh} family.

4.2.2 The T-IE{logistic} Family

Assuming Y follows Logistic distribution with quantile function $Q_Y(u) = -\frac{1}{\alpha}\log(u^{-1}-1)$. If $u = \exp(-\frac{\lambda}{x})$, then, the CDF of T-IE{Logistic} is given by;

$$G_X(x) = G_T[-\tfrac{1}{\alpha}\log(\exp(\lambda/x)-1)], \tag{4.7}$$

where the random variable $T \in (-\infty,\infty)$.

The corresponding PDF is given by

$$g_X(x) = \frac{\lambda \exp(\lambda/x)}{\alpha x^2 (\exp(\lambda/x)-1)} g_T[-\tfrac{1}{\alpha}\log(\exp(\lambda/x)-1)]. \tag{4.8}$$

4.2.3 T-IE{Lomax} Family

Assuming Y follows Lomax distribution with the quantile function $Q_Y(u)=\alpha\{(1-u)^{-\frac{1}{\gamma}}-1\}$. If $T\in(0,\infty)$, $\alpha=1$ and $u=\exp(-\frac{\lambda}{x})$, then the CDF of the T-IE{Lomax}family of distributions is given by;

$$G_X(x)=G_T[(1-\exp(-\lambda/x))^{-\frac{1}{\gamma}}-1]. \tag{4.9}$$

The corresponding PDF is given by

$$g_X(x)=\frac{\lambda\exp(-\lambda/x)}{\gamma x^2(1-\exp(-\lambda/x))^{\frac{1}{\gamma}-1}}g_T\left[(1-\exp(-\lambda/x))^{-\frac{1}{\gamma}}-1\right]. \tag{4.10}$$

4.3 Statistical Properties of the T-IE{Y} Family

In this section, a detailed investigation of some statistical properties of the T-IE{Y} family of distributions has been done. The relationship between random variable T and X for some cases which can be used to simulate X from T is given in Lemma 4.4, 4.5 and 4.6.

Lemma 4.4. If T is a random variable with CDF $G_T(x)$, then the random variable $X=-\dfrac{\lambda}{\log(1-\exp^{-T^\gamma})}$ follows the T-IE{Weibull} family of distributions.

Proof. Substituting the CDF of Weibull distribution, $G_Y=1-\exp(-T^\gamma)$ into Lemma 4.2 yields $X=-\dfrac{\lambda}{\log[(1-\exp^{-T^\gamma})]}$. Thus, the proof is complete.

Lemma 4.5. If T is a random variable with CDF $G_T(x)$, then the random variable $X=-\dfrac{\lambda}{\log(1+\exp^{-\alpha T})^{-1}}$ follows the T-IE{Logistic} family of distributions.

Proof. Substituting the CDF of logistic distribution, $G_Y = \dfrac{1}{1+\exp(-\alpha T)}$ into Lemma 4.2 yields

$X = -\dfrac{\lambda}{\log(1+\exp^{-\alpha T})^{-1}}$. Thus, the proof is complete.

Lemma 4.6. If T is a random variable with CDF $G_T(x)$, then the random variable

$X = -\dfrac{\lambda}{\log(1-(1+T)^{-\gamma})}$ follows the T-IE(Lomax) family of distributions.

Proof. Substituting the CDF of Lomax distribution, $G_Y = 1-(1+T)^{-\gamma}$ into Lemma 4.2 yields

$X = -\dfrac{\lambda}{\log(1-(1+T)^{-\gamma})}$. Thus, the proof is complete.

Quantile functions are used to generate random samples from a given distribution. Quantile functions can be used to describe the characteristics such as skewness and kurtosis of a distribution. The following are quantile functions for T-IE{Weibull}, T-IE{Logistic}and T-IE{Lomax} families of distributions.

Lemma 4.7. The quantile functions for T-IE{Weibull}family is given by;

$$Q_X(u) = -\frac{\lambda}{\log[1-\exp(-(Q_T(u))^{\gamma})]},\ u \in (0,1).$$

Proof. Equating the CDF of T-IE{Weibull} $G_X(x) = G_T\left[-\log\left(1-\exp\left(-\dfrac{\lambda}{x_u}\right)\right)\right]^{\frac{1}{\gamma}}$ to u and solving for x_u completes the proof.

Lemma 4.8. The quantile functions for T-IE{Logistic}family is given by;

$$Q_X(u) = \frac{\lambda}{\log[\exp(-\alpha Q_T(u)) + 1]},\ u \in (0,1).$$

Proof. Equating the quantile function for T-IE{Logistic} family of distributions. $G_X(x) = G_T\left[-\frac{1}{\alpha}\log\left(\exp\left(\frac{\lambda}{x_u}\right) - 1\right)\right]$ to u and solving for x_u completes the proof.

Lemma 4.9. The quantile functions for T-IE{Lomax}family is given by;

$$Q_x(u) = -\frac{\lambda}{\log[1 - (Q_T(u) + 1)^{-\gamma}]},\ u \in (0,1).$$

Proof. Equating the CDF of T-IE{Lomax} family $G_X(x) = G_T\left[\left(1 - \exp\left(-\frac{\lambda}{x_u}\right)\right)^{-\frac{1}{\gamma}} - 1\right]$ to u and solving for x_u completes the proof.

4.3.1 Mode

The most commonly occurring value in a set of observation is termed the mode. The mode could either be unimodal or multimodal. A distribution with one mode is termed unimodal while those with more than one mode are termed multimodal (bimodal, trimodal and so on).

Proposition 4.1. If $g_X(x)$ is the PDF of T-IE$\{Y\}$ family of distributions, then, the mode of the T-IE{Y} family is obtained by finding the solution to the equation

$$\zeta[Q_Y'(\exp(-\lambda/x))] + \zeta[g_T(Q_Y(\exp(-\lambda/x)))] + \frac{\lambda}{x^2} - \frac{2}{x} = 0, \quad (4.11)$$

where $\zeta(q) = \dfrac{q'}{q}$.

Proof. Recall that the PDF of the T-IE{Y} family of distributions is

$$g_X(x) = \tfrac{\lambda}{x^2}\exp(-\lambda/x)Q_Y'(\exp(-\lambda/x))g_T\{Q_Y(\exp(-\lambda/x))\}.$$

Taking logarithms of both sides, we have

$$\log g_X(x) = \log(\lambda) - 2\log(x) - \tfrac{\lambda}{X} + \log Q_Y'(\exp(-\lambda/x)) + \log g_T\{Q_Y(\exp(-\lambda/x))\}. \quad (4.12).$$

Differentiating equation (4.12) with respect to x and equating to zero, we have

$$\frac{d}{dx}g_x(x) = \tfrac{\lambda}{x^2} - \tfrac{2}{x} + \frac{Q_Y''(\exp(-\lambda/x))}{Q_Y'(\exp(-\lambda/x))} + \frac{g_t'[Q_Y'(\exp(-\lambda/x))]}{g_t[Q_Y(\exp(-\lambda/x))]} = 0.$$

Rearranging and simplifying the terms, we have

$$\frac{Q_Y''(\exp(-\lambda/x))}{Q_Y'(\exp(-\lambda/x))} + \frac{g_t'[Q_Y'(\exp(-\lambda/x))]}{g_t[Q_Y(\exp(-\lambda/x))]} + \tfrac{\lambda}{x^2} - \tfrac{2}{x} = 0.$$

Simplifying further, we have

$\zeta[Q_Y'(\exp(-\lambda/x)] + \zeta[g_T(Q_Y(\exp(-\lambda/x)))] + \tfrac{\lambda}{x^2} - \tfrac{2}{x} = 0$. Hence, the proof is complete.

4.3.2 Moments

In statistical analysis, moments are very essential especially in application. They are used for finding skewness, kurtosis, measures of variation, measures of central tendency among others.

Proposition 4.2. If the moments exist, then the n^{th} non-central moment of the T-IE{Y} family of distributions is given as

$$\mu_n' = (-1)^n \lambda^n E[\log G_Y(T)]^{-n}, n = 1, 2, \ldots \qquad (4.13)$$

Proof. By definition $\mu' = E(X^n)$. Substituting $X = -\dfrac{\lambda}{\log G_Y(T)}$ from Lemma 4.2 and simplifying results in $\mu_n' = (-1)^n \lambda^n E[\log G_Y(T)]^{-n}$. Hence, the proof is complete.

Corollary 4.1. The moments of the T-IE{Weibull}family of distributions is given by

$$\mu_n' = E\left(X^n\right) = \left(-1\right)^n \lambda^n E\left[\log\left(1-\exp\left(-T^{\gamma}\right)\right)\right]^{-n}, n = 1, 2, \ldots \quad (4.14)$$

Proof. Substituting the CDF of the baseline Weibull distribution $G_X(t) = 1 - \exp(-T^{\gamma})$ into equation (4.13) completes the proof.

Corollary 4.2. The moments of the T-IE{Logistic} family of distributions is given by

$$\mu_n' = E(X^n) = (-1)^n \lambda^n E[\log(1+\exp(-\alpha T))]^{-n}, n = 1, 2, \ldots \quad (4.15)$$

Proof. Substituting the CDF of the baseline logistic distribution $G_X(t) = 1 + \exp(-\alpha T)$ into equation (4.13) completes the proof.

Corollary 4.3. The moments of the T-IE{Lomax} family of distributions is given by

$$\mu_n' = (-1)^n \lambda^n E[\log(1-(1+T))^{\gamma}]^{-n}, n = 1, 2, \ldots \quad (4.16)$$

Proof. Substituting the CDF of the baseline Lomax distribution $G_X(t) = (1-(1+T))^{\gamma}$ into (4.13) completes the proof.

4.3.3 Shannon Entropy

Entropy is a measure of variation or uncertainty. Shannon entropy (1948) has been used in several applications in engineering, machine learning and information theory. It provides a way to estimate the mean number of bits needed to encode a string of symbols. The entropy of random variable X with PDF $g_X(x)$ is given by $\vartheta_X = -E[\log g_X(X)]$. The following proposition gives the Shannon entropy of T-IE{Y} family.

Proposition 4.3. The Shannon entropy of T-IE{Y} family of distributions is given by

$$\vartheta_X = \vartheta_T + E(\log g_Y(T)) - \log(\lambda) + 2E(\log(X)) + \lambda E(\tfrac{1}{X}), \qquad (4.17)$$

where ϑ_T is the Shannon entropy for the random variable T.

Proof. As $X \overset{d}{=} Q_R(G_Y(T))$, then, $T \overset{d}{=} Q_Y(G_R(X))$, hence, from equation (3.6) we can write

$$g_X(x) = \frac{g_T(t)}{g_Y(t)} g_R(x). \qquad (4.18)$$

Taking logarithms of both sides of equation (4.18) and multiplying through by negative one gives

$$-\log g_X(x) = -\log g_T(t) - (-\log g_Y(t)) + (-\log g_R(x)). \quad (4.19)$$

Taking expectation of both sides of equation (4.19), we have

$$E(-\log g_X(X)) = E(-\log g_T(T)) - E(-\log g_Y(T)) + E(-\log g_R(X)). \quad (4.20)$$

But by definition,

$$\vartheta_X = E[-\log g_X(X)].$$

This implies that,

$$\vartheta_X = \vartheta_T + E(\log g_Y(T)) - E(\log g_R(X)). \qquad (4.21)$$

Recall that, $g_R(x) = \frac{\lambda}{x^2}\exp(\frac{\lambda}{x})$.

Taking logarithms of both sides of $g_R(x) = \frac{\lambda}{x^2}\exp(\frac{\lambda}{x})$ and substituting into equation (4.21) gives

$$\vartheta_x = \vartheta_T + E(\log g_Y(T)) - E(\log(\lambda) - 2\log(X) + \tfrac{\lambda}{X}).$$

Simplifying further results in

$\vartheta_X = \vartheta_T + E(\log g_Y(T)) - \log(\lambda) + 2E(\log(X)) + \lambda E(\frac{1}{X})$. Hence, the proof is complete.

Corollary 4.4. The Shannon entropy for the T-IE{Weibull}family of distributions is given by

$$\vartheta_x = \vartheta_T + \log(\alpha/\lambda) + (\alpha - 1)E(\log(T)) - E(T^{\alpha}) + 2E(\log(X)) + \lambda E(1/X). \quad (4.22)$$

Proof. For the Shannon entropy of T-IE{Weibull}family of distributions,

The density function of Weibull distribution is $g_Y(T) = \alpha T^{\alpha-1}\exp(-T^{\alpha})$.

Taking logarithm of the density function of the Weibull distribution, we have $\log g_Y(T) = \log(\alpha) + (\alpha - 1)\log(T) - T^{\alpha}$.

Now, substituting the above into equation (4.17) and simplifying results in

$$\vartheta_x = \vartheta_T + \log(\alpha/\lambda) + (\alpha - 1)E(\log(T)) - E(T^{\alpha}) + 2E(\log(X)) + \lambda E(1/X).$$

Hence, the proof is complete.

Corollary 4.5. The Shannon entropy for the T-IE{Logistic}family of distributions is given by

$$\vartheta_x = \vartheta_T + \log(\alpha/\lambda) - \alpha E(T) - 2E(\log(1 + \exp(-\alpha T))) + 2E(\log(X)) + \lambda E(1/X). (4.23)$$

Proof. For the Shannon entropy of the T-IE{Logistic} distribution,

Let the density function of Logistic be $g_Y(T) = \dfrac{\alpha\exp(-\alpha t)}{(1+\exp(-\alpha t))^2}$. Taking logarithms of both sides of the density function, we have

$$\log g_Y(T) = \log(\alpha)\text{-}\alpha\text{T-}2\log(1+\exp(-\alpha T))$$

Now, taking expectation of the log-density, substituting into equation (4.17) and simplifying, we have

$$\vartheta_X = \vartheta_T + \log(\alpha/\lambda) - \alpha E(T) - 2E(\log(1+\exp(-\alpha T))) + 2E(\log(X)) + \lambda E(1/X).$$

Hence, the proof is complete.

Corollary 4.6. The Shannon entropy for the T-IE{Lomax}family of distributions is given by

$$\vartheta_x = \vartheta_T + \log(\gamma/\lambda) - (\gamma+1)E(\log(1+T)) + 2E(\log(X)) + \lambda E(1/X). \qquad (4.24)$$

Proof. Let the density function of Lomax distribution be $g_Y(T) = \gamma(1+T)^{-(\gamma+1)}$.

Taking logarithm of both sides of the density function, we have

$$\log g_Y(T) = \log(\gamma(1+T)^{-(\gamma+1)}) = \log(\gamma)\text{-}(\gamma+1)\log(1+T).$$

Now, taking expectation and substituting into equation (4.17), we have

$$\vartheta_x = \vartheta_T + \log(\gamma) - (\gamma+1)E(\log(1+T)) - \log(\lambda) + 2E(\log(X)) + \lambda E(1/X),$$

and simplifying further gives

$$\vartheta_X = \vartheta_T + \log(\gamma/\lambda) - (\gamma+1)E(\log(1+T)) + 2E(\log(X)) + \lambda E(1/X).$$

Hence, the proof is complete.

4.3.4 Moment Generating Function

When the distribution of the random variable is heavy tail, moment generating function may not exist as the moment does not exist. Let X be a random variable with $E|X|^r < \infty$, for $r \in N$. The moment generating function for X is $M_X(z) = E(e^{zX})$.

Proposition 4.4. The Moment generating function of the T-IE{Y} family of distributions is defined as

$$M_X(z) = \sum_{r=0}^{\infty} \frac{z^r (-1)^r \lambda^r}{r!} E[\log G_Y(T)]^{-r}. \qquad (4.25)$$

Proof. By definition

$$M_X(z) = E(e^{zx}) = E\left[\sum_{r=0}^{\infty} \frac{z^r X^r}{r!}\right] = \sum_{r=0}^{\infty} \frac{z^r}{r!} E(X^r).$$

Substituting Lemma 4.2, $X = -\dfrac{\lambda}{[\log G_Y(T)]}$ into $M_X(z)$ and simplifying gives

$M_X(z) = \sum_{r=0}^{\infty} \frac{z^r}{r!} (-1)^r \lambda^r E[\log G_Y(T)]^{-r}$. Hence, the proof is complete.

The following are the moment generating functions for T-IE{Weibull}, T-IE{Logistic} and T-IE{Lomax} family of distributions.

Corollary 4.7. Moment generating function for T-IE{Weibull} family of distributions is defined as

$$M_X(z) = \sum_{r=0}^{\infty} \frac{z^r (-1)^r \lambda^r}{r!} E[\log(1 - \exp(-T^{\gamma}))]^{-r}. \qquad (4.26)$$

Proof. If $M_X(z)=\sum_{r=0}^{\infty}\frac{z^r}{r!}\mu_r^{'}$ and $\mu_r^{'}=(-1)^r\lambda^r E[\log(1-\exp(-T^{\gamma}))]^{-r}$, then by substitution, we have

$$M_X(z)=\sum_{r=0}^{\infty}\frac{z^r(-1)^r\lambda^r}{r!}E[\log(1-\exp(-T^{\gamma}))]^{-r}.$$

Hence, the proof is complete.

Corollary 4.8. Moment generating function for T-IE{Logistic} family of distributions is defined as

$$M_X(z)=\sum_{r=0}^{\infty}\frac{z^r(-1)^r\lambda^r}{r!}E[\log(1+\exp(-\alpha T))]^{-r}. \quad (4.27)$$

Proof. If $M_X(z)=\sum_{r=0}^{\infty}\frac{z^r}{r!}\mu_r^{'}$ and $\mu_r^{'}=(-1)^r\lambda^r E[\log(1+\exp(-\alpha T))]^{-r}$, then, by substitution, we have $M_X(z)=\sum_{r=0}^{\infty}\frac{z^r(1)^r\lambda^r}{r!}E[\log(1+\exp(-\alpha T))]^{-r}$. Hence, the proof is complete.

Corollary 4.9. Moment generating function for T-IE{Lomax} family of distributions is defined as

$$M_X(z)=\sum_{r=0}^{\infty}\frac{z^r(-1)^r\lambda^r}{r!}E[\log(1-(1+T)^{-\gamma})]^{-r}. \quad (4.28)$$

Proof. If $M_X(z)=\sum_{r=0}^{\infty}\frac{z^r}{r!}\mu_r^{'}$ and $\mu_r^{'}=(-1)^r\lambda^r E[\log(1-(1+T)^{-\gamma})]^{-r}$, then, by substitution, we have

$M_X(z)=\sum_{r=0}^{\infty}\frac{z^r(-1)^r\lambda^r}{r!}E[\log(1-(1+T)^{-\gamma})]^{-r}$. Hence, the proof is complete.

4.4 Special Distributions

In this section, three special distributions in the T-IE{Y} family of distributions have been developed. These three distributions are Log-Logistic-IE{Weibull}(LLIEW), Gumbel-IE{Logistic}(GIEL) and Weibull-IE{Lomax}(WIEL).

4.5 The Log-Logistic-IE{Weibull} Distribution

Let $T\sim$ Log-Logistic(α,θ) distribution with the CDF $G_T(x)=1-[1+(x/\alpha)^{\theta}]^{-1}$ and the PDF $g_T(x)=\frac{\theta}{\alpha\theta}x^{\theta-1}[1+(x/\alpha)^{\theta}]^{-2}$ with θ as a shape parameter and α as a scale parameter. Using CDF of T-IE{Weibull} family of distribution, the CDF of the Log-Logistic-IE{Weibull} (LLIEW) distribution is defined as

$$G_X(x)=1-[1+(-\alpha^{-1}\{\log(1-\exp(-\lambda/x))\}^{\frac{1}{\gamma}})^{\theta}]^{-1}, x>0, \quad (4.29)$$

where $\alpha>0,\gamma>0,\theta>0,\lambda>0.$ Figure 4.1 shows the plots of the CDF of the LLIEW distribution. The parameters γ,θ determine the shape of the distribution while α,λ are the scale parameters. When the difference between the values of θ and γ increases, the curve sharply approaches its maximum. Also, as the value of α increases, the curve takes longer time to reach the maximum value.

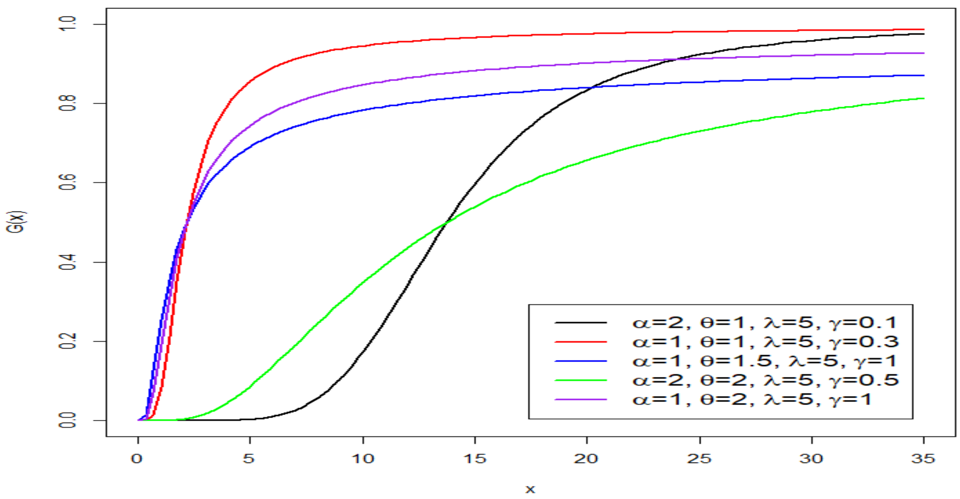

Figure 4.1: CDF of LLIEW distribution

From equation (4.28), the PDF of LLIEW distribution is given by

$$g_X(x)=\frac{\lambda\theta\exp(-\lambda/x)\{-\log[1-\exp(-\lambda/x)]\}^{\frac{1}{\gamma}-1}[\{-\log[1-\exp(-\lambda/x)]\}^{\frac{1}{\gamma}}]^{\theta-1}}{\alpha^{\theta}\gamma x^{2}(1-\exp(-\lambda/x))[1+[-\alpha^{-1}\{\log(1-\exp(-\lambda/x))\}^{\frac{1}{\gamma}}]^{\theta}]^{2}},x>0. \qquad (4.30)$$

Figure 4.2 shows the plots of the PDF of LLIEW distribution. The PDF can be positively skewed, negatively skewed, J-shape and reversed-J shape. In addition, it can be observed that, the distribution can be unimodal and almost symmetrical with varied degree of skewness and kurtosis for different parameter values. When $\lambda<1$ and the values of α is greater than the values of γ, we have reversed-J shape. When the values of the parameters are more than or equal one, we have a unimodal.

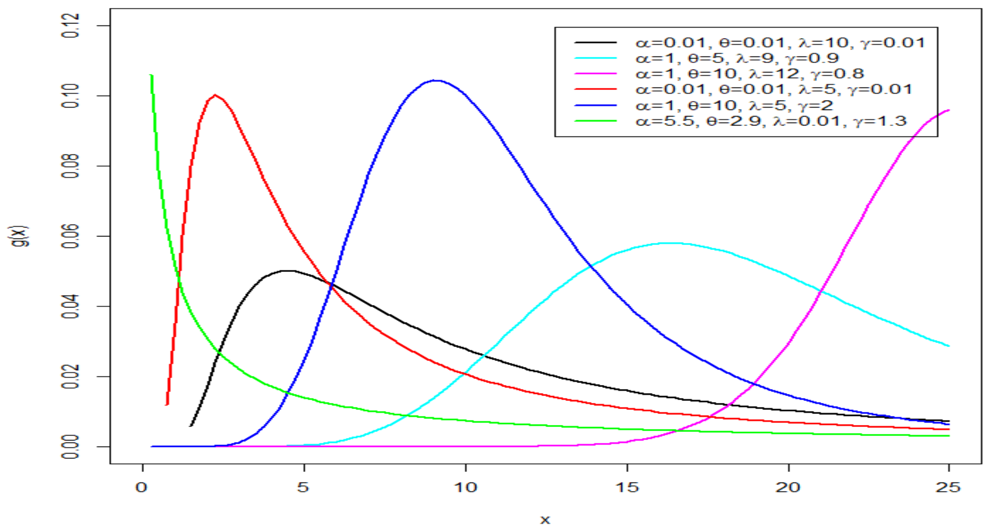

Figure 4.2: PDF of LLIEW distribution

Also, the survival function for LLIEW is given as:

$$S_X(x) = \{1 + (-\alpha^{-1}\{\log(1 - \exp(-\lambda/x))\}^{\frac{1}{\gamma}})^{\theta}\}^{-1}, \; x > 0. \qquad (4.31)$$

Figure 4.3 shows the plots of the survival function of LLIEW distribution. When gamma is less than one and theta is more than or equal to one, the curve is constant for some time and decreases sharply towards zero. When alpha, theta, and gamma are all less than or equal to one and lambda more than one, the curve move slowly towards zero.

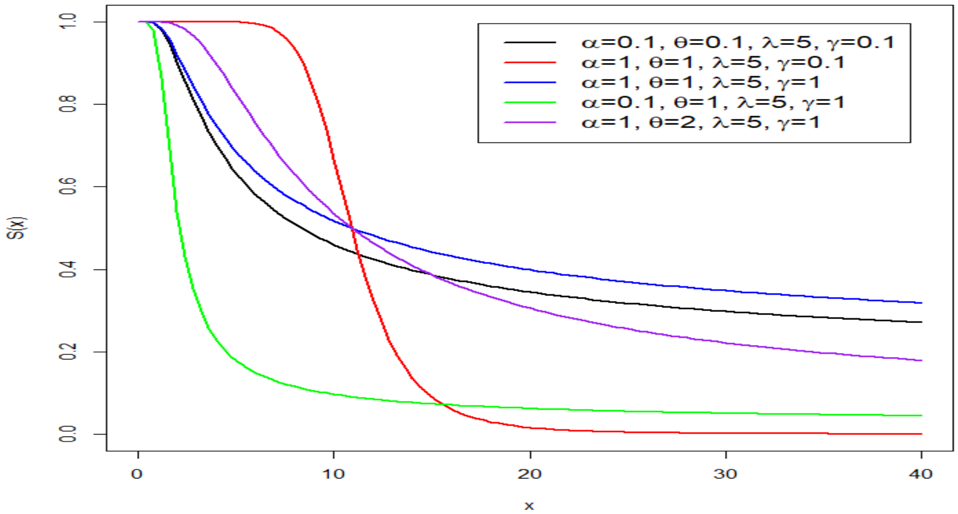

Figure 4.3: Survival function of LLIEW distribution

The hazard function is

$$h_X(x) = \frac{\lambda\theta \exp(-\lambda/x)\{-\log(1-\exp(-\lambda/x))\}^{\frac{1}{\gamma}-1}\,[\{-\log(1-\exp(-\lambda/x))\}^{\frac{1}{\gamma}}]^{\theta-1}}{\alpha^{\theta}\gamma x^{2}(1-\exp(-\lambda/x))[1+\{-\alpha^{-1}[\log(1-\exp(-\lambda/x))]^{\frac{1}{\gamma}}\}^{\theta}]}, x>0. \qquad (4.32)$$

Figure 4.4 represents plots of the hazard function of LLIEW distribution. The plots of hazard rate function give various shapes such as decreasing, increasing and inverted bathtub. This makes the LLIEW distribution suitable for modelling failure rates that are monotonic and non-monotonic in real life.

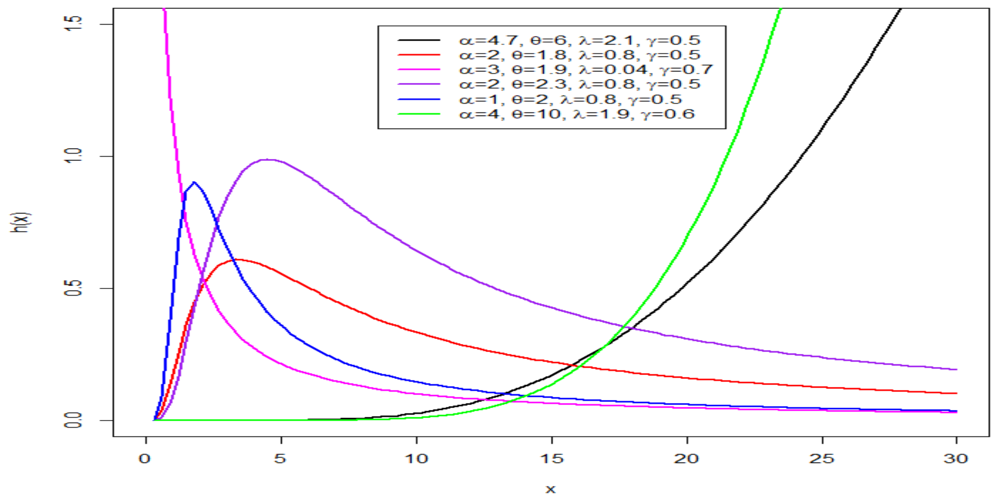

Figure 4.4: Hazard rate function of LLIEW distribution

Distributions of random variables can be described by their quantile functions. The quantile function is useful in computing the characteristics of a distribution such as median, skewness and kurtosis. When $u = 0.5$, $u = 0.25$ and $u = 0.75$ we get the median, lower quartile and upper quartile respectively. The quantile function of the LLIEW distribution for $u \in (0,1)$ is given by

$$Q_X(u) = -\frac{\lambda}{\log[1+\exp\{\alpha[(1-u)^{-1}-1]^{\frac{1}{\theta}}\}^{\gamma}]}. \qquad (4.33)$$

Random samples from LLIEW distribution can be generated using equation (4.33).

4.5.1 Methods of Parameter Estimation of LLIEW Distribution

This section presents five (5) parameter estimation procedures for estimating the parameters of LLIEW distribution. These estimators are: Maximum Likelihood estimators (MLE), Ordinary Least Square estimators, Weighted Least Square estimators, Percentile based estimators and Cramér-von Mises estimators.

4.5.2 Maximum Likelihood Estimation Method for Complete Sample

In estimating the parameters of LLIEW distribution, the MLE was employed. Arguably, the MLE is the most commonly used parameter estimation. Given that $X \sim LLIEW(\alpha,\theta,\lambda,\gamma)$, $\Theta=(\lambda,\alpha,\theta,\gamma)^T$ and $z_i=-\log(1-\exp(-\frac{\lambda}{x}))$. Let X be a random sample from size n of LLIEW distribution, then the likelihood function is given by:

$$L(x_i;\Theta)=\prod_{i=1}^{n}\left[\frac{\lambda\theta\exp(-\lambda/x_i)(z_i)^{\frac{1}{\gamma}-1}[\{z_i\}^{\frac{1}{\gamma}}]^{\theta-1}}{\alpha^{\theta}\gamma x_i^2(1-\exp(-\lambda/x_i))[1+[-\alpha^{-1}(z_i)^{\frac{1}{\gamma}}]^{\theta}]^2}\right].$$

Taking the natural logarithm of $L(x_i;\Theta)$ and also, let $\ell=\log L(x_i;\Theta)$, the total log-likelihood function is given by:

$$\ell=n\log(\lambda\theta)-n\theta\log\alpha-n\log\gamma-2\sum_{i=1}^{n}\log x_i-\lambda\sum_{i=1}^{n}\tfrac{1}{x_i}-\sum_{i=1}^{n}\log(1-\exp(-\tfrac{\lambda}{x_i}))+$$
$$(\tfrac{1}{\gamma}-1)\sum_{i-1}^{n}\log(z_i)+(\theta-1)\sum_{i=1}^{n}\log(z_i)^{\frac{1}{\gamma}}-2\sum_{i=1}^{n}\log(1+(\alpha^{-1}z_i^{\frac{1}{\gamma}})^{\theta}). \tag{4.34}$$

Taking partial derivative of equation (4.34) with respect to the parameters λ,θ,γ and α, the score functions are;

$$\frac{\partial\ell}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\tfrac{1}{x_i}-\sum_{i=1}^{n}\frac{\exp(-\frac{\lambda}{x_i})}{\left(1-\exp(-\frac{\lambda}{x_i})\right)x_i}, \tag{4.35}$$

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}-n\log\alpha+\sum_{i=1}^{n}\log(z_i)^{\frac{1}{\gamma}}-2\sum_{i=1}^{n}\frac{\log\left(\frac{(z_i)^{\frac{1}{\gamma}}}{\alpha}\right)\left(\frac{(z_i)^{\frac{1}{\gamma}}}{\alpha}\right)^{\theta}}{1+\left(\frac{(z_i)^{\frac{1}{\gamma}}}{\alpha}\right)^{\theta}}, \tag{4.36}$$

$$\frac{\partial \ell}{\partial \gamma}=-\frac{n}{\gamma}-\frac{\sum_{i=1}^{n}\log z_i}{\gamma^2}+(\theta-1)\sum_{i=1}^{n}-\frac{\log(\log z_i)\log(z_i)^{\frac{1}{\gamma}}}{\gamma^2}-2\sum_{i=1}^{n}-\frac{\theta\log z_i (z_i)^{\frac{1}{\gamma}}\left(\frac{(z_i)^{\frac{1}{\gamma}}}{\alpha}\right)^{\theta-1}}{\alpha\gamma^2\left(1+\left(\frac{(z_i)^{\frac{1}{\gamma}}}{\alpha}\right)^{\theta}\right)},\qquad(4.37)$$

and

$$\frac{\partial \ell}{\partial \alpha}=-\frac{n\theta}{\alpha}+2\sum_{i=1}^{n}\frac{\theta (z_i)^{\frac{1}{\gamma}}\left(\frac{(z_i)^{\frac{1}{\gamma}}}{\alpha}\right)^{\theta-1}}{\alpha^2\left(1+\left(\frac{(z_i)^{\frac{1}{\gamma}}}{\alpha}\right)^{\theta}\right)},\qquad(4.38)$$

respectively.

The estimates of the parameters are obtained by equating the score functions to zero and solving the system of equations simultaneously. It can be observed however that, the resulting system of equations are not tractable and have to be solved numerically to obtain the estimates of the parameters.

To determine the confidence intervals for the parameters of LLIEW distribution, we use the observed information matrix $\Psi(\Theta)$ given by

$$\Psi(\Theta)=-\begin{bmatrix}\frac{\partial^2 l}{\partial\lambda^2} & \frac{\partial^2 l}{\partial\lambda\partial\alpha} & \frac{\partial^2 l}{\partial\lambda\partial\theta} & \frac{\partial^2 l}{\partial\lambda\partial\gamma}\\ & \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\theta} & \frac{\partial^2 l}{\partial\alpha\partial\gamma}\\ & & \frac{\partial^2 l}{\partial\theta^2} & \frac{\partial^2 l}{\partial\beta\partial\gamma}\\ & & & \frac{\partial^2 l}{\partial\gamma^2}\end{bmatrix},$$

The elements of $\Psi(\Theta)$ are given in appendix A1. The multivariate normal $N_4(0, I^{-1}(\Theta))$, where $I(\Theta)$ is the expected information matrix can be employed to construct approximate confidence interval for the parameters under the regularity conditions. When $I(\Theta)$ is replaced by the observed information matrix evaluated at $\Psi(\hat{\Theta})$, the asymptotic behaviour remains the same. The $100(1-\eta)\%$ asymptotic confidence interval (CI) for each parameter Θ_η is given by:

$$CI = (\Theta_\eta - z_{\eta/2} \times \sigma_{\Theta_\eta}, \Theta_\eta + z_{\eta/2} \times \sigma_{\Theta_\eta}),$$

where σ_{Θ_η} is the standard error of the estimated parameter and $z_{\eta/2}$ is the upper quantile of the standard normal distribution.

4.5.3 Parameter Estimation of the LLIEW Distribution for Incomplete Sample

Suppose we observed the first r failed items $x_1, x_2, ..., x_r$ and $\Theta = (\alpha, \lambda, \theta, \gamma)^T$. Then, the likelihood function is given by

$$L(x_,;\Theta) = \prod_{i=1}^{r} [g(x_i;\Theta)]^{r_i} \times [S_X(x)]^{1-r_i}. \qquad (4.39)$$

Substituting the CDF and the survival functions of LLIEW distribution into (4.39) and let $z_i = -\log(1-\exp(-\frac{\lambda}{x}))$, then the total log-likelihood function for LLIEW distribution in case of censored samples becomes

$$\ell = r_i[n\log(\lambda\theta) + n\theta\log\alpha - n\log\gamma - 2\sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}\tfrac{1}{x_i} - \sum_{i=1}^{n}\log(1-\exp(-\tfrac{\lambda}{x_i})) +$$
$$(\tfrac{1}{\gamma}-1)\sum_{i-1}^{n}\log(z_i) + (\theta-1)\sum_{i=1}^{n}\log(z_i)^{\frac{1}{\gamma}} - 2\sum_{i=1}^{n}\log(1+(\alpha^{-1}z_i^{\frac{1}{\gamma}})^{\theta}] - \sum_{i=1}^{n}(1-r_i)\log[1+\alpha^{-1}(z_i)^{\frac{1}{\gamma}}]^{\theta}. \quad (4.40)$$

Taking partial derivative of ℓ with respect to the parameters λ, θ, γ and α, the score functions are;

$$\frac{\partial \ell}{\partial \lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\frac{1}{x_i}-\sum_{i=1}^{n}\frac{\exp(-\frac{\lambda}{x_i})}{\left(1-\exp(-\frac{\lambda}{x_i})\right)x_i}\mathrm{r}_i^{'}[-n\theta\log\alpha-n\log\gamma+n\log(\theta\lambda)-\sum_{i=1}^{n}\log[1-\exp(-\tfrac{\lambda}{x_i})]-2\sum_{i=1}^{n}\log x_i+$$

$$(\tfrac{1}{\gamma}-1)\sum_{i=1}^{n}\log z_i+(\theta-1)\sum_{i=1}^{n}\log(z_i)^{\frac{1}{\gamma}}-2\sum_{i=1}^{n}\log\left[1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right]-\lambda\sum_{i=1}^{n}\tfrac{1}{x_i}], \qquad (4.41)$$

$$\frac{\partial \ell}{\partial \theta}=\left(\tfrac{n}{\theta}-n\log\alpha+\sum_{i=1}^{n}\log(z_i)^{\frac{1}{\gamma}}-2\sum_{i=1}^{n}\frac{\log\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}}{1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}}\right)r_i^{'}[-n\theta\log\alpha-n\log\gamma+n\log\theta\lambda$$

$$-\sum_{i=1}^{n}\log[1-\exp(-\tfrac{\lambda}{x_i})]-2\sum_{i=1}^{n}\log x_i+(\tfrac{1}{\gamma}-1)\sum_{i=1}^{n}\log z_i+(\theta-1)\sum_{i=1}^{n}\log(z_i)^{\frac{1}{\gamma}}-2\sum_{i=1}^{n}\log[1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}]$$

$$-\lambda\sum_{i=1}^{n}\tfrac{1}{x_i}]-\sum_{i=1}^{n}\frac{\log\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)(1-r_i)\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}}{1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}}, \qquad (4.42)$$

$$\frac{\partial \ell}{\partial \gamma}=\left[-\frac{n}{\gamma}-\frac{\sum_{i=1}^{n}\log z_i}{\gamma^2}+(\theta-1)\sum_{i=1}^{n}-\frac{\log(\log z_i)\log(z_i)^{\frac{1}{\gamma}}}{\gamma^2}-2\sum_{i=1}^{n}-\frac{\theta\log z_i(z_i)^{\frac{1}{\gamma}}\left(\alpha^{-1}((z_i)^{\frac{1}{\gamma}})\right)^{\theta-1}}{\alpha\gamma^2\left(1+(\alpha^{-1}((z_i)^{\frac{1}{\gamma}})^{\theta}\right)}\right]\times$$

$$\mathrm{r}_i^{'}[-n\theta\log\alpha-n\log\gamma+n\log(\theta\lambda)-\sum_{i=1}^{n}\log(1-\exp(-\tfrac{\lambda}{x_i}))-2\sum_{i=1}^{n}\log x_i+(\tfrac{1}{\gamma}-1)\sum_{i=1}^{n}\log z_i+(\theta-1)\sum_{i=1}^{n}\log(z_i)^{\frac{1}{\gamma}}-$$

$$2\sum_{i=1}^{n}\log[1+(\alpha^{-1}(z_i)^{\frac{1}{\gamma}})^{\theta}]-\lambda\sum_{i=1}^{n}\frac{1}{x_i}]-\sum_{i=1}^{n}-\frac{\theta\log z_i(1-r_i)(z_i)^{\frac{1}{\gamma}}\left(\alpha^{-1}((z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha\gamma^2\left(1+(\alpha^{-1}((z_i)^{\frac{1}{\gamma}})^{\theta}\right)}, \qquad (4.43)$$

$$\frac{\partial \ell}{\partial \alpha}=\left(-\frac{n\theta}{\alpha}-2\sum_{i=1}^{n}-\frac{\theta(z_i)^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)}\right)\mathrm{r}_i^{'}[-n\theta\log\alpha-n\log\gamma+n\log\theta\lambda-\sum_{i=1}^{n}\log(1-\exp(-\tfrac{\lambda}{x_i}))-$$

$$2\sum_{i=1}^{n}\log x_i+(\tfrac{1}{\gamma}-1)\sum_{i=1}^{n}\log z_i+(\theta-1)\sum_{i=1}^{n}\log(z_i)^{\frac{1}{\gamma}}-2\sum_{i=1}^{n}\log(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta})-\lambda\sum_{i=1}^{n}\tfrac{1}{x_i}-$$

$$\sum_{i=1}^{n}-\frac{\theta(1-r)(z_i)^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)}, \qquad (4.44)$$

respectively.

Equating $\frac{\partial \ell}{\partial \lambda}$, $\frac{\partial \ell}{\partial \alpha}$, $\frac{\partial \ell}{\partial \beta}$ and $\frac{\partial \ell}{\partial \gamma}$ to zero and solving the system of equations simultaneously for the parameters will give the maximum likelihood estimates.

4.5.4 Ordinary and Weighted Least-square Estimators of LLIEW Distribution

The ordinary least squares (OLS) estimators and weighted least squares (WLS) estimators were proposed by Swain et al. (1988) to estimate the parameters of beta distribution. Given the ordered statistics $x_{(1)}, x_{(2)}, ..., x_{(n)}$ of the random sample of size n from LLIEW distribution. The OLS estimates $\alpha_{LSE}, \theta_{LSE}, \lambda_{LSE}$ and γ_{LSE} for the parameters $\Theta = (\alpha, \theta, \lambda, \gamma)^T$ of LLIEW distribution can be obtained by minimizing

$$L(x_{(i)} \mid \Theta) = \sum_{i=1}^{n} [\{1 - (\alpha^{-1}(-\log(1 - \exp(-\tfrac{\lambda}{x_{(i)}})))^{\frac{1}{\gamma}})^{\theta}\}^{-1} - \tfrac{i}{n+1}]^2, \qquad (4.45)$$

with respect to α, θ, λ and γ. Also, they can be obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n} [\{1 - (\alpha^{-1}(-\log(1 - \exp(-\tfrac{\lambda}{x_{(i)}})))^{\frac{1}{\gamma}})^{\theta}\}^{-1} - \tfrac{i}{n+1}]\Psi_1(x_{(i)} \mid \alpha, \theta, \lambda, \gamma) = 0, \qquad (4.46)$$

$$\sum_{i=1}^{n} [\{1 - (\alpha^{-1}(-\log(1 - \exp(-\tfrac{\lambda}{x_{(i)}})))^{\frac{1}{\gamma}})^{\theta}\}^{-1} - \tfrac{i}{n+1}]\Psi_2(x_{(i)} \mid \alpha, \theta, \lambda, \gamma) = 0, \qquad (4.47)$$

$$\sum_{i=1}^{n} [\{1 - (\alpha^{-1}(-\log(1 - \exp(-\tfrac{\lambda}{x_{(i)}})))^{\frac{1}{\gamma}})^{\theta}\}^{-1} - \tfrac{i}{n+1}]\Psi_3(x_{(i)} \mid \alpha, \theta, \lambda, \gamma) = 0, \qquad (4.48)$$

and

$$\sum_{i=1}^{n} [\{1 - (\alpha^{-1}(-\log(1 - \exp(-\tfrac{\lambda}{x_{(i)}})))^{\frac{1}{\gamma}})^{\theta}\}^{-1} - \tfrac{i}{n+1}]\Psi_4(x_{(i)} \mid \alpha, \theta, \lambda, \gamma) = 0, \qquad (4.49)$$

where

$$\Psi_1(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=\frac{\partial}{\partial\alpha}G_X(x_{(i)})=\frac{\theta\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}}(-\alpha^{-1}(\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}}))^{\theta-1}}{\alpha^2(1+(-\alpha^{-1}(\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}}))^{\theta})^2},\quad(4.50)$$

$$\Psi_2(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=\frac{\partial}{\partial\theta}G_X(x_{(i)})=\frac{(-\alpha^{-1}\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}})^{\theta}\log(-\alpha^{-1}\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}}}{(1+(-\alpha^{-1}\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}})^{\theta})^2},\quad(4.51)$$

$$\Psi_3(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=\frac{\partial}{\partial\lambda}G_X(x_{(i)})=\frac{(\exp(\frac{\lambda}{x_{(i)}}))\theta\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}-1}(-\alpha^{-1}\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}})^{\theta-1}}{x_{(i)}\alpha\gamma(1-\exp(\frac{\lambda}{x_{(i)}}))(1+(-\alpha^{-1}\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}})^{\theta})^2},\quad(4.52)$$

and

$$\Psi_4(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=\frac{\partial}{\partial\gamma}G_X(x_{(i)})=\frac{\theta\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}}(-\alpha^{-1}\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}})^{\theta-1}\log(\log(1-\exp(\frac{\lambda}{x_{(i)}})))}{\alpha\gamma^2(1+(-\alpha^{-1}\log(1-\exp(\frac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}})^{\theta})^2}.\quad(4.53)$$

The WLS estimates $\alpha_{LSE},\theta_{LSE},\lambda_{LSE}$ and γ_{LSE} of the LLIEW distribution parameters are obtained by minimizing the function:

$$L(x_{(i)}\,|\,\Theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left(1-\left(-\alpha^{-1}\log(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}}\right)^{\theta}\right)^{-1}-\frac{i}{n+1}\right]^2,\qquad(4.54)$$

with respect to α,θ,λ and γ. The estimates can also be obtained by solving the nonlinear equations

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\{1-(-\alpha^{-1}\log(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}})^{\theta}\}^{-1}-\frac{i}{n+1}\right]\Psi_1(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=0,\qquad(4.55)$$

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\{1-(-\alpha^{-1}\log(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}})^{\theta}\}^{-1}-\frac{i}{n+1}\right]\Psi_2(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=0, \qquad (4.56)$$

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\{1-(-\alpha^{-1}\log(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}})^{\theta}\}^{-1}-\frac{i}{n+1}\right]\Psi_3(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=0, \qquad (4.57)$$

and

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\{1-(-\alpha^{-1}\log(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{\frac{1}{\gamma}})^{\theta}\}^{-1}-\frac{i}{n+1}\right]\Psi_4(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=0, \qquad (4.58)$$

where $\Psi_1(.), \Psi_2(.), \Psi_3(.)$ and $\Psi_4(.)$ are given from equation (4.50)-(4.53).

4.5.5 Estimators Based on Percentiles of LLIEW Distribution

The percentile based estimators (PCE) was introduced by Kao (1958). The PCE are mainly obtained by minimizing the Euclidean distance between the sample percentile and population percentile points. Let $x_{(1)},\ldots,x_{(n)}$ be a sample ordered statistics and $u_i=\frac{i}{(n+1)}$ be unbiased estimator of $G(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)$. Then, the PCE of the LLIEW distribution are obtained by minimizing the function:

$$p(x_{(i)}\,|\,\Theta)=\sum_{i=1}^{n}\left(x_{(i)}-\left[-\frac{\lambda}{\log[1+\exp(\alpha[(1-u_i)^{-1}-1]^{\frac{1}{\theta}})^{\gamma}]}\right]\right)^2, \qquad (4.59)$$

with respect to α,θ,λ and γ. Minimizing equation (4.59) with respect to the parameters gives the following functions:

$$\frac{\partial}{\partial\lambda}p(x_{(i)}\,|\,\Theta)=\frac{2n}{\lambda}=0, \qquad (4.60)$$

$$\frac{\partial}{\partial\alpha}p(x_{(i)}|\Theta)=-2\sum_{i=1}^{n}\frac{\gamma\exp\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma}\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma-1}}{\left(1+\exp\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma}\right)\log\left(1+\exp\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma}\right)}=0, (4.61)$$

$$\frac{\partial}{\partial\theta}p(x_{(i)}|\Theta)=2\sum_{i=1}^{n}\frac{\exp\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma}\alpha\gamma\log\left(\frac{1}{1-u_i}-1\right)\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma-1}}{\left(1+\exp\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma}\right)\theta^{2}\log\left(1+\exp\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma}\right)}=0,\ (4.62)$$

and

$$\frac{\partial}{\partial\gamma}p(x_{(i)}|\Theta)=2\sum_{i=1}^{n}-\frac{\exp\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma}\log\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma}}{\left(1+\exp\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma}\right)\log\left(1+\exp\left(\alpha\left(\frac{1}{1-u_i}-1\right)^{\frac{1}{\theta}}\right)^{\gamma}\right)}=0.\ (4.63)$$

Equating $\frac{\partial}{\partial\lambda}p(x_{(i)}|\Theta), \frac{\partial}{\partial\alpha}p(x_{(i)}|\Theta), \frac{\partial}{\partial\theta}p(x_{(i)}|\Theta)$ and $\frac{\partial}{\partial\gamma}p(x_{(i)}|\Theta)$ to zero and solving the system of equations simultaneously gives the estimates.

4.5.6 The Cramér-von Mises Minimum Distance Estimators of LLIEW Distribution

The Cramér- von Mises estimator (CVM) is a method that is based on the difference between the estimate of the CDF and the empirical distribution function (Louzada et al., 2016). The Cramér-von Mises estimates $\alpha_{CME}, \theta_{CME}, \lambda_{CME}$ and γ_{CME} of LLIEW distribution are obtained by minimizing the function:

$$c(\alpha,\theta,\lambda,\gamma)=\frac{1}{12n}+\sum_{i=1}^{n}[\{1-(\alpha^{-1}(-\log(1-\exp(-\tfrac{\lambda}{x_{(i)}})))^{\frac{1}{\gamma}})^{\theta}\}^{-1}-\tfrac{2i-1}{2n}]^{2}, \quad (4.64)$$

with respect to α,θ,λ and γ or equivalently solving the following nonlinear equations

$$\sum_{i=1}^{n}[\{1-(\alpha^{-1}(-\log(1-\exp(-\tfrac{\lambda}{x_{(i)}})))^{\frac{1}{\gamma}})^{\theta}\}^{-1}-\tfrac{2i-1}{2n}]\Psi_{1}(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=0, \quad (4.65)$$

$$\sum_{i=1}^{n}[\{1-(\alpha^{-1}(-\log(1-\exp(-\tfrac{\lambda}{x_{(i)}})))^{\frac{1}{\gamma}})^{\theta}\}^{-1}-\tfrac{2i-1}{2n}]\Psi_{2}(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=0, \quad (4.66)$$

$$\sum_{i=1}^{n}[\{1-(\alpha^{-1}(-\log(1-\exp(-\tfrac{\lambda}{x_{(i)}})))^{\frac{1}{\gamma}})^{\theta}\}^{-1}-\tfrac{2i-1}{2n}]\Psi_{3}(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=0, \quad (4.67)$$

and

$$\sum_{i=1}^{n}[\{1-(\alpha^{-1}(-\log(1-\exp(-\tfrac{\lambda}{x_{(i)}})))^{\frac{1}{\gamma}})^{\theta}\}^{-1}-\tfrac{2i-1}{2n}]\Psi_{4}(x_{(i)}\,|\,\alpha,\theta,\lambda,\gamma)=0, \quad (4.68)$$

where $\Psi_{1}(.),\Psi_{2}(.),\Psi_{3}(.)$ and $\Psi_{4}(.)$ are provided from equation (4.50)-(4.53).

4.6 The Gumbel-IE{Logistic}(GIEL) Distribution

In this section, detailed results of GIEL distribution are provided.

If $T\sim\text{Gumbel}(0,1)$ with CDF $G_{T}(x)=\exp(-\exp(-x))$ and PDF $g_{T}(x)=\exp[-x-\exp(-x)]$, then using the CDF of T-IE{Logistic} family of distributions, the CDF of the GIEL distribution is given as

$$G_{X}(x)=\exp[-(\exp(\lambda/x)-1)^{\frac{1}{\alpha}}], x>0, \quad (4.69)$$

where $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. A random variable in equation (4.69) is denoted by X ~GIEL(α, λ).

The plots of CDF of GIEL distribution are presented in Figure 4.5. When the values of both the shape and the scale parameters are less than or equal to one, the curve approaches maximum faster than when both values are more than one.

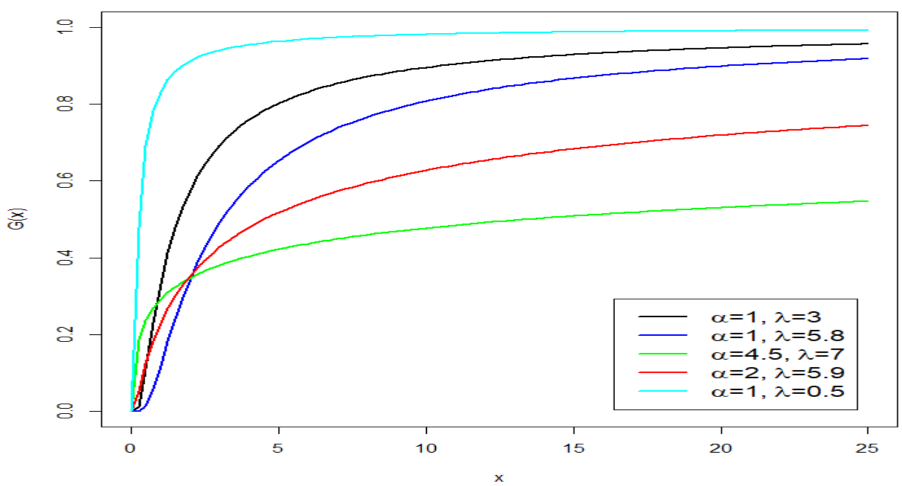

Figure 4.5: CDF of GIEL distribution

The corresponding PDF of GIEL distribution can be expressed as:

$$g_X(x) = \frac{\lambda \exp(\lambda/x)}{\alpha x^2}(\exp(\lambda/x) - 1)^{\frac{1}{\alpha}-1} \exp[-(\exp(\lambda/x) - 1)^{\frac{1}{\alpha}}], x > 0, \alpha > 0, \lambda > 0. \quad (4.70)$$

At chosen parameter values, the PDF is presented in Figure 4.6. The PDF can be positively skewed and reverse J-shaped. Also, the GIEL distribution is unimodal. When the value of the shape parameter is more than the value of the scale parameter, the distribution has reversed-J shape. However, when the value of the shape parameter is less than the value of the scale parameter, the distribution is positively skewed shape.

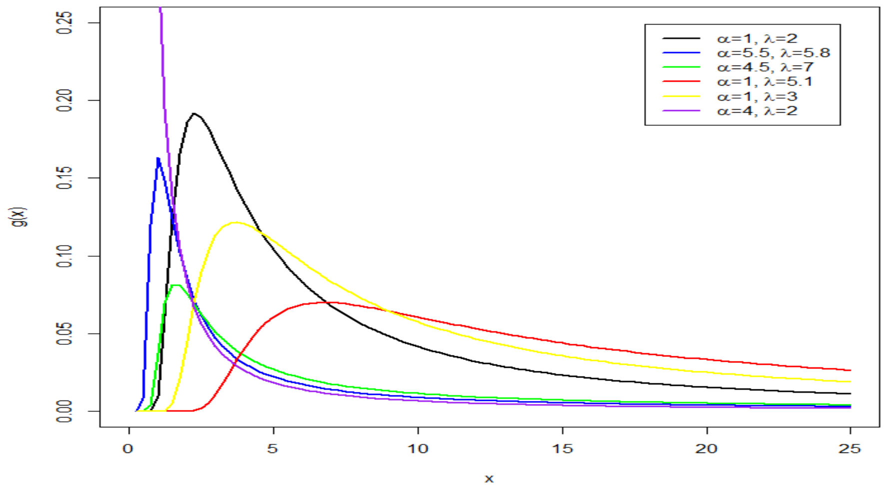

Figure 4.6: PDF of GIEL distribution

The survival function for GIEL distribution can be expressed as

$$S_X(x)=1\text{-}\exp[-(\exp(\lambda/x)-1)^{\frac{1}{\alpha}}], x>0. \tag{4.71}$$

The survival function of GIEL distribution is graphically shown in Figure 4.7 at selected parameter values. When the values of both the shape and scale parameters are less than or equal to one, the curve sharply approaches zero. When the values of both are more than one, the curve slowly approaches zero.

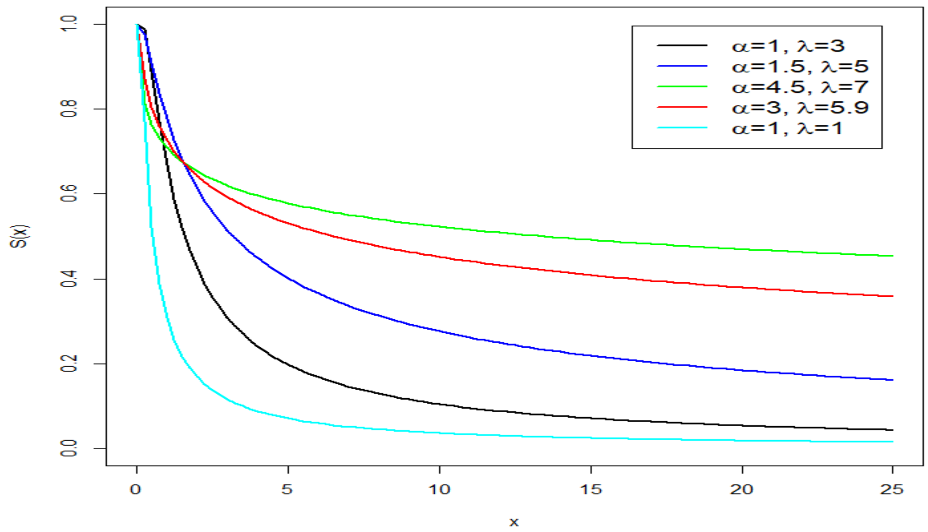

Figure 4.7: Survival function of GIEL distribution

The hazard rate function of GIEL distribution can be expressed as

$$h_X(x) = \frac{\lambda \exp(\lambda/x)\{\exp(\lambda/x)-1\}^{\frac{1}{\alpha}-1} \exp\{-[\exp(\lambda/x)-1]^{\frac{1}{\alpha}}\}}{\alpha x^2 (1-(\exp\{-\{\exp(\lambda/x)-1\}^{\frac{1}{\alpha}}\}))}, x > 0. \qquad (4.72)$$

The shape of hazard rate function of GIEL distribution was determined graphically at a selected parameter values. For brevity purpose, some values were selected and the plots presented in Figure 4.8. The hazard rate function of the GIEL distribution shows decreasing, increasing and inverted bathtub shape. When $\lambda \le 1$ and $\alpha > 1$, the hazard rate function is monotonically decreasing. Also, when value of the scale parameter is more than one and the shape parameter is less than or equal to one, the hazard rate function increases for some time and either remains constant or decreases.

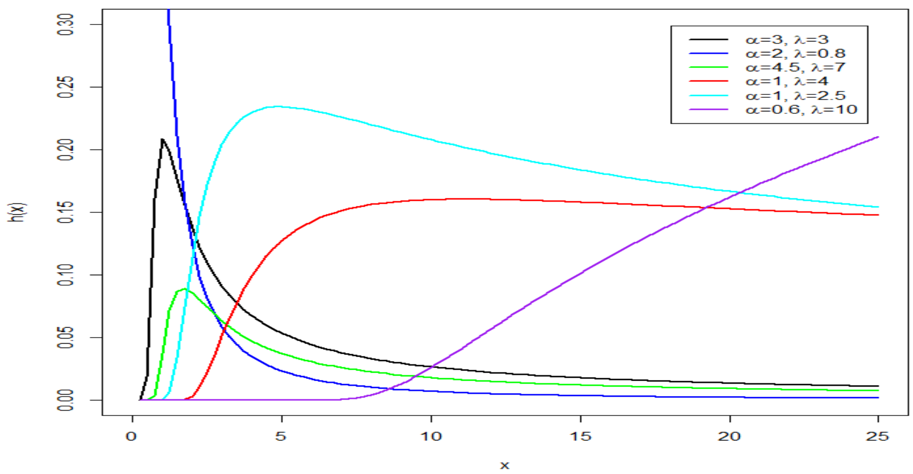

Figure 4.8: Hazard rate function of GIEL distribution.

In order to simulate random samples from GIEL distribution, it is necessary to develop its quantile function. The quantile function of GIEL distribution is given by

$$Q_X(u) = \frac{\lambda}{\log[1-\alpha\log u]}, u \in (0,1). \tag{4.73}$$

The median, lower quartile and upper quartiles can be obtained by substituting $u = 0.5, u = 0.25$ and $u = 0.75$ respectively.

4.6.1 Methods of Parameter Estimation of GIEL Distribution

This section presents five (5) parameter estimation procedures for estimating the parameters of GIEL distribution. These estimators are: Maximum Likelihood estimators, Ordinary Least Square estimators, Weighted Least Square estimators, Percentile based estimators and Cramér-von Mises.

4.6.1.1 Maximum Likelihood Estimation of GIEL Distribution for Complete Sample

In this section, we estimated the parameters $\Theta = (\lambda, \alpha)^T$, using the method of maximum likelihood estimation. Let $x_1, x_2, ..., x_n$ be a random sample of size n from a GIEL distribution. The total log-likelihood function is given by

$$\ell = n\log(\tfrac{\lambda}{\alpha}) + \sum_{i=1}^{n}\tfrac{\lambda}{x_i} - 2\sum_{i=1}^{n}\log x_i + (\tfrac{1}{\alpha}-1)\sum_{i=1}^{n}\log(\exp(\tfrac{\lambda}{x_i})-1) - \sum_{i=1}^{n}(\exp(\tfrac{\lambda}{x_i})-1)^{\frac{1}{\alpha}}. \quad (4.74)$$

Differentiating equation (4.74) with respect to the parameters λ and α respectively gives;

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\frac{1}{x_i} + (\tfrac{1}{\alpha}-1)\sum_{i=1}^{n}\frac{\exp(\frac{\lambda}{x_i})}{(\exp(\frac{\lambda}{x_i})-1)x_i} - \sum_{i=1}^{n}\frac{\exp(\frac{\lambda}{x_i})(\exp(\frac{\lambda}{x_i})-1)^{\frac{1}{\alpha}-1}}{\alpha x_i}, \quad (4.75)$$

and

$$\frac{\partial \ell}{\partial \alpha} = -\frac{n}{\alpha} - \sum_{i=1}^{n}\frac{\log(\exp(\frac{\lambda}{x_i})-1)}{\alpha^2} + \sum_{i=1}^{n}\frac{(\exp(\frac{\lambda}{x_i})-1)^{\frac{1}{\alpha}}\log(\exp(\frac{\lambda}{x_i})-1)}{\alpha^2}. \quad (4.76)$$

Equating $\frac{\partial \ell}{\partial \lambda}$ and $\frac{\partial \ell}{\partial \lambda}$ to zero and solving for the parameters λ and α in the system of equations yields the maximum likelihood estimates of the parameters.

To determine the confidence intervals for the parameters of GIEL distribution, we use the observed information matrix $\Psi(\Theta)$ given by

$$\Psi(\Theta) = -\begin{bmatrix} \frac{\partial^2 l}{\partial \lambda^2} & \frac{\partial^2 l}{\partial \lambda \partial \alpha} \\ & \frac{\partial^2 l}{\partial \alpha^2} \end{bmatrix}.$$

The elements of $\Psi(\Theta)$ are given in appendix A2. The multivariate normal $N_2(0, I^{-1}(\Theta))$, where $I(\Theta)$ is the expected information matrix can be employed to construct approximate confidence interval for the parameters under the regularity conditions. When $I(\Theta)$ is replaced by the observed information matrix evaluated at $\Psi(\hat{\Theta})$, the asymptotic behaviour remains the valid. The $100(1-\eta)\%$ asymptotic confidence interval (CI) for each parameter Θ_η is given by:

$$CI = (\Theta_\eta - z_{\eta/2} \times \sigma_{\Theta_\eta}, \Theta_\eta + z_{\eta/2} \times \sigma_{\Theta_\eta}),$$

where σ_{Θ_η} is the standard error of the estimated parameter and $z_{\eta/2}$ is the upper quantile of the standard normal distribution.

4.6.1.2 Maximum likelihood Estimation of the GIEL Distribution for Incomplete Samples

Suppose we observed the first r failed items $x_1, x_2, \ldots, x_r$. Let $z_i = \exp(\frac{\lambda}{x_i}) - 1$, then, the total log-likelihood function for GIEL distribution with right censored data is given by

$$\ell = \sum_{i=1}^{n} r_i \times \log\left[\frac{\lambda \exp(\frac{\lambda}{x_i})}{\alpha \times x_i^2} \times (z_i)^{\frac{1}{\alpha}-1} \times \exp(-z_i)^{\frac{1}{\alpha}}\right] + \sum_{i=1}^{n} (1-r_i) \times \log(1-\exp(-z_i)^{\frac{1}{\alpha}}). \quad (4.77)$$

Differentiating ℓ with respect to α and λ gives

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n} \frac{\exp(-z_i)^{\frac{1}{\alpha}} (-z_i)^{\frac{1}{\alpha}} \log(-z_i)^{\frac{1}{\alpha}} (1-r_i)}{(1-\exp(-z_i)^{\frac{1}{\alpha}})\alpha^2} - \sum_{i=1}^{n} \lambda^{-1} \alpha r_i (z_i)^{1-\frac{1}{\alpha}} \left\{ \frac{\lambda (z_i)^{\frac{1}{\alpha}-1} \left(\exp(\frac{\lambda}{x_i}) + (-z_i)^{\frac{1}{\alpha}} \right)}{\alpha^2 x_i^2} \right\} \quad (4.78)$$

$$- \sum_{i=1}^{n} \lambda^{-1} \alpha r_i (z_i)^{1-\frac{1}{\alpha}} \left\{ \frac{\lambda \log(z_i)(z_i)^{\frac{1}{\alpha}-1} \left(\exp(\frac{\lambda}{x_i}) + (-z_i)^{\frac{1}{\alpha}} \right)}{\alpha^3 x_i^2} + \frac{\lambda \log(-z_i)(-z_i)^{\frac{1}{\alpha}} (z_i)^{\frac{1}{\alpha}-1} \left(\exp(\frac{\lambda}{x_i}) + (-z_i)^{\frac{1}{\alpha}} \right)}{\alpha^3 x_i^2} \right\},$$

and

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \lambda^{-1} \alpha r_i x_i^2 (z_i)^{1-\frac{1}{\alpha}} \left(\exp\left((-\tfrac{\lambda}{x_i}) - (-z_i)^{\frac{1}{\alpha}} \right) \right) \left(\frac{\lambda (z_i)^{\frac{1}{\alpha}-1} \exp\left((\tfrac{\lambda}{x_i}) + (-z_i)^{\frac{1}{\alpha}} \right)}{\alpha x_i^2} + \frac{(z_i)^{\frac{1}{\alpha}-1} \exp\left((\tfrac{\lambda}{x_i}) + (-z_i)^{\frac{1}{\alpha}} \right)}{\alpha x_i^2} \right). \quad (4.79)$$

Equating equations (4.77), (4.78) and (4.79) to zero and solving simultaneously results in the estimates.

4.6.1.3 Ordinary and Weighted Least Square Estimators of GIEL Distribution

The OLS estimates α_{OLS} and λ_{OLS}, can be obtained by minimizing the function:

$$L(x_{(i)} \mid \Theta) = \sum_{i=1}^{n} [\exp[-(\exp(\lambda/x_{(i)}) - 1)^{\frac{1}{\alpha}}] - \tfrac{i}{n+1}]^2, \quad (4.80)$$

with respect to λ and α. Where $x_{(i)}$ is the ordered statistics of the random sample of size n.

Also, the estimates can be obtained by solving the following non-linear equations

$$\sum_{i=1}^{n} [\exp[-(\exp(\lambda/x_{(i)}) - 1)^{\frac{1}{\alpha}}] - \tfrac{i}{n+1}] \Psi_1(x_{(i)} \mid \lambda, \alpha) = 0, \quad (4.81)$$

and

$$\sum_{i=1}^{n} [\exp[-(\exp(\lambda/x_{(i)}) - 1)^{\frac{1}{\alpha}}] - \tfrac{i}{n+1}] \Psi_2(x_{(i)} \mid \lambda, \alpha) = 0, \quad (4.82)$$

where $\Psi_1(x_{(i)} \mid \lambda, \alpha) = \frac{\partial}{\partial \lambda} G_X(x_{(i)}) = -\frac{\exp(\frac{\lambda}{x_{(i)}}) \left(\exp(\frac{\lambda}{x_{(i)}}) - 1 \right)^{\frac{1}{\alpha}-1}}{\alpha x^2_{(i)}} \exp(-(\exp(\tfrac{\lambda}{x_{(i)}}) - 1)^{\frac{1}{\alpha}}), \quad (4.83)$

and

$$\Psi_2(x_{(i)} \mid \lambda, \alpha) = \frac{\partial}{\partial \alpha} G_X(x_{(i)}) = -(\exp(\tfrac{\lambda}{x_{(i)}}) - 1)^{\frac{1}{\alpha}} \log(\exp(\tfrac{\lambda}{x_{(i)}}) - 1) \exp(-(\exp(\tfrac{\lambda}{x_{(i)}}) - 1)^{\frac{1}{\alpha}}). \quad (4.84)$$

The WLS estimates can be obtained by minimizing the following function

$$L(x_{(i)}\,|\,\Theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\exp[-(\exp(\lambda/x_{(i)})-1)^{\frac{1}{\alpha}}-\frac{i}{n+1}\right]^2, \quad (4.85)$$

with respect to α and λ. Equivalently, the estimates can be obtained by solving the following non-linear functions

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\exp[-(\exp(\lambda/x_{(i)})-1)^{\frac{1}{\alpha}}-\frac{i}{n+1}\right]\Psi_1(x_{(i)}\,|\,\Theta)=0, \quad (4.86)$$

And

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\exp[-(\exp(\lambda/x_{(i)})-1)^{\frac{1}{\alpha}}-\frac{i}{n+1}\right]\Psi_2(x_{(i)}\,|\,\Theta)=0, \quad (4.87)$$

where, $\Psi_1(x_{(i)}\,|\,\Theta)$ and $\Psi_2(x_{(i)}\,|\,\Theta)$ are defined in equation (4.83) and (4.84) respectively.

4.6.1.4 Percentile Based Estimators of GIEL Distribution

As stated earlier, PCE was introduced by Kao (1958) and is mainly obtained by minimizing the Euclidean distance between the sample percentile and population percentile points. Again, assume $x_{(1)},\ldots,x_{(n)}$ to be a sample ordered statistics and $u_i=\frac{i}{(n+1)}$ to be unbiased estimator of $G(x_{(i)}\,|\,\alpha,\lambda)$. Then, the PCE of the parameters of GIEL distribution are obtained by minimizing the function:

$$p(\lambda,\alpha)=\sum_{i=1}^{n}\left(x_{(i)}-\left[\frac{\lambda}{\log[1-\alpha\log u_i]}\right]\right)^2, \quad (4.88)$$

with respect to λ and α. Minimizing equation (4.88) with respect to λ and α gives

$$\frac{\partial}{\partial \lambda} p(\lambda, \alpha) = -\frac{2n}{\lambda} = 0, \quad (4.89)$$

and

$$\frac{\partial}{\partial \alpha} p(\lambda, \alpha) = 2\sum_{i=1}^{n} -\frac{\log u_i}{\log[1-\alpha \log u_i](1-\alpha \log u_i)} = 0, (4.90)$$

respectively.

4.6.1.5 Cramér-von Mises Estimators of GIEL Distribution

The Cramer-von Mises estimators λ_{CME} and α_{CME} of GIEL distribution are obtained by minimizing the function:

$$c(\alpha, \lambda) = \tfrac{1}{12n} + \sum_{i=1}^{n} [\exp[-(\exp(\lambda / x_{(i)}) - 1)^{\frac{1}{\alpha}}] - \tfrac{2i-1}{2n}]^2, \quad (4.91)$$

with respect to α and λ or equivalently solving the following nonlinear equations

$$\sum_{i=1}^{n} [\exp - (\exp(\lambda / x_{(i)}) - 1)^{\frac{1}{\alpha}} - \tfrac{2i-1}{2n}] \Psi_1(x_{(i)} \mid \lambda, \alpha) = 0, \quad (4.92)$$

and

$$\sum_{i=1}^{n} [\exp - (\exp(\lambda / x_{(i)}) - 1)^{\frac{1}{\alpha}} - \tfrac{2i-1}{2n}] \Psi_2(x_{(i)} \mid \lambda, \alpha) = 0, \quad (4.93)$$

where $\Psi_1(x_{(i)} \mid \lambda, \alpha)$, and $\Psi_2(x_{(i)} \mid \lambda, \alpha)$ are given in equations (4.83) and (4.84).

4.7 Weibull-IE{Lomax} Distribution

Let T distributed as Weibull (α,β) distribution with CDF $G_T(x)=1-\exp(-\alpha x^{\beta})$ and PDF $g_T(x)=\alpha\beta x^{\beta-1}\exp(-\alpha x^{\beta})$. Using the CDF of T-IE{Lomax} family of distributions, then the CDF of WIEL distribution is given as

$$G_X(x)=1-\exp\{-\alpha[(1-\exp(-\lambda/x))^{-\frac{1}{\gamma}}-1]^{\beta}\}, x>0, \tag{4.94}$$

where $\alpha>0,\beta>0,\gamma>0,\lambda>0.$ The shape parameters are β and γ while the scale parameters are α and λ. Figure 4.9 shows the plots of the CDF of WIEL distribution with varying parameter values. When the values of all the parameters are less than or equal to one, the curve approaches the maximum faster than when the values are more than one. Also, when the value of alpha is less than or equal to one and the other three values are more than one, the curve takes longer time to reach maximum.

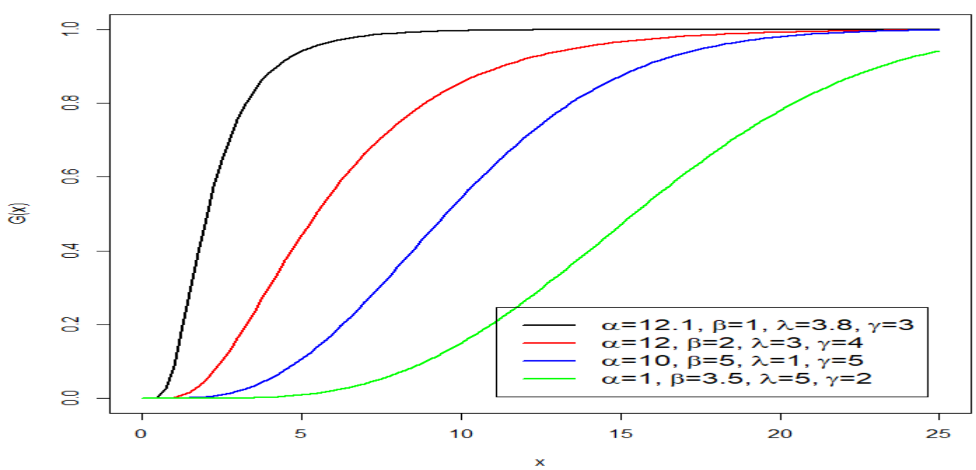

Figure 4.9: CDF of WIEL distribution

The corresponding PDF is given by

$$g_X(x)=\frac{\alpha\beta\lambda\exp(-\lambda/x)}{\gamma x^2[1-\exp(-\lambda/x)]^{\frac{1}{\gamma}-1}}[\{1-\exp(-\lambda/x)\}^{-\frac{1}{\gamma}}-1]^{\beta-1}\exp[-\alpha\{(1-\exp(-\lambda/x)^{-\frac{1}{\gamma}}-1\}^{\beta}], x>0.(4.95)$$

The PDF of WIEL distribution can be positively skewed, approximately symmetric and unimodal with varying degree of kurtosis. Figure 4.10 displays the various shapes of the PDF of WIEL distribution. When the values of alpha and gamma are less than or equal to one, the distribution is approximately symmetric with varying degree of kurtosis. Also, when the value of alpha is less than or equal to one and the rest are more than one, the distribution is positively skewed.

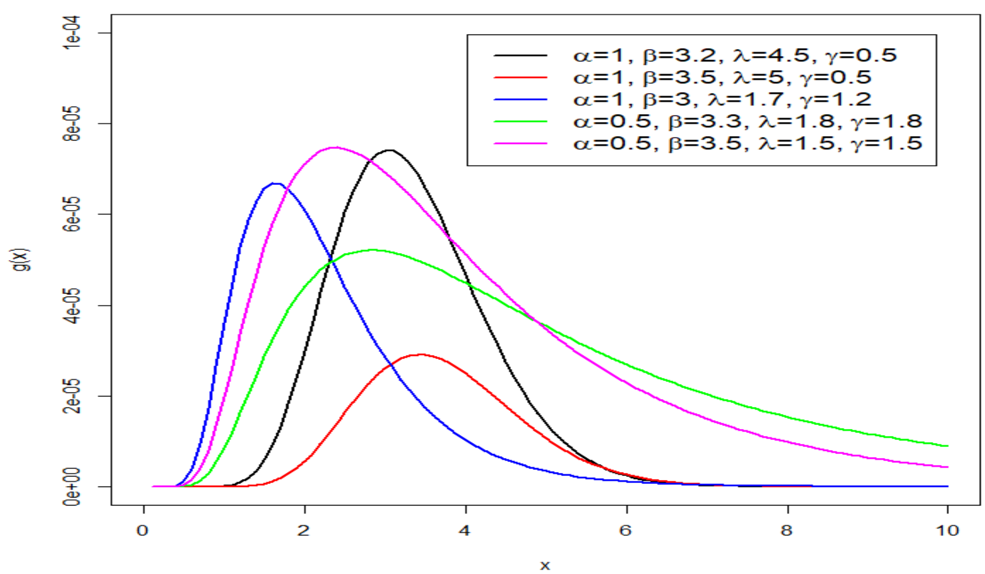

Figure 4.10: PDF of WIEL distribution

The Survival function is given by

$$S_X(x) = \exp\{-\alpha[[1-\exp(-\tfrac{\lambda}{x})]^{-\frac{1}{\gamma}}]-1\}^{\beta}, x > 0. \qquad (4.96)$$

Figure 4.11 shows the survival function of WIEL distribution. When the values of alpha is less than or equal to one and the others are more than one, the survival curve takes longer time to get to zero. Also, when all the values are more than one, the curve approaches zero faster.

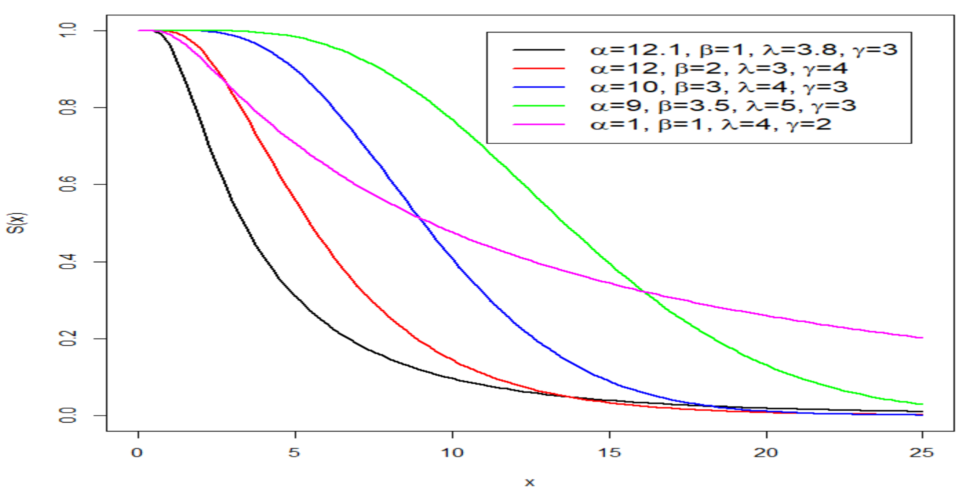

Figure 4.11: Survival function of WIEL distribution

The hazard rate function is given by

$$h_X(x) = \frac{\alpha\beta\lambda\exp(-\lambda/x)}{\gamma x^2 (1-\exp(-\lambda/x))^{\frac{1}{\gamma}-1}} (\{1-\exp(-\lambda/x)\}^{-\frac{1}{\gamma}}-1)^{\beta-1}, x > 0. \qquad (4.97)$$

Figure 4.12 shows the graphical presentation of the hazard rate function of WIEL distribution. It can be observed that, with varying parameter values, the hazard rate function has inverted bathtub and decreasing shape. When lambda value is less than one, the hazard rate function is decreasing and when all the parameter values are equal to or more than one, the distribution has an inverted hazard rate function.

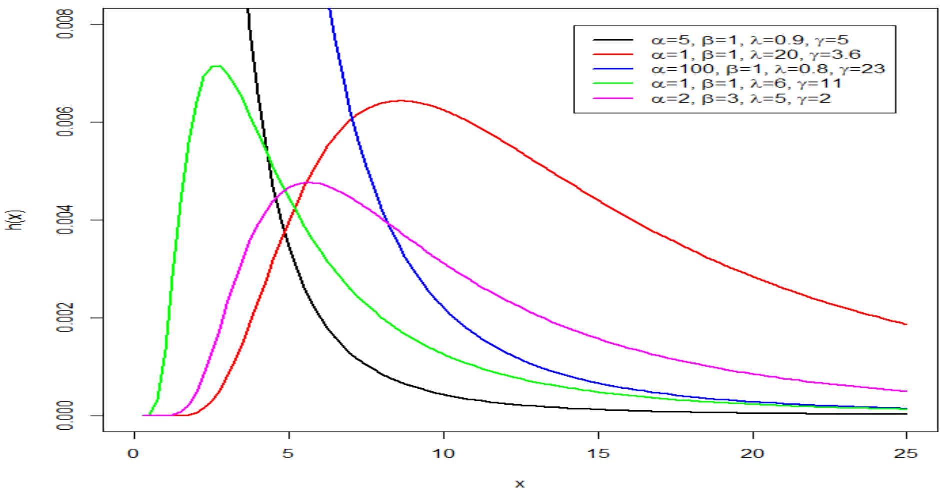

Figure 4.12: Hazard rate function of WIEL distribution

It is important to develop the quantile function in order to simulate random samples from WIEL distribution. The quantile function of WIEL distribution is given as

$$Q_X(u) = -\frac{\lambda}{\log(1+\{\alpha^{-1}(1+\log\left(1-u\right)^{\frac{1}{\beta}})\}^{\gamma})}, u \in (0,1). \tag{4.98}$$

Substituting $u = 0.5, u = 0.25$ and $u = 0.75$ gives median, first quartile and upper quartile of WIEL distribution respectively.

4.7.1 Methods of Parameter Estimation for WIEL Distribution

This section presents five (5) parameter estimation procedures for estimating the parameters of WIEL distribution. These estimators are: Maximum Likelihood estimators, Ordinary Least Square estimators, Weighted Least Square estimators, Percentile based estimators and Cramer-von Mises.

4.7.1.1 Maximum Likelihood Estimation Method for Complete Sample

In this section, we estimated the parameters $\Theta = (\lambda, \alpha, \beta, \gamma)^T$, using the method of maximum likelihood estimation. Let $x_1, x_2, ...x_n$ be a random sample of size n independently and identically random variables each distributed according to WIEL distribution. Let $z_i = 1 - \exp\left(-\frac{\lambda}{x_i}\right)$. The total log-likelihood function is given by

$$\ell = n\log\lambda + n\log\alpha + n\log\beta - n\log\gamma - 2\sum_{i=1}^{n} x_i - \lambda\sum_{i=1}^{n}\tfrac{1}{x_i} + (1\text{-}\tfrac{1}{\gamma})\sum_{i=1}^{n}\log(z_i) + (\beta-1)\sum_{i=1}^{n}\log((z_i)^{-\frac{1}{\gamma}} - 1) + \sum_{i=1}^{n}(-\alpha(z_i)^{-\frac{1}{\gamma}} - 1)^{\beta}. \tag{4.99}$$

Differentiating ℓ with respect to the parameters λ, α, β, and γ gives the following score functions;

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\frac{1}{x_i}, \tag{4.100}$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} -\beta(z_i)^{-\frac{1}{\gamma}}(-\alpha(z_i)^{-\frac{1}{\gamma}} - 1)^{\beta-1}, \tag{4.101}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log((z_i)^{-\frac{1}{\gamma}} - 1) + \sum_{i=1}^{n}\log(-\alpha(z_i)^{-\frac{1}{\gamma}} - 1)(-\alpha(z_i)^{-\frac{1}{\gamma}} - 1)^{\beta}, \tag{4.102}$$

and

$$\frac{\partial\ell}{\partial\gamma} = \frac{\sum_{i=1}^{n}\log(z_i)}{\gamma^2} + (\beta-1)\sum_{i=1}^{n}\frac{\log(z_i)(z_i)^{-\frac{1}{\gamma}}}{((z_i)^{-\frac{1}{\gamma}} - 1)\gamma^2} + \frac{\alpha\beta\log(z_i)(z_i)^{-\frac{1}{\gamma}}(-\alpha(z_i)^{-\frac{1}{\gamma}} - 1)^{\beta-1}}{\gamma^2} - n\log\gamma. \tag{4.103}$$

Equating $\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}$ and $\frac{\partial \ell}{\partial \gamma}$ to zero and solving the system of equations numerically for the parameters will give the maximum likelihood estimates. Determining the confidence intervals for the parameters of WIEL distribution, the following observed information matrix $\Psi(\Theta)$ was used;

$$\Psi(\Theta) = -\begin{bmatrix} \frac{\partial^2 l}{\partial \lambda^2} & \frac{\partial^2 l}{\partial \lambda \partial \alpha} & \frac{\partial^2 l}{\partial \lambda \partial \beta} & \frac{\partial^2 l}{\partial \lambda \partial \gamma} \\ & \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \alpha \partial \gamma} \\ & & \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \gamma} \\ & & & \frac{\partial^2 l}{\partial \gamma^2} \end{bmatrix}.$$

The elements of $\Psi(\Theta)$ are given in appendix A3. The multivariate normal $N_4(0, I^{-1}(\Theta))$, where $I(\Theta)$ is the expected information matrix can be employed to construct approximate confidence interval for the parameters under the regularity conditions. When $I(\Theta)$ is replaced by the observed information matrix evaluated at $\Psi(\hat{\Theta})$, the asymptotic behaviour remains the same. The $100(1-\eta)\%$ asymptotic confidence interval (CI) for each parameter Θ_η is given by:

$$CI = (\Theta_\eta - z_{\eta/2} \times \sigma_{\Theta_\eta}, \Theta_\eta + z_{\eta/2} \times \sigma_{\Theta_\eta}),$$

where σ_{Θ_η} is the standard error of the estimated parameter and $z_{\eta/2}$ is the upper quantile of the standard normal distribution.

4.7.1.2 Maximum Likelihood Estimation Method for Censored Data

Suppose we observed the first r failed items $x_1, x_2, ..., x_r$. Then, the likelihood function for GIEL distribution with right censored data is given by

$$\ell = \sum_{i=1}^{n} r_i \log\left(\frac{\alpha\beta\lambda \exp(-\frac{\lambda}{x_i})}{\gamma x_i} \{(z_i)^{-\frac{1}{\gamma}} - 1\}^{\beta} \right) + \sum_{i=1}^{n} (1-r_i) \log[\exp\{-\alpha\{(z_i)^{-\frac{1}{\gamma}} - 1\}^{\beta}]. (4.104)$$

Differentiating ℓ with respect to the parameters give the following functions:

$$\frac{\partial \ell}{\partial \lambda} = (\alpha\beta\lambda)^{-1}\gamma \mathrm{r}_i x_i \exp(\tfrac{\lambda}{x_i})\left(-\frac{\exp(-\frac{\lambda}{x_i})\alpha\beta\lambda\left((z_i)^{-\frac{1}{\gamma}}-1\right)^{\beta}}{\gamma x_i^2} + \frac{\exp(-\frac{\lambda}{x_i})\alpha\beta\left((z_i)^{-\frac{1}{\gamma}}-1\right)^{\beta}}{\gamma x_i} \right), \quad (4.105)$$

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n}\frac{r_i}{\alpha} + \sum_{i=1}^{n} -\beta(1-r_i)(z_i)^{-\frac{1}{\gamma}}\left(-1-\alpha(z_i)^{-\frac{1}{\gamma}}\right)^{\beta-1}, \quad (4.106)$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{n}\log\left(-1-\alpha(z_i)^{-\frac{1}{\gamma}}\right)(1\text{-r}_i)\left(-1-\alpha(z_i)^{-\frac{1}{\gamma}}\right)^{\beta} + \sum_{i=1}^{n}(\alpha\beta\lambda)^{-1}\gamma r_i x_i \exp(\tfrac{\lambda}{x_i})\left(-1+\alpha(z_i)^{-\frac{1}{\gamma}}\right)^{-\beta}$$
$$\times\left(\frac{\alpha\lambda\exp(-\frac{\lambda}{x_i})\left(-1+\alpha(z_i)^{-\frac{1}{\gamma}}\right)^{\beta}}{\gamma x_i} + \frac{\alpha\beta\lambda\exp(-\frac{\lambda}{x_i})\log\left(-1+\alpha(z_i)^{-\frac{1}{\gamma}}\right)\left(-1+\alpha(z_i)^{-\frac{1}{\gamma}}\right)^{\beta}}{\gamma x_i} \right), \quad (4.107)$$

and

$$\frac{\partial \ell}{\partial \gamma} = \sum_{i=1}^{n} -\alpha\beta\gamma^{-2}\log(z_i)(1-r_i)(z_i)^{-\frac{1}{\gamma}}\left(-1-\alpha(z_i)^{-\frac{1}{\gamma}}\right)^{\beta-1} + \sum_{i=1}^{n}(\alpha\beta\lambda)^{-1}\gamma r_i x_i\left(-1+(z_i)^{-\frac{1}{\gamma}}\right)^{-\beta}\times$$
$$\left(\frac{\alpha\beta^2\lambda\exp(-\frac{\lambda}{x_i})\log(z_i)(z_i)^{-\frac{1}{\gamma}}\left(-1+(z_i)^{-\frac{1}{\gamma}}\right)^{\beta-1}}{\gamma^3 x_i} - \frac{\alpha\beta\lambda\exp(-\frac{\lambda}{x_i})\left(-1+(z_i)^{-\frac{1}{\gamma}}\right)^{\beta}}{\gamma^2 x_i} \right). \quad (4.108)$$

Equating the functions to zero and solving yields the desired results.

4.7.1.3 Ordinary Least Square Estimators and Weighted Least Square Estimators

The OLS estimates $\lambda_{OLS}, \alpha_{OLS}, \beta_{OLS}$ and γ_{OLS}, can be obtained by minimizing the function:

$$L(x_{(i)} \mid \Theta) = \sum_{i=1}^{n}[1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]^{\beta}\} - \tfrac{i}{n+1}]^2. \quad (4.109)$$

With respect to λ, α, β and γ where $x_{(i)}$ is the ordered statistics of a random sample of size n.

Also, the estimates can be obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}[1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]^{\beta}\}-\tfrac{i}{n+1}]\Psi_1(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)=0, \qquad (4.110)$$

$$\sum_{i=1}^{n}[1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]^{\beta}\}-\tfrac{i}{n+1}]\Psi_2(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)=0, \qquad (4.111)$$

$$\sum_{i=1}^{n}[1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]^{\beta}\}-\tfrac{i}{n+1}]\Psi_3(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)=0, \qquad (4.112)$$

and

$$\sum_{i=1}^{n}[1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]^{\beta}\}-\tfrac{i}{n+1}]\Psi_4(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)=0, \qquad (4.113)$$

where $\Psi_{.}(x_{(.)}\,|\,\Theta)=\dfrac{\partial}{\partial\Theta}G_X(x_{(i)})$ and

$$\frac{\partial}{\partial\alpha}G_X(x_{(i)})=1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]^{\beta}\}[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]^{\beta}, \qquad (4.114)$$

$$\frac{\partial}{\partial\beta}G_X(x_{(i)})=\alpha[(1-\exp(\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]^{\beta}\log[(1-\exp(\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]\exp\{-\alpha[(1-\exp(\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]^{\beta}\}, \qquad (4.115)$$

$$\frac{\partial}{\partial\gamma}G_X(x_{(i)})=\alpha\beta\left(1-\exp(\tfrac{\lambda}{x_{(i)}})\right)^{-\frac{1}{\gamma}}\left[\left(1-\exp(\tfrac{\lambda}{x_{(i)}})\right)^{-\frac{1}{\gamma}}-1\right]^{\beta-1}\times$$
$$\log\left(1-\exp(-\tfrac{\lambda}{x_{(i)}})\right)\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}-1]^{\beta}\}, \qquad (4.116)$$

and

$$\frac{\partial}{\partial\lambda}G_X(x_{(i)})=\frac{\beta\exp(\frac{\lambda}{x_{(i)}})\left(1-\exp(\frac{\lambda}{x_{(i)}})\right)^{-1-\frac{1}{\gamma}}\left(\left(1-\exp(\frac{\lambda}{x_{(i)}})\right)^{-\frac{1}{\gamma}}-1\right)^{\beta-1}}{x\gamma}\exp\{-\alpha[(1-\exp(\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}}]^{\beta}\}.(4.117)$$

The WLS estimates (Swain et al., 1988), $\lambda_{WLS},\alpha_{WLS},\beta_{WLS}$ and γ_{WLS}, can be obtained by minimizing the function:

$$w(x_{(i)}\,|\,\Theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}[1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_i}))^{-\frac{1}{\gamma}}-1]^{\beta}\}-\tfrac{i}{n+1}]^2. \qquad (4.118)$$

Equivalently, the estimates can be obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}[1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_i}))^{-\frac{1}{\gamma}}-1]^{\beta}\}-\tfrac{i}{n+1}]\Psi_1(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)=0, \ (4.119)$$

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}[1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_i}))^{-\frac{1}{\gamma}}-1]^{\beta}\}-\tfrac{i}{n+1}]\Psi_2(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)=0, \ (4.120)$$

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}[1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_i}))^{-\frac{1}{\gamma}}-1]^{\beta}\}-\tfrac{i}{n+1}]\Psi_3(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)=0, \ (4.121)$$

and

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}[1-\exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_i}))^{-\frac{1}{\gamma}}-1]^{\beta}\}-\tfrac{i}{n+1}]\Psi_4(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)=0. \ (4.122)$$

where $\Psi_1(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)$, $\Psi_2(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)$, $\Psi_3(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)$ and $\Psi_3(x_{(i)}\,|\,\lambda,\alpha,\beta,\gamma)$ are from equation (4.114)-(4.117).

4.7.1.4 Percentile Based Estimators

Let $x_{(1)},\dots,x_{(n)}$ be a sample ordered statistics, $\Theta=(\alpha,\beta,\lambda,\gamma)^T$ and $u_i=\frac{i}{(n+1)}$ be unbiased estimator of $G(x_{(i)}\,|\,\alpha,\beta,\lambda,\gamma)$. Then, the PCE of the parameters of WIEL distribution are obtained by minimizing the function:

$$p(\Theta)=\sum_{i=1}^{n}\left(x_{(i)}-\left[-\frac{\lambda}{\ln(1+\{\alpha^{-1}(1+\ln\left(1-u\right)^{\frac{1}{\beta}})\}^{\gamma})}\right]\right)^{2}, \qquad (4.123)$$

with respect to α,β,λ and γ. Minimizing equation (4.116) gives the following functions:

$$\frac{\partial}{\partial\alpha}p(\Theta)=2\sum_{i=1}^{n}\frac{\gamma\left(1+\log(1-u_i)^{\frac{1}{\beta}}\right)\left(\alpha^{-1}\left(1+\log(1-u_i)^{\frac{1}{\beta}}\right)\right)^{\gamma-1}}{\alpha^{2}\log\left[1+\left(\alpha^{-1}\left(1+\log(1-u_i)^{\frac{1}{\beta}}\right)\right)^{\gamma}\right]\left[1+\left(\alpha^{-1}\left(1+\log(1-u_i)^{\frac{1}{\beta}}\right)\right)^{\gamma}\right]}=0, (4.124)$$

$$\frac{\partial}{\partial\beta}p(\Theta)=2\sum_{i=1}^{n}\frac{\gamma\log(\log(1-u_i))\log(1-u_i)^{\frac{1}{\beta}}\left(\alpha^{-1}(1+\log(1-u_i)^{\frac{1}{\beta}})\right)^{\gamma-1}}{\alpha\beta^{2}\log\left(1+\left(\alpha^{-1}(1+\log(1-u_i)^{\frac{1}{\beta}})\right)^{\gamma}\right)\left(1+\left(\alpha^{-1}(1+\log(1-u_i)^{\frac{1}{\beta}})\right)^{\gamma}\right)}=0, (4.125)$$

$$\frac{\partial}{\partial\gamma}p(\Theta)=-2\sum_{i=1}^{n}\frac{\log\left(\alpha^{-1}(1+\log(1-u_i)^{\frac{1}{\beta}})\right)\left(\alpha^{-1}(1+\log(1-u_i)^{\frac{1}{\beta}})\right)^{\gamma}}{\log\left[1+\left(\alpha^{-1}(1+\log(1-u_i)^{\frac{1}{\beta}})\right)^{\gamma}\right]\left(1+\left(\alpha^{-1}(1+\log(1-u_i)^{\frac{1}{\beta}})\right)^{\gamma}\right)}=0, (4.126)$$

and

$$\frac{\partial}{\partial\lambda}p(\Theta)=-\frac{2n}{\lambda}=0. \qquad (4.127)$$

Equating $\frac{\partial}{\partial \alpha} p(\Theta), \frac{\partial}{\partial \beta} p(\Theta), \frac{\partial}{\partial \gamma} p(\Theta)$ and $\frac{\partial}{\partial \lambda} p(\Theta)$ to zero and solving simultaneously gives the estimates.

4.7.1.5 The Cramér-von Mises Estimators

The Cramér-von Mises estimators $\alpha_{CME}, \beta_{CME}, \lambda_{CME}$ and γ_{CME} of WIEL distribution are obtained by minimizing the function:

$$c(\alpha,\beta,\lambda,\gamma) = \tfrac{1}{12n} + \sum_{i=1}^{n} [1 - \exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}} - 1]^{\beta}\} - \tfrac{2i-1}{2n}]^2, \qquad (4.128)$$

with respect to α, β, λ and γ or equivalently solving the following nonlinear equations

$$\sum_{i=1}^{n} [1 - \exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}} - 1]^{\beta}\} - \tfrac{2i-1}{2n}]\Psi_1(x_{(i)} \mid \alpha,\beta,\lambda,\gamma) = 0, \qquad (4.129)$$

$$\sum_{i=1}^{n} [1 - \exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}} - 1]^{\beta}\} - \tfrac{2i-1}{2n}]\Psi_2(x_{(i)} \mid \alpha,\beta,\lambda,\gamma) = 0, \qquad (4.130)$$

$$\sum_{i=1}^{n} [1 - \exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}} - 1]^{\beta}\} - \tfrac{2i-1}{2n}]\Psi_3(x_{(i)} \mid \alpha,\beta,\lambda,\gamma) = 0, \qquad (4.131)$$

and

$$\sum_{i=1}^{n} [1 - \exp\{-\alpha[(1-\exp(-\tfrac{\lambda}{x_{(i)}}))^{-\frac{1}{\gamma}} - 1]^{\beta}\} - \tfrac{2i-1}{2n}]\Psi_4(x_{(i)} \mid \alpha,\beta,\lambda,\gamma) = 0, \qquad (4.132)$$

where $\Psi_1(x_{(i)} \mid \lambda,\alpha,\beta,\gamma)$, $\Psi_2(x_{(i)} \mid \lambda,\alpha,\beta,\gamma)$, $\Psi_3(x_{(i)} \mid \lambda,\alpha,\beta,\gamma)$ and $\Psi_3(x_{(i)} \mid \lambda,\alpha,\beta,\gamma)$ are from equation (4.114)-(4.117).

CHAPTER FIVE

SIMULATION AND EMPIRICAL RESULTS

5.0 Introduction

This chapter presents the simulation results and applications of the newly developed distributions (GIEL, WIEL and LLIEW distribution) to real life data. As an illustration, GIEL distribution was chosen for the simulation.

5.1 Simulation

In this section, the properties of the five methods of parameter estimations (MLE, OLS, WLS, PCE and CVM) for the parameters of the GIEL distribution were examined using Monte Carlo simulation. The average bias (ABias) and mean square error (MSE) of the parameters were obtained. The quantile function of GIEL distribution was used to generate random samples. The simulation experiment was replicated for N=1500 each with sample sizes, $n = 20, 50, 100, 300$ and 600 and parameter values $(\alpha, \lambda) = (0.4, 4.4), (1.5, 1.2)$, $(2.3, 0.5)$ and $(1.2, 1.5)$. The ABias and MSE of all the five estimation procedures are recorded in Table 5.1 and Table 5.2.

The simulation study indicates that MLE, CVM, OLS and WLS methods of estimation are all consistent except PCE. The MSEs and ABias of MLE, CVM and OLS decreases as the sample size increases. Also, the MSEs and ABias of WLS decreased for some time as the sample size increase. The values of the PCE however fluctuate as the sample size increases. It can also be observed that, MLE has the minimum values of MSEs and ABias compare to the other methods with different combination of the parameter values. It can be concluded that, MLE is the best methods in estimating GIEL distribution.

Table 5.1 Simulation results for (α=0.4, λ=4.4) and (α=1.5, λ=1.2)

Parameter	n	MLE		OLS		WLS		CVM		PCE	
$\alpha = 0.4$		ABias	MSE	ABias	MSE	ABias	MSE	ABias	MSE	ABias	MSE
	20	0.122	0.023	0.190	0.050	0.194	0.050	0.224	0.065	0.717	0.535
	50	0.078	0.009	0.183	0.041	0.204	0.054	0.200	0.047	0.729	0.545
	100	0.054	0.004	0.188	0.039	0.203	0.046	0.197	0.043	0.740	0.559
	300	0.032	0.002	0.194	0.039	0.206	0.044	0.197	0.040	0.748	0.566
	600	0.022	8E-04	0.194	0.038	0.206	0.043	0.196	0.039	0.755	0.577
$\lambda = 4.4$	20	0.464	0.362	13.310	183.820	13.533	189.700	13.540	190.400	8.2784	499.000
	50	0.300	0.148	13.260	178.447	13.490	184.770	13.340	180.800	10.352	1487.000
	100	0.199	0.063	13.200	175.560	13.456	182.310	13.250	176.700	10.786	5147.000
	300	0.122	0.023	13.120	172.497	13.369	179.150	13.130	172.900	9.618	291.700
	600	0.083	0.011	13.140	172.743	13.388	179.440	13.140	172.900	10.533	1501.000
$\alpha = 1.5$	20	0.248	0.095	0.350	0.171	0.364	0.178	0.408	0.219	1.386	1.971
	50	0.156	0.039	0.325	0.133	0.363	0.156	0.356	0.155	1.389	1.962
	100	0.108	0.018	0.332	0.125	0.379	0.155	0.349	0.137	1.401	1.992
	300	0.064	0.006	0.344	0.124	0.397	0.161	0.350	0.128	1.409	2.004
	600	0.045	0.003	0.344	0.121	0.401	0.162	0.347	0.123	1.403	1.993
$\lambda = 1.2$	20	0.226	0.092	3.982	18.124	4.171	19.689	4.105	19.370	246.070	22342181
	50	0.141	0.034	3.889	15.951	4.109	17.756	3.935	16.330	162.890	1255784
	100	0.094	0.014	3.835	15.064	4.065	16.907	3.857	15.240	310.920	7520487
	300	0.056	0.005	3.775	14.376	4.016	16.258	3.782	14.430	181.270	379135
	600	0.038	0.002	3.782	14.358	4.027	16.282	3.785	14.390	181.900	212972

Table 5.2 Simulation results for (α=2.3, λ=0.5) and (α=1.2, λ=1.5)

Parameter	n	MLE		OLS		WLS		CVM		PCE	
		ABias	**MSE**	**ABias**	**MSE**	**ABias**	**MSE**	**ABias**	**MSE**	**ABias**	**MSE**
$\alpha = 2.3$	20	0.408	0.256	0.539	0.411	0.572	0.441	0.623	0.519	2.158	4.742
	50	0.268	0.109	0.491	0.309	0.579	0.396	0.537	0.358	2.155	4.713
	100	0.175	0.049	0.498	0.287	0.613	0.405	0.524	0.313	2.162	4.732
	300	0.103	0.017	0.517	0.281	0.653	0.435	0.526	0.290	2.160	4.705
	600	0.074	0.009	0.518	0.275	0.663	0.445	0.522	0.279	2.141	4.641
$\lambda = 0.5$	20	0.130	0.034	1.960	5.389	2.140	6.116	2.051	6.076	5557	17038217571
	50	0.081	0.011	1.841	3.822	2.057	4.737	1.872	3.954	15321	239952526233
	100	0.052	0.005	1.784	3.359	2.013	4.263	1.799	3.415	5467	6249850669
	300	0.030	0.001	1.733	3.062	1.975	3.972	1.738	3.078	15597	50889455990
	600	0.020	0.0007	1.735	3.038	1.985	3.973	1.737	3.046	12291	12385927330
$\alpha = 1.2$	20	0.192	0.057	0.281	0.110	0.290	0.113	0.329	0.142	1.095	1.244
	50	0.122	0.023	0.264	0.087	0.289	0.098	0.289	0.101	1.104	1.246
	100	0.084	0.011	0.271	0.083	0.300	0.097	0.284	0.090	1.117	1.278
	300	0.049	0.004	0.28	0.082	0.311	0.099	0.285	0.084	1.122	1.275
	600	0.035	0.002	0.281	0.080	0.319	0.108	0.283	0.082	1.120	1.269
$\lambda = 1.5$	20	0.232	0.094	4.753	24.569	4.901	26.091	4.872	25.890	27.218	36488
	50	0.147	0.037	4.689	22.750	4.867	24.481	4.734	23.190	62.480	656083
	100	0.098	0.015	4.647	21.926	4.832	23.693	4.669	22.140	38.704	44234
	300	0.059	0.005	4.594	21.217	4.786	23.028	4.601	21.280	48.754	61105
	600	0.040	0.003	4.601	21.224	4.799	23.093	4.605	21.260	48.088	37785

5.2 Applications

In this section applications of the special distributions to real data sets were done. In these applications, the MLE was used to estimate the parameters of the special distributions. The AIC, AICc, BIC, Cramér-Von Mises minimum distance (W^*), K-S value and the *P*-value of the K-S statistics were computed to compare the fitted models. Generally, the smaller the values of these statistics, the better the fit to the data. Also included are the plots of the CDFs, PDFs of both the empirical and the fitted distributions and their probability plots.

5.2.1 Application of Gumbel-IE{Logistic} distribution to real data set

The GIEL distribution was applied to three (3) real life data sets of which one (1) data set is censored. This application is to demonstrate how the GIEL distribution can be applied in real life and also to demonstrate its superiority over other competitive distributions.

5.2.1.1 Analgesic Data

This data set represent relief times of 20 patients receiving an analgesic. The data set was retrieved from Oguntunde et al. (2017) and is shown in Table 1, Appendix B. The GIEL distribution was compared with Kumaraswamy inverse exponential (KIE) distribution, IE distribution, Burr type XII (BXII) distribution and generalized inverse exponential (GIE) distribution using the analgesic data set. The descriptive statistics of the data set are given in Table 12, Appendix B. It can be observed that, the distribution of the data is positively skewed and platykurtic (skewness= 1.592 and kurtosis=2.347). The average relief time for patients receiving the medication is 1.700 minutes. The minimum and the maximum relief time are 1.100 minutes and 1.475 minutes respectively.

The data set has an increasing failure rate since the TTT-transform curve is concaved above the 45^0 line as shown in Figure 5.1.

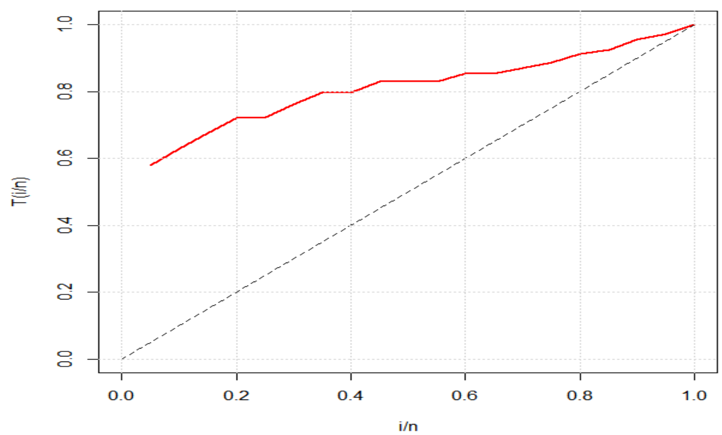

Figure 5.1: TTT-transform plot for analgesic data set

The maximum likelihood estimates for the parameters and the standard errors are given in Table 5.3. All the parameters of the fitted distributions except the parameters of Burr type XII distribution were significant at 5% level of significance.

Table 5.3: Maximum likelihood estimates for relief time of analgesic data

Model	Estimates	Standard error	z-value	*P*-value
GIEL	$\alpha = 0.3418$	6.0361×10^{-2}	5.6557×10^{0}	1.552×10^{-8} *
	$\lambda = 1.0728$	6.1835×10^{-2}	1.73500×10^{1}	$< 2.2\times10^{-16}$ *
KIE	$a = 96.8838$	4.2346×10^{-3}	2.2879×10^{4}	$< 2.2\times10^{-16}$ *
	$b = 20.7664$	1.0919×10^{1}	1.9018×10^{0}	5.719×10^{-2} *
	$\theta = 0.0637$	1.1193×10^{-2}	5.6913×10^{0}	$< 2.2\times10^{-16}$ *
GIE	$a = 20.8142$	1.0964×10^{1}	1.8985×10^{0}	5.763×10^{-2} *
	$\theta = 6.1757$	1.0865×10^{0}	5.6839×10^{0}	1.316×10^{-8} *
BXII	$\alpha = 25.1031$	2.7244×10^{1}	9.2140×10^{-1}	3.568×10^{-1}
	$\beta = 0.0676$	7.4788×10^{-2}	9.0360×10^{-1}	3.6620×10^{-1}
IE	$\lambda = 1.7248$	3.8567×10^{-1}	4.4721×10^{0}	7.744×10^{-6} *

*: means significant at $\leq 5\%$ level of significance

The log-likelihood values and the goodness of fit values are reported in Table 5.4. The results indicate that, GIEL distribution gives a better fit to this data set than KIE, IE, BXII and GIE distributions. This is because, for all the model selection criteria and goodness of fit statistics, GIEL distribution has the highest log-likelihood value and least goodness-of-fit values.

Table 5.4: Model selection criteria for analgesic data

Model	Log-likelihood	AIC	AICc	BIC	W^*	K-S	*P*-value
GIEL	-15.5400	35.0856	35.7915	37.0770	0.0327	0.1164	0.9491
GIE	-17.1000	38.2091	38.9150	40.2006	0.0806	0.1543	0.7278
KIE	-17.1050	40.2091	41.7091	43.1963	0.0805	0.1543	0.7280
BXII	-21.3100	46.6269	47.3328	48.6183	0.0373	0.2850	0.0776
IE	-32.6690	67.337	67.5596	68.3331	0.0490	0.3872	0.0050

Figure 5.2 presents the plots of the CDFs and the PDFs of both the empirical and the fitted distributions. The empirical density and the fitted densities are shown on the left of Figure 5.2. Also, the empirical CDF and the fitted CDFs are shown on the right of figure 5.2. The graph of GIE super impose KIE graph. The GIEL distribution mimics the shapes of the data set better than KIE, GIE, BXII and IE distributions.

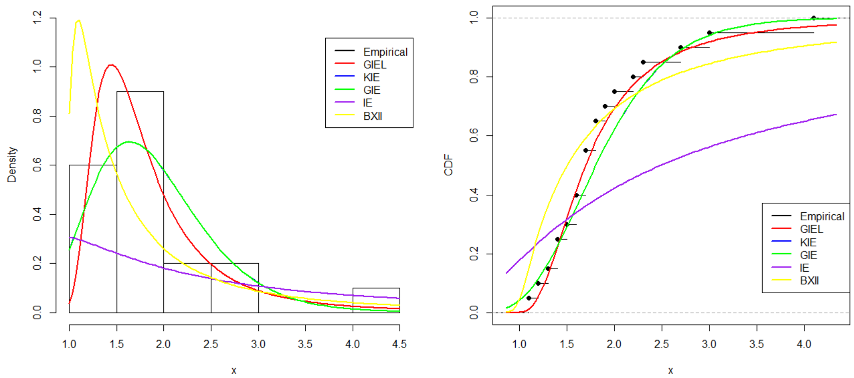

Figure 5.2: PDFs and CDFs for analgesic

Figure 5.3 presents the probability plots of the fitted distributions. It can be observed that, GIEL distribution provide a better fit to the data set than KIE, GIE and IE distributions. This is because, the plot of GIEL observed probability against the expected cluster closely along the diagonal as compare to the competing models.

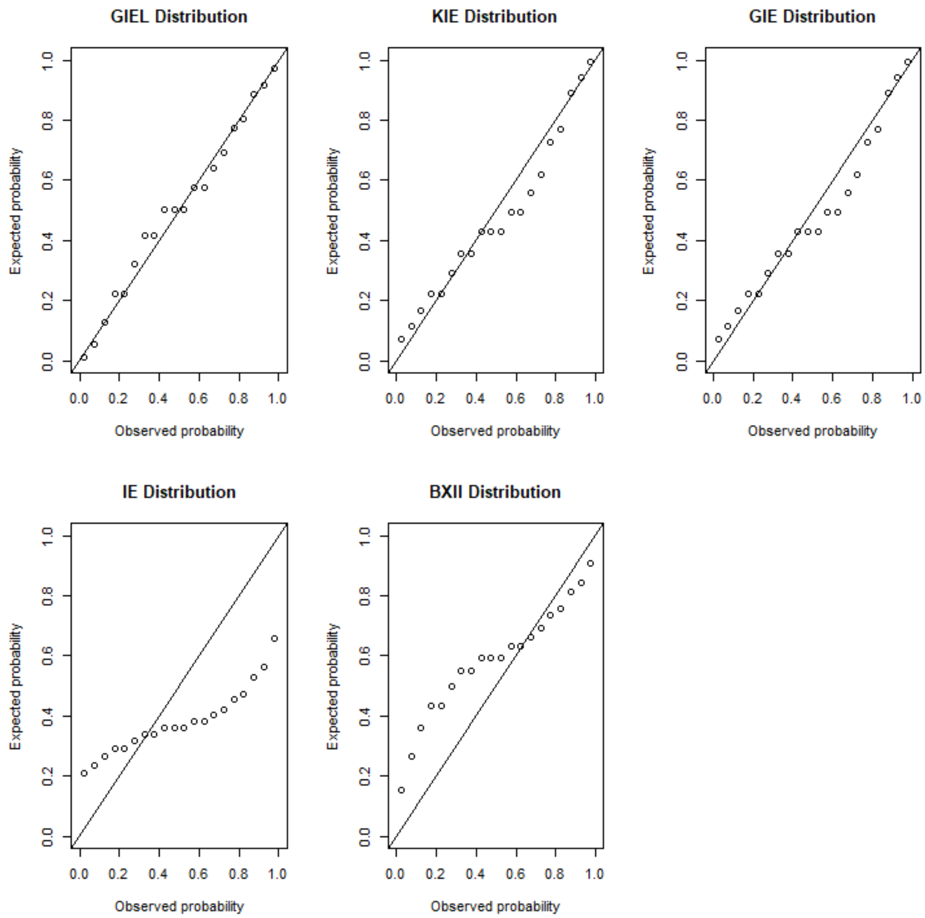

Figure 5.3: P-P plots for analgesic data

The asymptotic variance-covariance matrix for the estimated parameters of GIEL distribution is given by

$$v^{-1} = \begin{bmatrix} 3.6434\times10^{-3} & 1.5095\times10^{-3} \\ 1.5095\times10^{-3} & 3.8236\times10^{-3} \end{bmatrix}$$

Thus, the approximate 95% CI for the parameters α and λ are $[0.2235, 0.4601]$ and $[0.9516, 1.1940]$ respectively. From the estimated CI, it is clear that none of them contain zero. Hence, the estimated parameters of GIEL distribution were all significant at the 5% CI.

5.2.1.2 Roofing Sheet Data

This uncensored data set represent coating weight by chemical method on top center side (TCS) of roofing sheets. The data set was retrieved from Rao and Mbwambo (2019) and shown in Table 2, Appendix B. The performance of the GIEL distribution was compare to inverse Rayleigh (IRD) distribution, Weibull (WD) distribution, BXII and Rayleigh (RD) distributions.

The descriptive statistics of this data set is presented in Table 13, Appendix B. It can be observed that, the minimum and maximum coating weigt are 28.7000gm/m^2 and 61.2000gm/m^2 respectively. The average coating weight for the TCS procedure is 43.0900gm/m^2. The data set is positively skewed and platykurtic (coefficient of skewness=0.4150 and kurtosis=-0.6827).

The exploratory analysis indicates that, the data set has an increasing failure rate since the TTT-transform curve is concaved above the 45^0 line as shown in Figure 5.4.

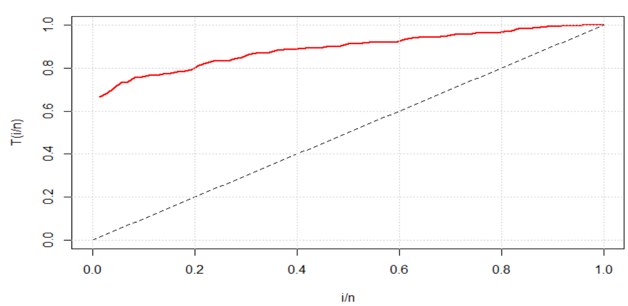

Figure 5.4: TTT-transform plot for the roofing sheet data set

The estimates for the parameters of the GIEL, IRD, WD, BXII and RD distributions are given in Table 5.5. All the parameters of the fitted distributions were significant at 5% level of significance.

Table 5.5: Maximum likelihood estimates for roofing sheet data

Model	Estimate	Standard error	z-value	*P*-value
GIEL	$\alpha = 0.2421$	2.1787×10^{-2}	1.1114×10^{1}	$<2.2\times10^{-16\,*}$
	$\lambda = 26.5815$	5.7754×10^{-1}	4.6025×10^{1}	$<2.2\times10^{-16\,*}$
WD	$\beta = 0.9825$	1.1579×10^{-1}	8.4853×10^{0}	$<2.2\times10^{-16\,*}$
	$\theta = 43.0989$	5.1252×10^{0}	8.4092×10^{0}	$<2.2\times10^{-16\,*}$
BXII	$\alpha = 2.7783$	4.9440×10^{0}	5.619×10^{-1}	5.742×10^{-1}
	$\beta = 0.0961$	1.7131×10^{-1}	5.608×10^{-1}	5.749×10^{-1}
IRD	$\sigma = 40.8748$	2.4086×10^{0}	1.697×10^{1}	$<2.2\times10^{-16\,*}$
RD	$\sigma = 31.0159$	1.8276×10^{0}	1.6971×10^{1}	$<2.2\times10^{-16\,*}$

*: means significant at $\leq 5\%$ level of significance

The log-likelihood values and the goodness of fit values are reported in Table 5.6. The results indicate that, GIEL distribution gives a better fit to this data set than IRD, WD, BXII and RD distributions as it has the highest log-likelihood value and the lowest goodness of fit statistics.

Table 5.6: Model selection criteria for the roofing sheet data

Model	Log-likelihood	AIC	AICc	BIC	W^*	K-S	*P*-value (K-S)
GIEL	-255.7300	515.4634	515.6373	520.0160	0.1378	0.1030	0.4296
IRD	-296.8000	595.6036	595.6607	597.8803	0.0363	0.3653	9.035×10^{-9}
RD	-296.8900	595.7888	595.8459	598.0655	0.0704	0.3632	1.122×10^{-8}
WD	-342.9500	689.8968	690.0708	694.4502	0.0496	0.4802	7.661×10^{-15}
BXII	-436.7600	877.5170	877.6909	882.0703	0.0335	0.5918	2.2×10^{-16}

Figure 5.5 presents the plots of the CDFs and the PDFs of both the empirical and the fitted distributions. The empirical density and the fitted densities are shown on the left of Figure 5.5. Also, the empirical CDF and the fitted CDFs are shown on the right of figure 5.5. The GIEL distribution provides a better fit to the data set as compare to IRD, WD, BXII and RD.

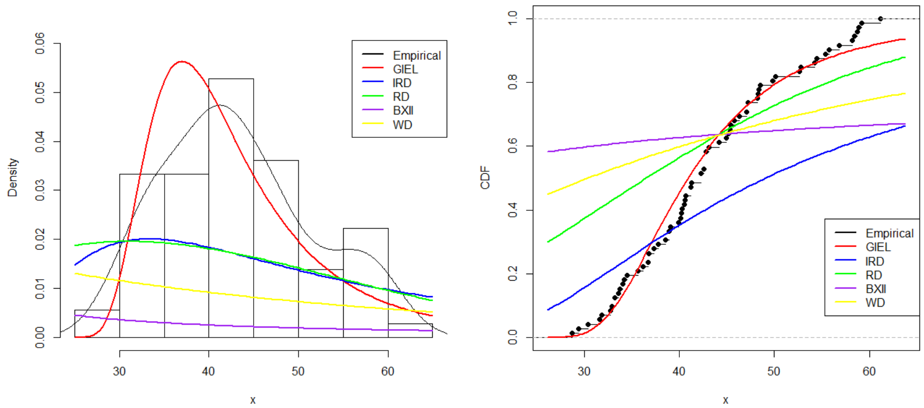

Figure 5.5: PDFs and CDFs for roofing sheet data

Figure 5.6 displays the probability plots of the fitted distributions. It can be observed that, GIEL distribution provide a better fit to the data set than IRD, WD, BXII and RD. This is because, the plot of GIEL observed probability against the expected cluster closely along the diagonal as compare to the competing models.

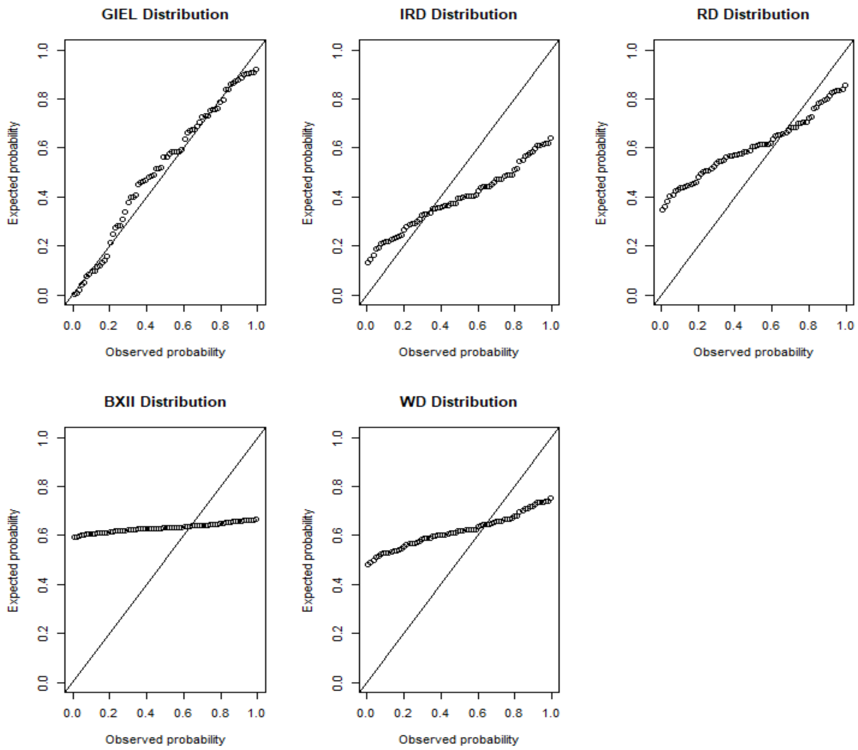

Figure 5.6: P-P plots for the roofing sheet data

The asymptotic variance-covariance matrix for the estimated parameters of the GIEL distribution for the TCS data is given by

$$v^{-1} = \begin{bmatrix} 4.7467\times10^{-4} & 4.9598\times10^{-3} \\ 4.9598\times10^{-3} & 3.3356\times10^{-1} \end{bmatrix}.$$

The estimated 95% CI for the parameters α and λ of the GIEL distribution are respectively given as $[0.1994,\ 0.2851]$ and $[25.4495,\ 27.7135]$. It can be seen that, none of the CI for the estimated parameters contain zero. This implies that, the estimated parameters of the GIEL distribution were all significant at the 5% significance level.

5.2.1.3 Dialysis Data

This data set represents lifetime data relating to recurrent times to infection at the point of inserting of a catheter for patients undergoing kidney dialysis. The data was retrieved from Lawless (2003) and is shown in Table 3, Appendix B.

In this section, an application of the GIEL distribution to censored data set has been provided. The AIC, AICc and BIC statistics were used to compare the performance of GIEL distribution to IRD, RD, and BXII. The data set consist of 38 persons undergoing kidney dialysis.

The data set has a decreasing failure rate since the TTT-transform curve is convex below the 45^0 line as shown in Figure 5.7.

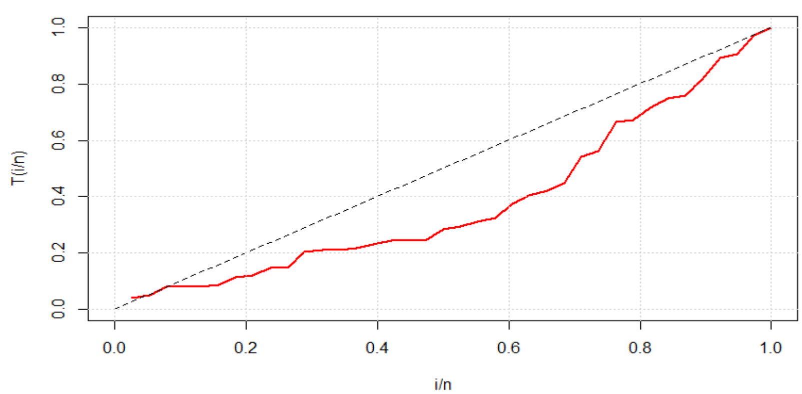

Figure 5.7: TTT-transform plot for dialysis data

The maximum likelihood estimates for the parameters and the standard errors are given in Table 5.7. All the parameters of the fitted distributions were significant at 5% level of significance.

Table 5.7: Maximum likelihood estimates for the dialysis data

Model	Estimate	Standard error	z-value	*p*-value
GIEL	α=1.7009	$2.8971\text{x}10^{-1}$	$5.8711\text{x}10^{0}$	$4.3290\text{x}10^{-9*}$
	λ=22.5010	$4.1303\text{x}10^{0}$	$5.4478\text{x}10^{0}$	$5.0990\text{x}10^{-8}$
BXII	α=3.3910	$7.5129\text{x}10^{0}$	$4.3800\text{x}10^{-1}$	$6.6140\text{x}10^{-1}$
	β=0.0545	$1.2474\text{x}10^{-1}$	$4.3670\text{x}10^{-1}$	$6.6240\text{x}10^{-1}$
RD	σ=139.8074	$1.3731\text{x}10^{1}$	$1.0182\text{x}10^{1}$	$<2.2\text{x}10^{-16*}$
IRD	σ=25.1519	$2.1962\ \text{x}10^{0}$	$1.1452\ \text{x}10^{1}$	$2.2\ \text{x}10^{-16*}$

*: means significant at $\leq 5\%$ level of significance

The log-likelihood values and the goodness of fit values are reported in Table 5.8. The results indicate that, GIEL distribution gives a better fit to this data set than IRD, BXII and RD distributions as it has the highest log-likelihood value and the lowest information criteria statistics.

Table 5.8: Model selection criteria statistics for the dialysis data

Model	Log-likelihood	AIC	AICc	BIC
GIEL	-152.2300	316.4553	316.7981	319.7304
RD	-175.6100	353.2284	353.3395	354.8659
BXII	-177.91	359.8266	360.1695	363.1018
IRD	-180.3600	362.7103	362.8214	364.3479

The asymptotic variance-covariance matrix for the estimated parameters for GIEL distribution is given by

$$v^{-1} = \begin{bmatrix} 8.3881\times10^{-2} & 7.6179\times10^{-1} \\ 7.6179\times10^{-1} & 1.7047\times10^{1} \end{bmatrix}$$

Thus, the estimated 95% confidence interval for the parameters α and λ of the GIEL distribution are respectively given as $[1.1331, 2.2687]$ and $[14.4052, 30.5964]$. It can be observed that, none of the CI for the estimated parameters contain zero. This implies that, the estimated parameters of the GIEL distribution were all significant at the 5% significance level.

5.2.2 Application of Weibull-IE{Lomax} distribution to real data set

The WIEL distribution was applied to three (3) data sets. This is to demonstrate the application of the distribution to real life data sets.

5.2.2.1 Bank data

The data set is an uncensored data on the waiting time of customers of a bank. This data set has 100 observations and was retrieved from Oguntunde et al. (2017). The data set is shown in Table 4, Appendix B. It was used to compare the performance of WIEL distribution to exponentiated inverse Rayleigh (EIRD) distribution, inverse Rayleigh (IRD) distribution and Rayleigh (RD) distribution. Table 14, Appendix B presents the descriptive statistics for the data set. The mean waiting time was 13.02 minutes and the minimum and maximum waiting time were 0.8000 minutes and 38.5000 minute respectively. This data set is positively skewed and platykurtic.

The data set has an increasing failure rate since the TTT-transform curve is concaved above the 45^0 line as shown in Figure 5.8.

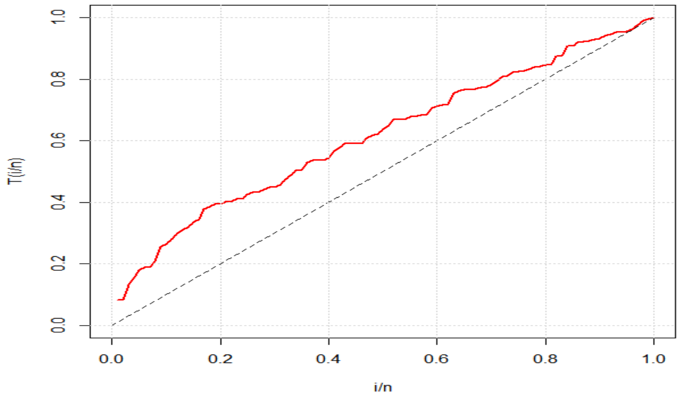

Figure 5.8: TTT-transform plot for the bank data

The maximum likelihood estimates for the parameters and the corresponding standard errors are given in Table 5.9. All the parameters of the fitted distributions were significant at 5% level of significance.

Table 5.9: Maximum likelihood estimates for Bank data set

Model	**Estimate**	**Standard error**	**z-value**	***p*-value**
WIEL	α=0.8905	$1.8550\text{x}10^{-1}$	$4.8004 \text{ x}10^{0}$	$1.58 \text{ x}10^{-6*}$
	λ=26.0955	$1.4496 \text{ x}10^{0}$	$1.8002 \text{ x}10^{1}$	$<2.2 \text{ x}10^{-16*}$
	β=0.2684	$4.1409 \text{ x}10^{-2}$	$6.4820 \text{ x}10^{0}$	$9.051 \text{ x}10^{-11*}$
	γ=0.0964	$3.0082 \text{ x}10^{-2}$	$3.2054 \text{ x}10^{0}$	$1.349 \text{ x}10^{-3*}$
EIRD	σ=2.5796	$2.2843 \text{ x}10^{-1}$	$1.1393 \text{ x}10^{1}$	$<2.2 \text{ x}10^{-16*}$
	α=0.4298	$5.1423 \text{ x}10^{-2}$	$8.3585 \text{ x}10^{0}$	$<2.2 \text{ x}10^{-16*}$
IRD	σ=3.6190	$1.8095\text{x}10^{-1}$	$2.0000\text{x}10^{1}$	$<2.2 \text{ x}10^{-16*}$

*: means significant at $\leq 5\%$ level of significance

Table 5.10 presents the model selection criteria and the values of the goodness of fit statistics. From Table 5.10, it can be observed that, WIEL distribution has the highest log-likelihood value and the least values of the goodness of fit statitics. This implies that, WIEL distribution better fit this data set than the competing models.

Table 5.10: Model selection criteria for the bank data set

Model	Log-likelihood	AIC	AICc	BIC	W^*	K-S	*P*-value (K-S)
WIEL	-334.0500	696.0942	696.5153	706.5149	0.1888	0.0920	0.3664
EIRD	-350.0900	704.1704	704.2941	709.3808	0.8029	0.2120	0.0003
IRD	-379.7800	761.5627	761.6035	764.1679	0.7796	0.3533	$2.880\text{x}10^{-11}$

Figure 5.9 presents the plots of the CDFs and the PDFs of both the empirical and the fitted distributions for the bank data. The empirical density and the fitted densities are shown on the left of Figure 5.9. Also, the empirical CDF and the fitted CDFs are shown on the right of figure 5.9. The WIEL distribution provides a better fit to the data set as compare to the other distributions as the graphs of the WIEL distribution mimic the empirical CDF and the empirical density.

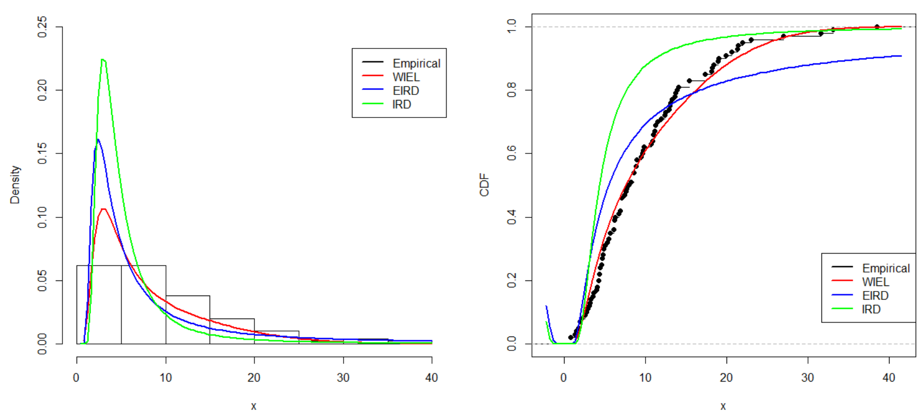

Figure 5.9: PDFs and CDFs for Bank data

The probability plots for the fitted distributions are shown in Figure 5.10. It can be observed that, WIEL distribution provide a better fit to the data set than EIRD and IRD. This is because, the plot of WIEL observed probability against the expected cluster closely along the diagonal as compare to the competing models.

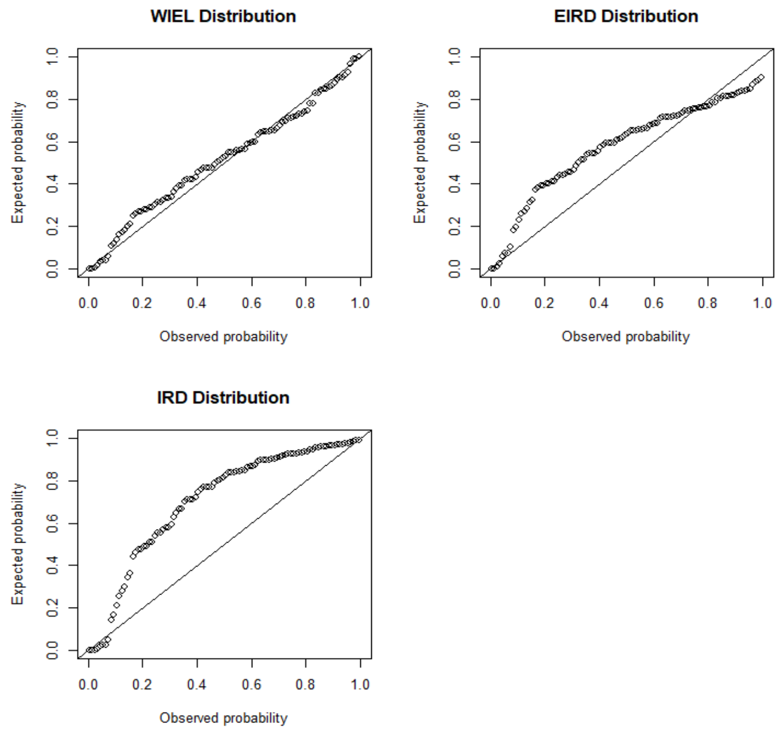

Figure 5.10: P-P plots of the fitted distributions for Bank data

The variance-covariance matrix for the estimated parameters of WIEL distribution is given by

$$v^{-1} = \begin{bmatrix} 3.4411\times10^{-2} & 4.9896\times10^{-2} & 4.2647\times10^{-3} & 4.3063\times10^{-3} \\ 4.9896\times10^{-2} & 2.1014\times10^{0} & -1.4368\times10^{-2} & 9.9860\times10^{-3} \\ 4.2647\times10^{-3} & -1.4368\times10^{-2} & 1.7147\times10^{-3} & 1.0750\times10^{-3} \\ 4.3063\times10^{-3} & 9.9860\times10^{-3} & 1.0750\times10^{-3} & 9.049\times10^{-4} \end{bmatrix}.$$

Hence, the approximate 95% CI for the parameters α, λ, β and γ are respectively given as $[0.5269, 1.2541]$, $[23.2543, 28.9367]$, $[0.1872, 0.3496]$ and $[0.0374, 0.1554]$. From the estimates, it can be observed that, none of them contain zero. This means that, the estimated parameters of WIEL distribution were all significant at 5% CI.

5.2.2.2 Skin Folds Data

The uncensored data is on the sum of skin folds among 202 athletes. The data set has 202 observations and was retrieved from Alzaatreh et al. (2016) and it is presented in Table 5, Appendix B. The data set was fitted to WIEL, IRD, modified Burr III (MBIII) distribution and Weibull Lomax (WLx) distributions. Table 15, Appendix B presents the descriptive statistics for the skin folds data set. The data set is positively skewed and platykurtic (skewness=1.171952 and kurtosis=1.3228). Also, the variance (1067) is far more than the mean (69.07). Thus, the data is over-dispersed.

The data set has an increasing failure rate since the TTT-transform curve is concaved above the 45^0 line as shown in Figure 5.11.

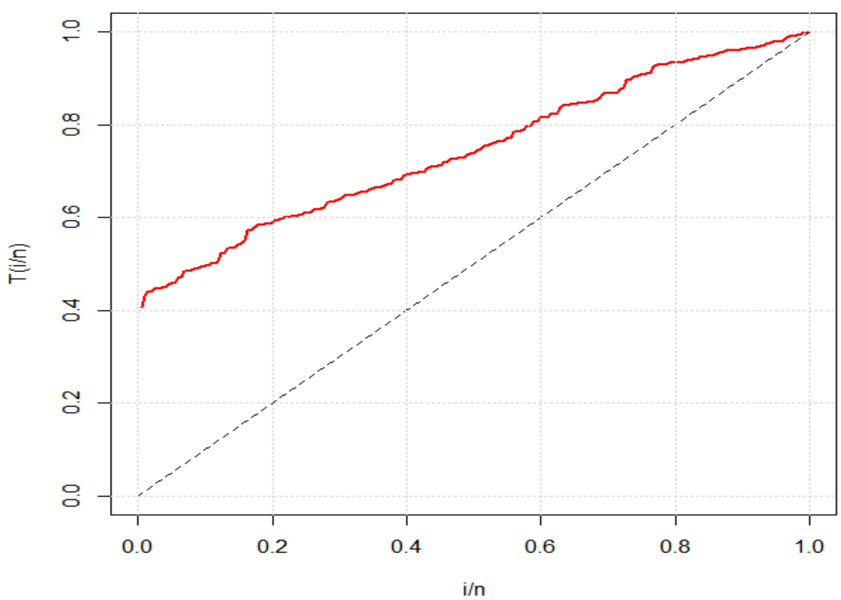

Figure 5.11: TTT-transform plot for skin folds data

The maximum likelihood estimates of the parameters, the corresponding standard errors, z-values and the *P*-values of the distributions are given in Table 5.11. All the parameters of the fitted distributions were significant at 5% significance level.

Table 5.11: Maximum likelihood estimates for skin folds data set

Model	Estimate	Standard error	z-value	*p*-value
WIEL	α=39.8855	$7.6318\text{x}10^{0}$	$5.2262\text{x}10^{0}$	$1.73\text{x}10^{-7*}$
	λ=544.4251	$7.7988\text{x}10^{-1}$	$6.8152\text{x}10^{2}$	$<2.2\text{x}10^{-16*}$
	β=0.3089	$1.7239\text{x}10^{-2}$	$1.7920\text{x}10^{1}$	$<2.2\text{x}10^{-16*}$
	γ=71.3602	$1.4655\text{x}10^{0}$	$4.8694\text{x}10^{1}$	$<2.2\text{x}10^{-16*}$
WLx	a=0.0038	$1.2188\text{x}10^{-3}$	$3.0944\text{x}10^{0}$	$1.9720\text{x}10^{-3*}$
	b=0.6206	$1.3030\text{x}10^{-1}$	$4.7633\text{x}10^{0}$	$1.9050\text{x}10^{-6*}$
	α=3.9101	$8.2553\text{x}10^{-1}$	$4.7365\text{x}10^{0}$	$2.1740\text{x}10^{-6*}$
	β=8.7555	$7.1524\text{x}10^{-4}$	$1.2241\text{x}10^{4}$	$<2.2\text{x}10^{-16*}$
MBIII	α=5100.6646	$1.0494\text{x}10^{-6}$	$4.8607\text{x}10^{9}$	$<2.2\text{x}10^{-16*}$
	β=2.1530	$2.0116\text{x}10^{-2}$	$1.0703\text{x}10^{2}$	$<2.2\text{x}10^{-16*}$
	γ=650.5115	$4.2941\text{x}10^{-7}$	$1.51494\text{x}10^{9}$	$<2.2\text{x}10^{-16*}$
IRD	σ=52.6076	$1.8507\text{x}10^{0}$	$2.8425\ \text{x}10^{1}$	$<2.2\text{x}10^{-16*}$

*: means significant at $\leq 5\%$ level of significance

Table 5.12 presents the statistics of the goodness of fit and the total log-likelihood values. From the Table 5.12, the WIEL distribution has the highest log-likelihood value of -958.2900 and the lowest goodness of fit values. This implies that, WIEL distribution provides a better fit to the data set than IRD, WLx and MBIII distributions.

Table 5.12: Model selection criteria for the skin fold data

Model	Log-likelihood	AIC	AICc	BIC	W^*	K-S	*P*-value (K-S)
WIEL	-958.2900	1924.5760	1924.7790	1937.8090	0.2475	0.0775	0.1762
IRD	-966.6600	1935.3100	1935.3300	1938.6190	0.1791	0.1145	0.0100
MBIII	-966.3300	1938.6670	1938.7880	1948.5910	0.1876	0.0954	0.0506
WLx	-981.2900	1970.5890	1970.7920	1983.8220	0.7432	0.1226	0.0046

Figure 5.12 presents the histogram, the empirical CDFs, fitted densities and fitted CDFs of the distributions for the bank data. From the plots, the WIEL distribution provides a better fit to the data set as compared to the other distributions as the WIEL distribution mimic the shapes of the data set well than IRD, WLx and MBIII distributions.

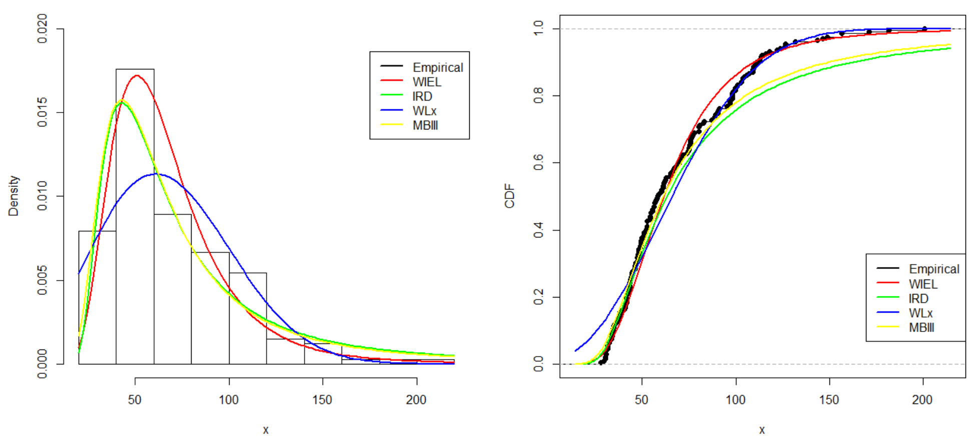

Figure 5.12: PDFs and CDFs for skin folds data

The probability plots for the fitted distributions are shown in Figure 5.13. It can be observed that, WIEL distribution provide a better fit to the data set than IRD, WLx and MBIII distribution. This is because, the plot of WIEL observed probability against the expected cluster closely along the diagonal as compare to the competing distributions.

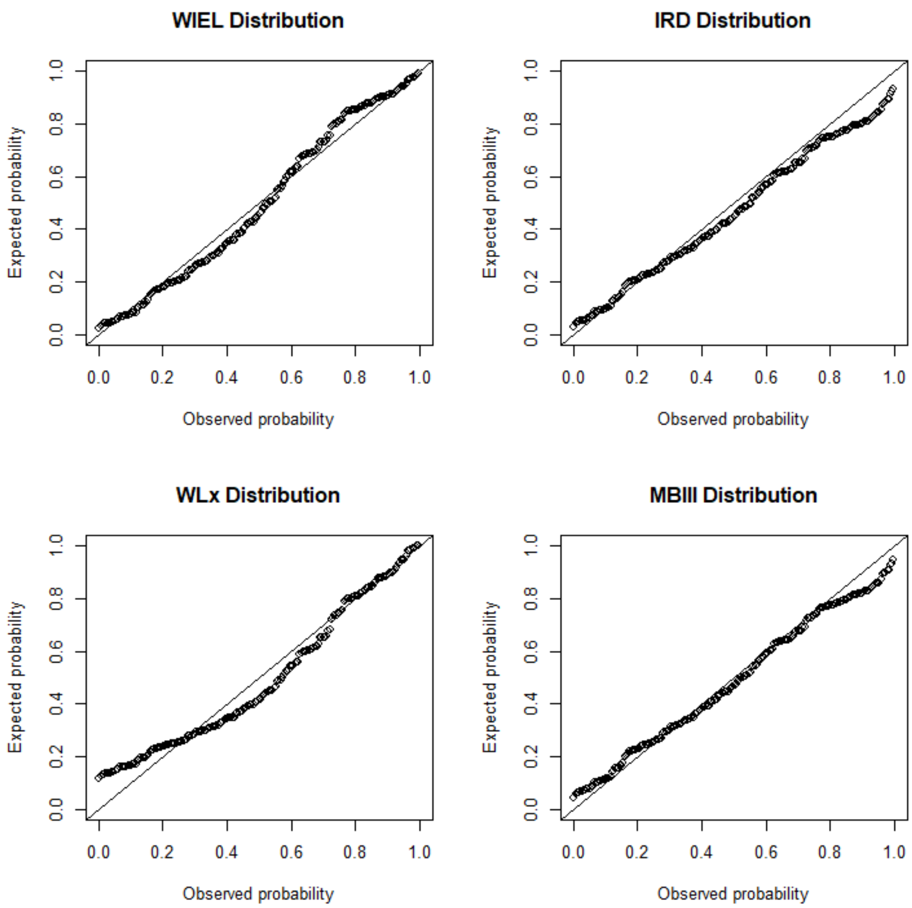

Figure 5.13: P-P plots of the fitted distributions for skin folds data

The variance-covariance matrix for the estimated parameters of WIEL distribution is given by

$$\begin{bmatrix} 58.2441 & 6.0966 & 1.2269\times10^{-1} & -11.1844 \\ 6.0966 & 6.3814\times10^{-1} & 1.2843\times10^{-2} & -1.1707 \\ 1.2269\times10^{-1} & 1.2843\times10^{-2} & 2.9719\times10^{-4} & -2.3560\times10^{-2} \\ -11.1844 & -1.1707 & -2.3560\times10^{-2} & 2.1477 \end{bmatrix}.$$

Thus, the approximate 95% CI for the parameters α, λ, β and γ are respectively given as $[24.9271, 54.8438]$, $[542.8965, 545.9537]$, $[0.2751, 0.3427]$ and $[68.4878, 74.2326]$. From the estimates, it is clear that none of them contain zero. This implies that, the estimated parameters of WIEL distribution were all significant at 5% CI.

5.2.2.3 Height data

This section presents an application of WIEL distribution to a bimodal data. The data set consist of height of 126 students. It was retrieved from Cruz-Medina (2001) and it is presented in Table 6, Appendix B. The data set was fitted to WIEL, MBIII and WLx distributions. Table 16, Appendix B presents the descriptive statistics for the data set. It can be observed that, the average height of a student was 68.55 inches and the maximum height was 78.5 inches. The data is negatively skewed and platykurtic.

The data set has an increasing failure rate since the TTT-transform curve is concaved above the 45^0 line as shown in Figure 5.14.

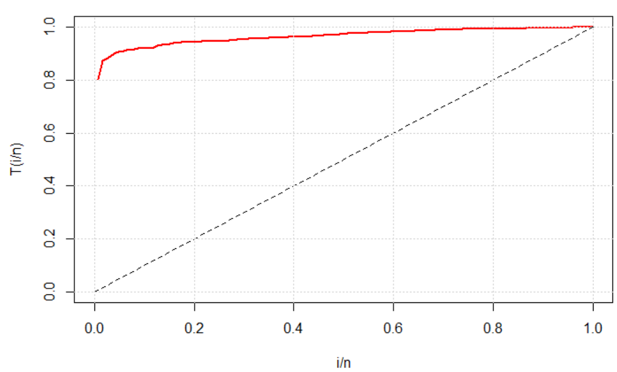

Figure 5.14: TTT-transform plot for the height data

The maximum likelihood estimates of the parameters, the corresponding standard errors, z-values and the *P*-values of the distributions are given in Table 5.13. All the parameters of the WIEL distribution were significant at 5% significance level.

Table 5.13: Maximum likelihood estimates for height data

Model	Estimate	Standard error	z-value	*p*-value
WIEL	α=107.2042	$1.4912\text{x}10^{-4}$	$7.1889\text{x}10^{5}$	$<2.2\text{x}10^{-16*}$
	λ=335.7372	$1.6411\text{x}10^{-3}$	$2.0446\text{x}10^{5}$	$<2.2\text{x}10^{-16*}$
	β=3.2730	$1.9421\text{x}10^{-1}$	$1.6853\text{x}10^{1}$	$<2.2\text{x}10^{-16*}$
	γ=0.0398	$2.8177\text{x}10^{-3}$	$1.4112\text{x}10^{1}$	$<2.2\text{x}10^{-16*}$
WLx	a=0.0011	$8.5441\text{x}10^{-4}$	$1.2736\text{x}10^{0}$	$2.028\text{x}10^{-1}$
	b=1.4633	$1.7384\text{x}10^{-1}$	$8.4173\text{x}10^{0}$	$<2.2\text{x}10^{-16*}$
	α=5.0566	$3.2570\text{x}10^{-2}$	$1.5525\text{x}10^{2}$	$<2.2\text{x}10^{-16*}$
	β=45.3160	$2.4297\text{x}10^{-3}$	$1.8651\text{x}10^{4}$	$<2.2\text{x}10^{-16*}$
MBIII	α=5100.6740	$1.0235\text{x}10^{-6}$	$4.9834\text{x}10^{9}$	$<2.2\text{x}10^{-16*}$
	β=2.0510	$2.2391\text{x}10^{-2}$	$9.1601\text{x}10^{1}$	$<2.2\text{x}10^{-16*}$
	γ=65.3810	$1.0381\text{x}10^{-7}$	$6.2984\text{x}10^{8}$	$<2.2\text{x}10^{-16*}$

*: means significant at $\leq 5\%$ level of significance

Table 5.14 presents the model selection criteria. From the Table 5.14, the WIEL distribution has the highest log-likelihood value of -364.1400 and the lowest goodness of fit values. This implies that, WIEL distribution provides a better fit to the data set than WLx and MBIII distributions.

Table 5.14: Model selection criteria for the height data

Model	Log-likelihood	AIC	AICc	BIC	W^*	K-S	*P*-value (K-S)
WIEL	-364.1400	736.2822	736.6128	747.6273	0.1700	0.0807	0.3852
WLx	-473.7400	955.4857	955.4857	966.8308	0.0810	0.4332	$2.2\text{x}10^{-16}$
MBIII	-570.6000	1147.1920	1147.3910	1155.7040	0.0688	0.4841	$2.2\text{x}10^{-16}$

Figure 5.15 presents the histogram, empirical density, the empirical CDFs, fitted densities and fitted CDFs of the distributions for the height data. From the plots, the WIEL distribution provides a better fit to the data set as compare to the competing distributions as the WIEL distribution mimic the shapes of the data set well than WLx and MBIII distributions.

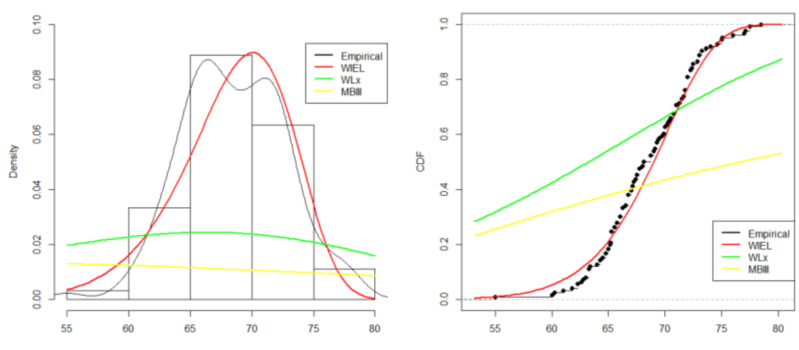

Figure 5.15: PDFs and CDFs for the height data

The probability plots for the fitted distributions are shown in Figure 5.16. It can be observed that, WIEL distribution provide a better fit to the data set than WLx and MBIII distribution. This is because, the plot of WIEL observed probability against the expected cluster closely along the diagonal as compare to the other distributions.

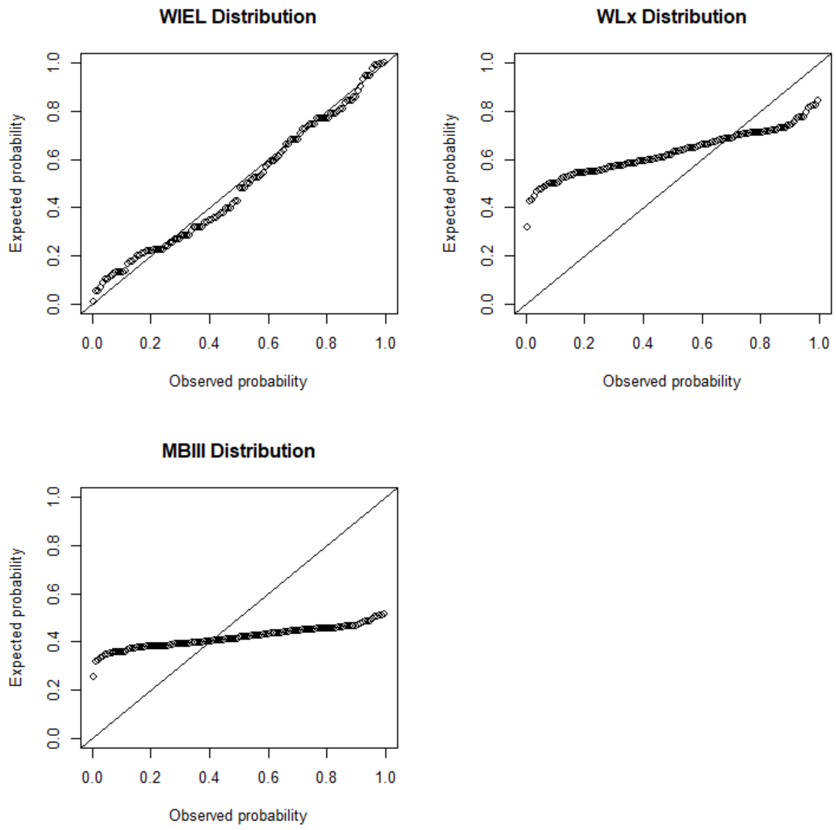

Figure 5.16: P-P plots of the fitted distributions for the height data

The variance-covariance matrix for the estimated parameters of WIEL distribution is given by

$$v^{-1} = \begin{bmatrix} 2.2238\times10^{-8} & -2.4473\times10^{-7} & -2.8961\times10^{-5} & 3.9510\times10^{-7} \\ -2.4473\times10^{-7} & 2.6933\times10^{-6} & 3.1872\times10^{-4} & -4.3486\times10^{-6} \\ -2.8961\times10^{-5} & 3.1872\times10^{-4} & 3.7716\times10^{-2} & -5.1465\times10^{-4} \\ 3.9510\times10^{-7} & -4.3486\times10^{-6} & -5.1465\times10^{-4} & 7.9395\times10^{-6} \end{bmatrix}.$$

Hence, the approximate 95% CI for the parameters α, λ, β and γ are respectively given as $[107.2039,\ 107.2045]$, $[335.7340,\ 335.7404]$, $[2.8923,\ 3.6537]$ and $[0.0343,\ 0.0453]$. From the estimates, it is clear that none of them contain zero. This implies that, the estimated parameters of WIEL distribution were all significant at 5% CI.

5.2.3 Applications of Log-logistic –IE{Weibull} Distribution

The LLIEW distribution was applied to two real life data sets. This application was to demonstrate how the LLIEW distribution can be applied in real life and also to demonstrate its superiority over other competitive distributions.

5.2.3.1 Kevlar 373/Epoxy Data

This complete data set on the life of the fatigue fracture of Kevlar 373/epoxy strands exposed to a persistent pressure of 90% stress level until all failed was used to compare the performance of LLIEW distribution to inverse Weibull (IW), GIE and IE distributions. The data set can be found in Oguntunde (2017) and shown in Table 7, Appendix B. Table 17, Appendix B presents the descriptive statistics of the data set. The data set is positively skewed and leptokurtic (skewness=1.9401 and kurtosis=4.94643). From the value of the kurtosis, it implies that, this data set might have heavy tails.

The exploratory analysis indicates that, the data set has an increasing failure rate since the TTT-transform curve is concaved above the 45^0 line as shown in Figure 5.17.

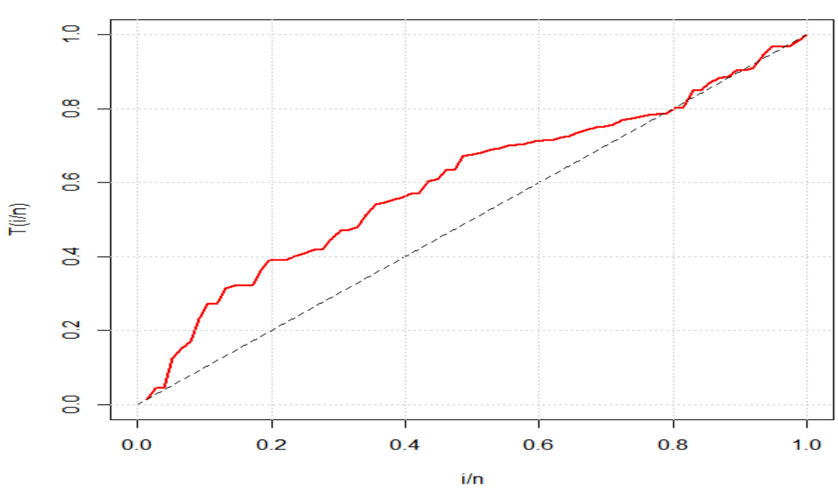

Figure 5.17: TTT-transform plot for Kevlar 373/epoxy data set

The MLEs of the parameters, standard errors, z-values and the *P*-values of the LLIEW, IW, GIE, IE and KIE distributions are shown in Table 5.15. All the parameters of the fitted distributions were significant at 5% significance.

Table 5.15: Maximum likelihood estimates of the Kevlar 373/epoxy data set

Model	**Estimate**	**Standard error**	**z-value**	***p*-value**
LLIEW	α=24.5161	$8.4248\text{x}10^{-5}$	$2.9100\ \text{x}10^{5}$	$<2.2\ \text{x}10^{-16*}$
	λ=0.00425	$1.4705\ \text{x}10^{-3}$	$2.8932\ \text{x}10^{0}$	$3.813\ \text{x}10^{-3*}$
	θ=6.0826	$5.6462\ \text{x}10^{-4}$	$1.0773\ \text{x}10^{4}$	$<2.2\ \text{x}10^{-16*}$
	γ=0.5519	$1.8415\ \text{x}10^{-2}$	$2.9971\ \text{x}10^{1}$	$<2.2\ \text{x}10^{-16*}$
IW	α=0.8211	$1.3232\ \text{x}10^{-1}$	$6.2052\ \text{x}10^{0}$	$5.4630\ \text{x}10^{-10*}$
	β=0.7588	$5.4084\ \text{x}10^{-2}$	$1.4029\ \text{x}10^{1}$	$<2.2\text{x}10^{-16*}$
GIE	a=0.7902	$1.2507\ \text{x}10^{-1}$	$6.3184\ \text{x}10^{0}$	$2.643\ \text{x}10^{-10*}$
	θ=0.5226	$9.2317\ \text{x}10^{-2}$	$5.6611\ \text{x}10^{0}$	$1.504\ \text{x}10^{-8*}$
IE	λ=0.6249	$7.1683\ \text{x}10^{-2}$	$8.7178\ \text{x}10^{0}$	$<2.2\ \text{x}10^{-16*}$

*: means significant at $\leq 5\%$ level of significance

Table 5.16 presents the statistics of the goodness of fit and the total log-likelihood values. From Table 5.16, the LLIEW distribution has the highest log-likelihood value of -131.92 and the lowest goodness of fit values. This implies that, LLIEW distribution provides a better fit to the data set than IW, GIE and IE distributions.

Table 5.16: Model selection criteria for Kevlar 373/epoxy data set

Model	Log-likelihood	AIC	AICc	BIC	W^*	K-S	*P*-value (K-S)
LLIEW	-131.8700	271.7367	272.2998	281.0596	0.3352	0.0965	0.4505
IW	-153.5600	311.1235	311.2879	315.7850	0.9173	0.1892	0.0074
GIE	-161.9900	327.9897	328.1541	332.6511	1.2380	0.2706	$2.091\text{x}10^{-5}$
IE	-163.1300	328.2586	328.3126	330.5893	1.2063	0.2900	$3.731\text{x}10^{-6}$

The empirical density and the fitted densities are shown on the left of Figure 5.18. Also, the empirical CDF and the fitted CDFs are shown on the right of the graph. The LLIEW distribution provide a better fit to the data set as compare to the other distributions.

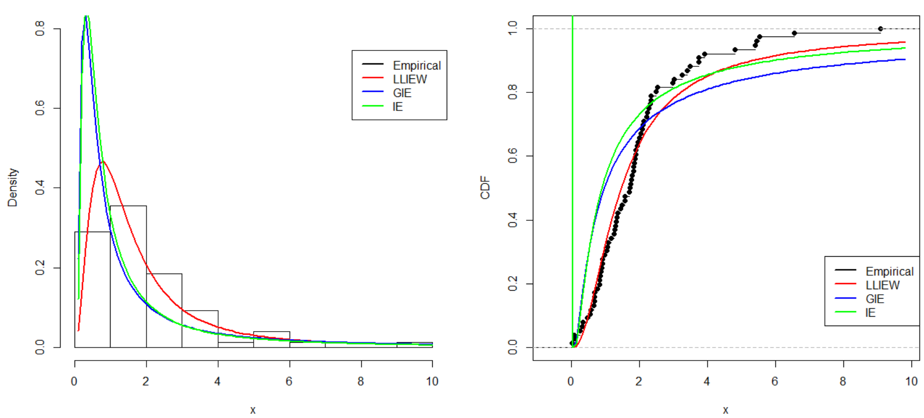

Figure 5.18: PDFs and CDFs for the Kevlar 373/epoxy data

Figure 5.19 presents the probability plots for the fitted distributions. It can be seen that, LLIEW distribution provide a better fit to the data set than the other distributions. This is because, the plot of LLIEW observed probability against the expected cluster closely along the diagonal as compare to the GIE and IE distributions.

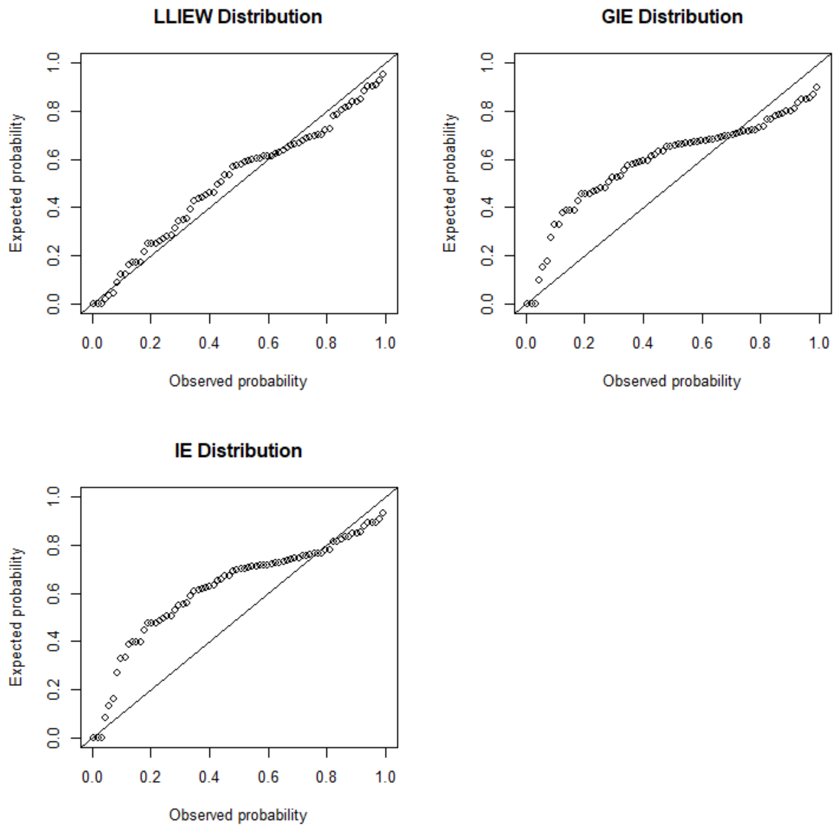

Figure 5.19: P-P plots of the fitted distributions for Kevlar 373/epoxy data

The variance-covariance matrix for the estimated parameters of the LLIEW distribution is given by

$$v^{-1} = \begin{bmatrix} -1.8101\times10^{-2} & 6.3767\times10^{-4} & -2.1582\times10^{-1} & -7.4258\times10^{-3} \\ 6.3767\times10^{-4} & -1.9685\times10^{-5} & 7.6061\times10^{-3} & 2.3007\times10^{-4} \\ -2.1582\times10^{-1} & 7.6061\times10^{-3} & -2.5734\times10^{0} & -8.8577\times10^{-2} \\ -7.4258\times10^{-3} & 2.3007\times10^{-4} & -8.8577\times10^{-2} & -2.6584\times10^{-3} \end{bmatrix}.$$

Thus, the estimated 95% CI for the parameters α, λ, θ and γ are respectively given as $[24.5159, 24.5163]$, $[0.0014, 0.00715]$, $[6.0815, 6.0837]$ and $[0.5158, 0.5880]$. From the approximates, it can be observed that, none of them contain zero. This implies that, the estimated parameters of LLIEW distribution were all significant at 5% CI.

5.2.3.2 Air Condition System Data

This uncensored data set on the failure and running time of air-conditioning system of an airplane was used to compare the performance of LLIEW distribution to KIE, exponentiated-exponential Fréchet distribution (EEFr) and IE distributions. The data set was retrieved from Nasiru (2018) and shown in Table 8, Appendix B.

Table 18, Appendix B presents the descriptive statistics of the failure time of air conditioning system of an aircraft. This data is positively skewed and leptokurtic (coefficient of skewness=1.609 and kurtosis=4.967). This implies that, the data set has fat tail. Also, the data set is over-dispersed as the variance is far more than the mean (this might be due to outliers in the data set).

The failure rate of the data set is bathtub shape since the TTT-transform curve is first convex below the 45^0 line and then concave above the 45^0 line as shown in Figure 5.20.

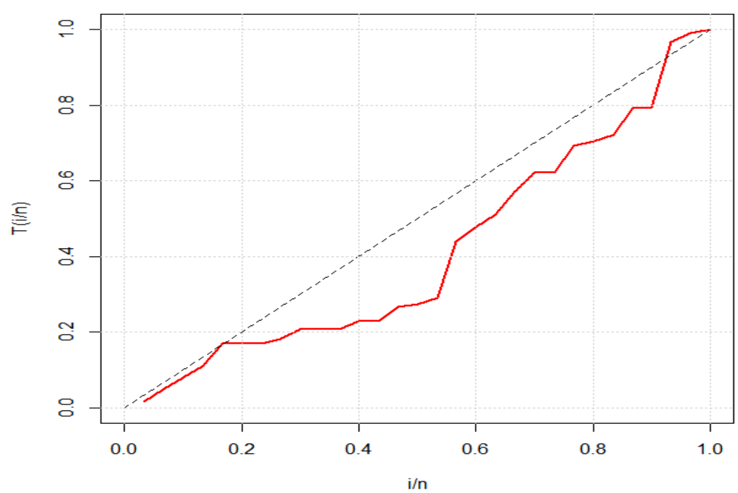

Figure 5.20: TTT-transform plot for air condition data

The maximum likelihood estimates for the parameters and their corresponding standard errors, z-values and *P*-values are given in Table 5.17. All the parameters of LLIEW distribution except λ were significant at 5% significance level. The parameters of both KIE and IE were significant at 5% significance level.

Table 5.17: Maximum likelihood estimates for air condition data

Model	Estimate	Standard error	z-value	*p*-value
LLIEW	α=31.3160	$1.129\text{x}10^{-4}$	$2.774\ \text{x}10^{5}$	$<2.2\ \text{x}10^{-16*}$
	λ=0.0076	$6.316\ \text{x}10^{-3}$	$1.206\text{x}10^{0}$	$2.277\ \text{x}10^{-1}$
	θ=6.6646	$9.637\ \text{x}10^{-4}$	$6.573\ \text{x}10^{3}$	$<2.2\ \text{x}10^{-16*}$
	γ=0.6110	$2.988\ \text{x}10^{-2}$	$2.045\ \text{x}10^{1}$	$<2.2\ \text{x}10^{-16*}$
EEFr	a=0.7902	$2.6548\text{x}10^{0}$	$1.4183\text{x}10^{0}$	$1.5611\text{x}10^{-1}$
	β=2.9295	$4.8322\text{x}10^{0}$	$6.0620\text{x}10^{-1}$	$5.4436\text{x}10^{-1}$
	σ=31.6321	$4.8284\ \text{x}10^{1}$	$6.5510\text{x}10^{-1}$	$5.1239\ \text{x}10^{-1}$
	θ=0.3525	$2.0132\ \text{x}10^{-1}$	$1.7510\ \text{x}10^{0}$	$7.9940\text{x}10^{-2}$
KIE	a=0.2161	$6.1072\text{x}10^{-2}$	$3.5379\text{x}10^{0}$	$4.033\text{x}10^{-4*}$
	b=0.6490	$1.5051\text{x}10^{-1}$	$4.3120\text{x}10^{0}$	$1.618\text{x}10^{-5*}$
	θ =37.5810	$3.5087\text{x}10^{-4}$	$1.0711\text{x}10^{0}$	$<2.2\ \text{x}10^{-16*}$
IE	λ=11.1800	$2.0412\text{x}10^{0}$	$5.4772\text{x}10^{0}$	$4.320\text{x}10^{-8*}$

*: means significant at $\leq 5\%$ level of significance

Table 5.18 presents the statistics of the goodness of fit and the log-likelihood. The LLIEW distribution has the highest log-likelihood value and lowest goodness of fit values. This implies that, the LLIEW distribution provides a better fit to the data set than KIE, EEFr and IE distributions.

Table 5.18: Model selection criteria for the air condition data

Model	Log-likelihood	AIC	AICc	BIC	W^*	K-S	*P*-value
LLIEW	-152.7700	313.5482	315.1482	319.1530	0.0819	0.1079	0.8763
EEFr	-152.95	313.89	315.4951	319.4999	0.0754	0.1299	0.6924
KIE	-157.1900	320.3859	321.3089	324.5895	0.1497	0.2108	0.1389
IE	-159.0600	320.1239	320.2668	321.5251	0.1436	0.2330	0.0771

The empirical density and the fitted densities are shown on the left of Figure 5.21. Also, the empirical CDF and the fitted CDFs are shown on the right of the graph. The LLIEW distribution provide a better fit to the data set as compare to the other distributions.

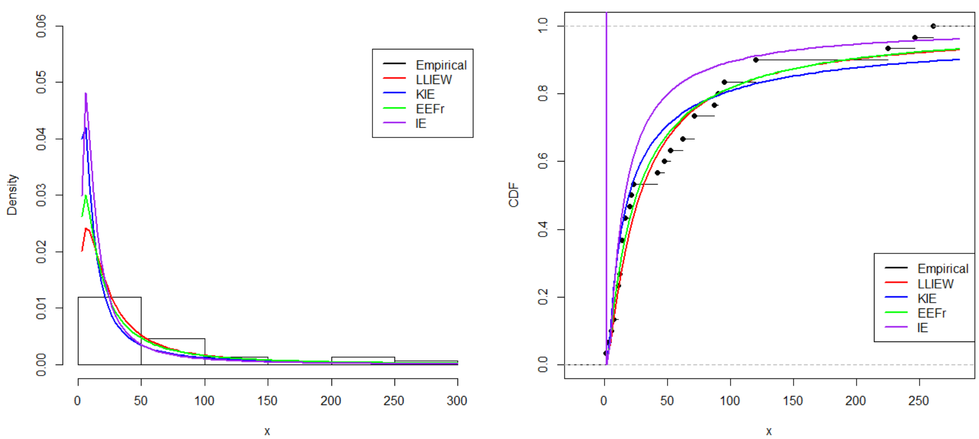

Figure 5.21: PDF and CDF for the air condition data

The probability plots for the fitted distributions are shown in Figure 5.22. It can be observed that, LLIEW distribution provide a better fit to the data set than the other distributions. This is because, the plot of LLIEW observed probability against the expected cluster closely along the diagonal as compare to the competing models.

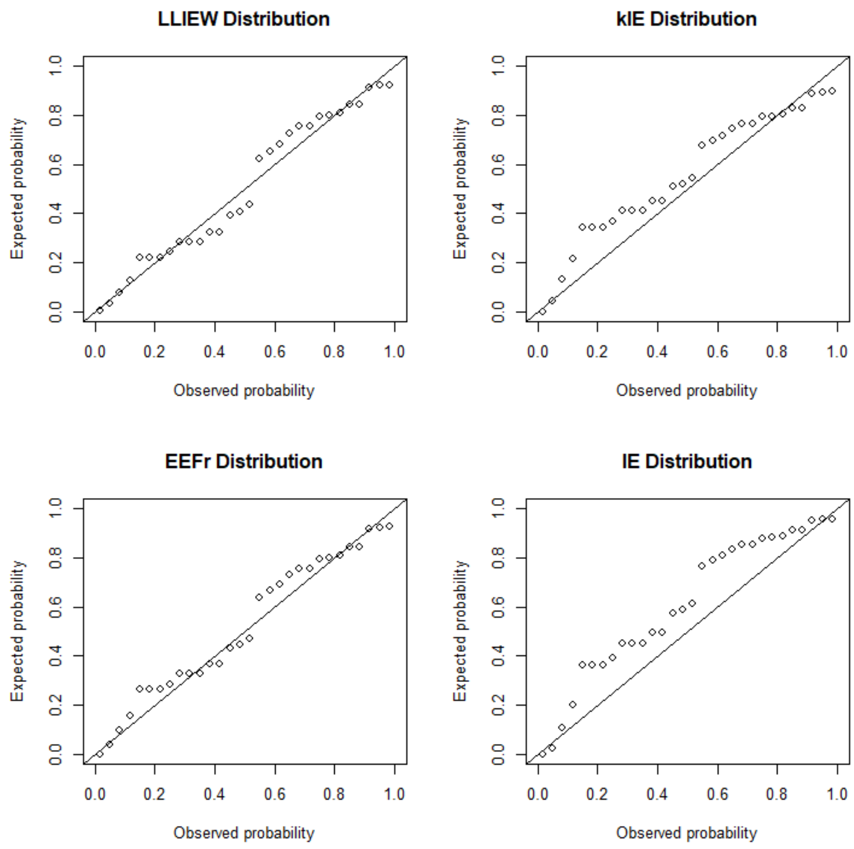

Figure 5.22: P-P plot for air condition data

The asymptotic variance-covariance matrix for the estimated parameters of the LLIEW distribution is given by

$$v^{-1}=\begin{bmatrix} 1.2770\times10^{-8} & -6.7776\times10^{-7} & -1.0860\times10^{-7} & 3.3744\times10^{-6} \\ -6.7776\times10^{-7} & 3.9609\times10^{-5} & 5.7990\times10^{-6} & -1.7946\times10^{-4} \\ -1.0860\times10^{-7} & 5.7990\times10^{-6} & 9.2391\times10^{-7} & -2.8701\times10^{-5} \\ 3.3744\times10^{-6} & -1.7946\times10^{-4} & -2.8701\times10^{-5} & 8.9172\times10^{-4} \end{bmatrix}.$$

Therefore, the estimated 95% CI for the parameters α, λ, θ and γ are respectively given as $[31.3158,\ 31.3162]$, $[0,\ 0.0200]$, $[6.3324,\ 6.3362]$ and $[0.5524,\ 0.6696]$. From the estimates it can be observed that, λ contains zero. This implies that, the estimated parameters of LLIEW distribution were significant at 5% CI except λ.

5.2.3.3 Endurance deep-groove ball bearing data

The uncensored data set on the endurance deep-groove ball bearing data was used to compare the performance of LLIEW distribution to Weibull Burr XII (WBXII) and Weibull Lomax (WLx) distributions. The data set was retrieved from Ghosh (2013) and it is Presented in Table 9, Appendix B.

Table 19, Appendix B presents the descriptive statistics of the data set. From Table 5.26, the data set is positively skewed and platykurtic (skewness=0.9424 and kurtosis=0.3514). Also, the minimum and the maximum revolutions before fatigue failure are 17.8800 and 173.4000.

The data set has an increasing failure rate since the TTT-transform curve is concaved above the 45^0 line as shown in Figure 5.23.

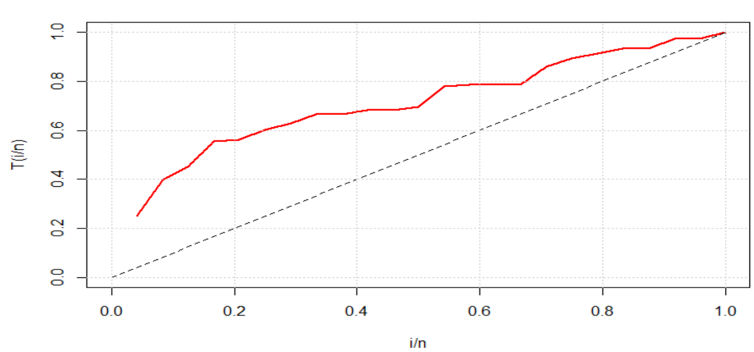

Figure 5.23: TTT-transform plot for endurance deep-groove ball bearing data

Table 5.19 presents the MLEs of the parameters, standard errors, z-values and the *P*-values of the LLIEW, WBXII and WLx distributions. All the parameters of LLIEW distribution were significant at 5% significance level except λ. Three out of four parameters of WLx distribution were significant at 5% level of significance. For, WBXII distribution, only the parameter b was significant at the 5% level of significance.

Table 5.19: Maximum likelihood estimates for endurance deep-groove ball bearing data

Model	Estimate	Standard error	z-value	*p*-value
LLIEW	α=29.2697	6.2432×10^{-5}	4.6883×10^{5}	$<2.2 \times 10^{-16*}$
	λ=0.0437	3.5618×10^{-2}	1.2263×10^{0}	2.201×10^{-1}
	θ=14.6307	9.5570×10^{-4}	1.5309×10^{4}	$<2.2 \times 10^{-16*}$
	γ=0.5876	3.3328×10^{-2}	1.7629×10^{1}	$<2.2 \times 10^{-16*}$
WBXII	α =0.7445	5.6715×10^{-1}	1.3127×10^{0}	1.893×10^{-1}
	β=0.3896	3.9601×10^{-1}	9.8380×10^{-1}	3.252×10^{-1}
	a=0.0061	2.3361×10^{-2}	2.6040×10^{-1}	7.945×10^{-1}
	b=5.3096	4.4503×10^{-2}	1.1791×10^{2}	$<2.2 \times 10^{-16*}$
WLx	a=0.0037	4.410×10^{-3}	8.4590×10^{-1}	3.9760×10^{-1}
	b=0.2550	4.8320×10^{-2}	5.2783×10^{0}	$1.3040 \times 10^{-7*}$
	α =8.6024	1.4465×10^{-3}	5.9473×10^{3}	$<2.2 \times 10^{-16*}$
	β=6.8251	5.9428×10^{-5}	1.1485×10^{5}	$<2.2 \times 10^{-16*}$

*: means significant at $\leq 5\%$ level of significance

Table 5.20 presents the goodness of fit statistics and the log-likelihood values for the endurance deep-groove ball data. The LLIEW distribution has the highest log-likelihood value and lowest goodness of fit values. This implies that, the LLIEW distribution provides a better fit to the data set than WBXII and WLx distributions.

Table 5.20: Model selection criteria for endurance deep-groove ball bearing data

Model	Log-likelihood	AIC	AICc	BIC	W^*	K-S	*P*-value (K-S)
LLIEW	-117.7300	243.4557	245.5609	248.1679	0.0362	0.1041	0.9573
WBXII	-117.96	243.9139	246.0191	248.6261	0.0699	0.1434	0.7075
WLx	-118.71	245.4268	247.5321	250.1390	0.0881	0.1459	0.6864

The empirical density and the fitted densities are shown on the left of Figure 5.24. Also, the empirical CDF and the fitted CDFs are shown on the right of the graph. The LLIEW distribution provide a better fit to the data set as compare to the other distributions.

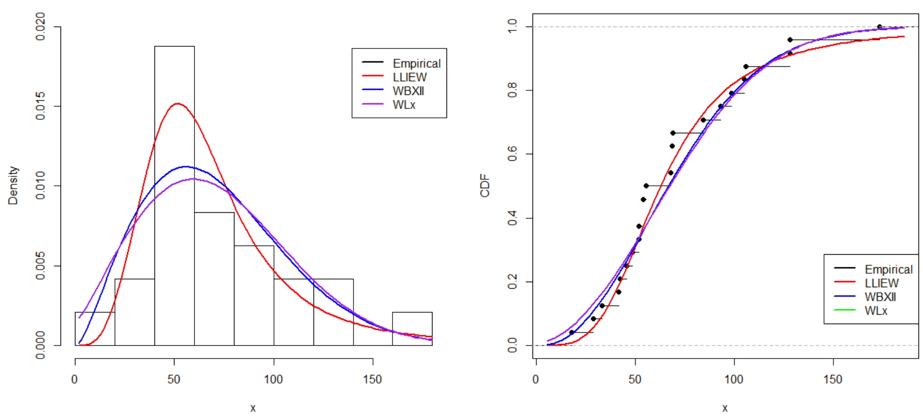

Figure 5.24: PDF and CDF for endurance deep-groove ball bearing data

Figure 5.25 presents the probability plots for the fitted distributions. It can be seen that, LLIEW distribution provide a better fit to the data set than the WBXII and WLx distributions.

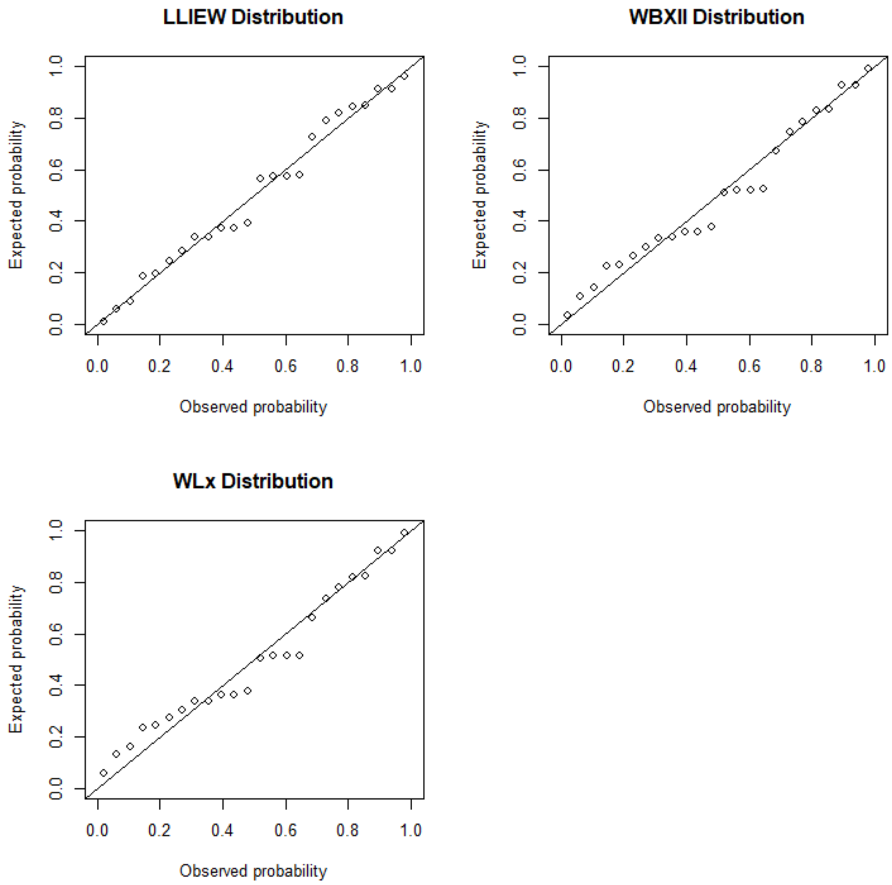

Figure 5.25: P-P plots for the fitted distributions

Hence, the approximate 95% CI for the parameters α, λ, θ and γ are $[29.2696,\ 29.2698]$, $[0,\ 0.1135]$, $[14.6288,\ 14.6326]$ and $[0.5224,\ 0.6528]$ respectively. From the estimates it can be observed that, λ contains zero. This implies that, the estimated parameters of LLIEW distribution were all significant at 5% CI except λ.

5.2.3.4 6061-T6 aluminum data

The uncensored data set on the fatigue life of 6061-T6 aluminum coupons data was used to compare the performance of LLIEW distribution to Lomax-Weibull {Log-logistic} (LWLL) distribution, WBXII, exponentiated-exponential Fréchet (EEFr) and WLx distributions. The data set was retrieved from Jamal (2017) and shown in Table 10, Appendix B.

Table 20, Appendix B presents the descriptive statistics of the data set. The data set is positively skewed and platykurtic (skewness=0.3256 and kurtosis=0.9730).

The data set has an increasing failure rate since the TTT-transform curve is concaved above the 45^0 line as shown in Figure 5.26.

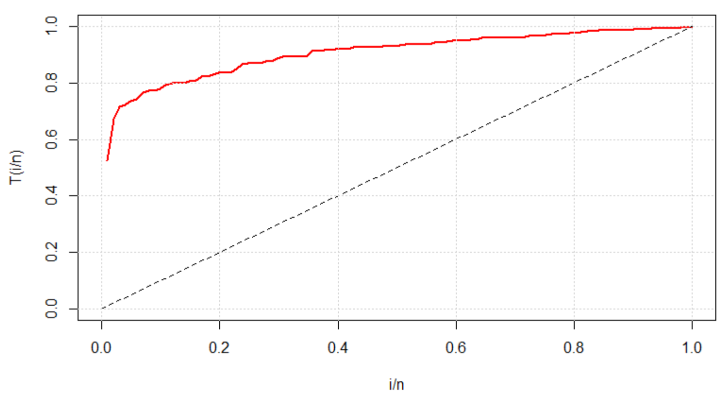

Figure 5.26: TTT-transform plot for fatigue life of 6061-T6 aluminum data

Table 5.21 presents the MLEs of the parameters, standard errors, z-values and the *P*-values of the LLIEW, LWLL, EEFr, WBXII and WLx distributions. All the parameters of LLIEW, LWLL, and WBXII distributions were significant at 5% significance. Most of the parameters for WBXII and EEFr were significant at the 5% level of significance.

Table 5.21: Maximum likelihood estimates for fatigue life of 6061-T6 aluminum data

Model	Estimate	Standard error	z-value	*p*-value
LLIEW	α=0.9848	$3.1506\text{x}10^{-4}$	$3.1258\text{x}10^{3}$	$<2.2\text{ x}10^{-16*}$
	λ=464.9026	$6.9009\text{x}10^{-9}$	$6.7369\text{x}10^{10}$	$<2.2\text{ x}10^{-16*}$
	θ=672.4969	$5.2453\text{x}10^{-11}$	$1.2821\text{x}10^{13}$	$<2.2\text{ x}10^{-16*}$
	γ=229.5022	$1.4183\text{x}10^{-8}$	$1.6182\text{x}10^{10}$	$<2.2\text{ x}10^{-16*}$
LWLL	k=60.9704	$3.5191\text{ x}10^{-4}$	$1.7325\text{ x}10^{5}$	$<2.2\text{ x}10^{-16*}$
	β=0.1626	$2.2701\text{ x}10^{-2}$	$7.1611\text{ x}10^{0}$	$8.002\text{ x}10^{-13*}$
	c=0.8203	$7.0713\text{ x}10^{-2}$	$1.1601\text{ x}10^{1}$	$<2.2\text{ x}10^{-16*}$
	γ =419.3146	$3.1412\text{ x}10^{-4}$	$1.2249\text{ x}10^{6}$	$<2.2\text{ x}10^{-16*}$
EEFr	α=0.2262	$1.806\text{ x}10^{-1}$	$1.2525\text{ x}10^{0}$	$2.1039\text{ x}10^{-1}$
	β=46.5429	$2.3511\text{ x}10^{1}$	$1.9796\text{ x}10^{0}$	$4.7750\text{ x}10^{-2*}$
	σ=54.1631	$4.9520\text{ x}10^{0}$	$1.0938\text{ x}10^{1}$	$2.0\text{ x}10^{-16*}$
	θ=21.3447	$1.6971\text{ x}10^{1}$	$1.2577\text{ x}10^{0}$	$2.0849\text{ x}10^{-1}$
WBXII	α=0.6063	$1.0585\text{ x}10^{-1}$	$5.7274\text{ x}10^{0}$	$1.020\text{ x}10^{-8*}$
	β=0.3316	$1.4337\text{ x}10^{-1}$	$2.3121\text{ x}10^{0}$	$2.071\text{ x}10^{-2*}$
	a=0.0024	$2.1884\text{ x}10^{-2}$	$1.1180\text{ x}10^{-1}$	$9.1095\text{ x}10^{-1}$
	b=10.8948	$1.5840\text{ x}10^{0}$	$6.8779\text{ x}10^{0}$	$6.074\text{ x}10^{-12*}$
WLx	a=0.0015	$1.1836\text{ x}10^{-3}$	$1.2420\text{ x}10^{0}$	$<2.142\text{ x}10^{-1}$
	b=0.2796	$3.3180\text{x}10^{-2}$	$8.4255\text{ x}10^{0}$	$<2.2\text{ x}10^{-16*}$
	α=12.5070	$1.9273\text{ x}10^{-5}$	$6.4893\text{ x}10^{5}$	$<2.2\text{ x}10^{-16*}$
	β=25.1074	$6.8153\text{ x}10^{-5}$	$3.6307\text{ x}10^{5}$	$<2.2\text{ x}10^{-16*}$

*: means significant at $\leq 5\%$ level of significance

Table 5.22 presents the goodness of fit statistics and the log-likelihood values for fatigue life of 6061-T6 aluminum coupons data. The LLIEW distribution has the highest log-likelihood value and lowest goodness of fit values. This implies that, the LLIEW distribution provides a better fit to the data set than the competing distributions.

Table 5.22: Model selection criteria for the fatigue life of 6061-T6 aluminum coupons

Model	Log-likelihood	AIC	AICc	BIC	W*	K-S	*P*-value (K-S)
LLIEW	-458.9600	925.9163	926.3330	936.3768	0.1470	0.0796	0.5437
LWLL	-462.9800	933.9587	934.3754	944.4192	0.1571	0.0997	0.2674
EEFr	-477.1600	962.3133	962.7299	972.7738	0.4533	0.1403	0.0375
WBXII	-488.4800	984.9532	985.3698	995.4137	0.0580	0.2567	$3.305\text{x}10^{-6}$
WLx	-497.4500	1002.8920	1003.3080	1013.3520	0.0623	0.2956	$4.312\text{x}10^{-8}$

From figure 5.27, it can be seen that both the PDF and the CDF of LLIEW distribution mimic the empirical PDF and the empirical CDF. Hence, LLIEW fit the data set well.

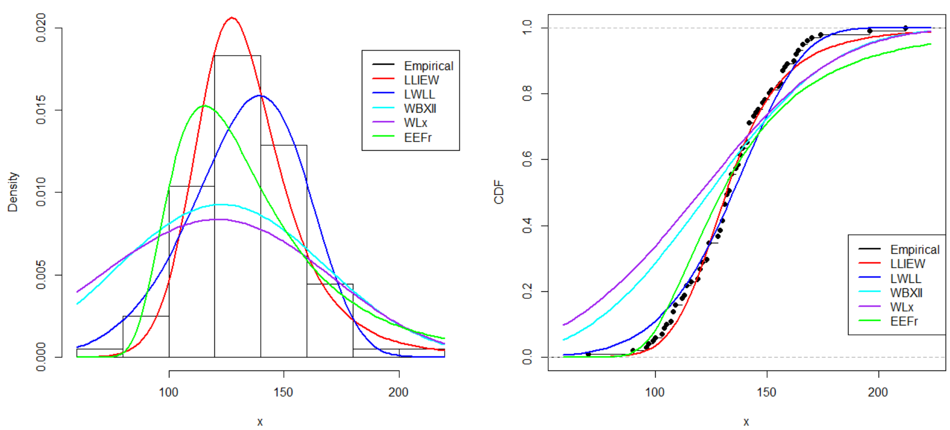

Figure 5.27: PDFs and CDFs of the fitted distributions

Figure 5.28 presents the probability plots for the fitted distributions. It can be seen that, LLIEW distribution provide a better fit to the data set than the competing distributions. This is because, the plot of expected probability against the observed probability indicates that, the data cluster closely along the diagonal than the competing distributions.

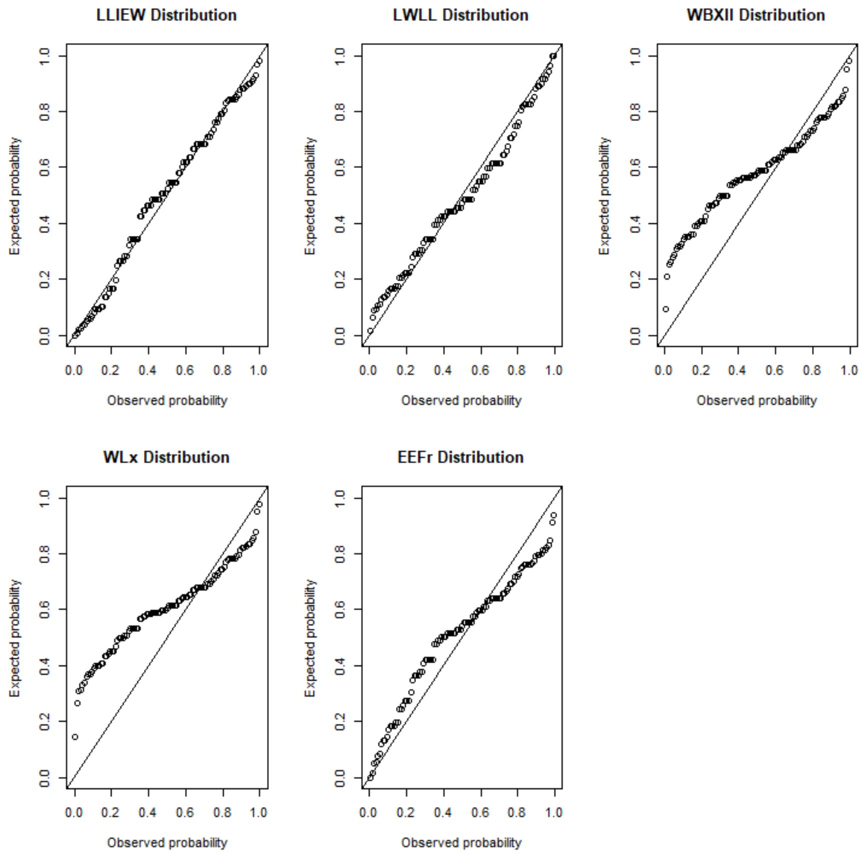

Figure 5.28: P-P plots of the fitted distributions

The variance-covariance matrix for the estimated parameters of LLIEW distribution is given by

$$v^{-1} = \begin{bmatrix} 9.9262\times10^{-8} & 2.1742\times10^{-12} & 1.6526\times10^{-14} & -4.4683\times10^{-12} \\ 2.1742\times10^{-12} & 4.7622\times10^{-17} & 3.6197\times10^{-19} & -9.7872\times10^{-17} \\ 1.6526\times10^{-14} & 3.6197\times10^{-19} & 2.7513\times10^{-21} & -7.4392\times10^{-19} \\ -4.4683\times10^{-12} & -9.7872\times10^{-17} & -7.4392\times10^{-19} & 2.0115\times10\times-16 \end{bmatrix}.$$

Hence, the approximate 95% CI for the parameters α, λ, θ and γ are $[0.9842,\ 0.9854]$, $[464.9026,\ 464.9026]$, $[672.4969,\ 672.4969]$ and $[229.5022,\ 229.5022]$ respectively. From the estimates, it can be observed that, none of them contain zero. This means that, the estimated parameters of LLIEW distribution were all significant at 5% CI.

5.2.3.5 New treatment data

An application of LLIEW distribution on censored data set is given. The AIC, AICc and BIC were used to compare LLIEW distribution with WBXII, WLx, MBIII and EEFr distributions. The data represent 30 people who were administered a new treatment and compare with another set who were given an old treatment. This was to compare the efficacy and safety of the new treatment. The event of interest for the study was death. The data was retrieved from Peat and Barton (2005) and shown in Table 11, Appendix B.

The data set has an increasing failure rate since the TTT-transform curve is concaved above the 45^0 line as shown in Figure 5.29.

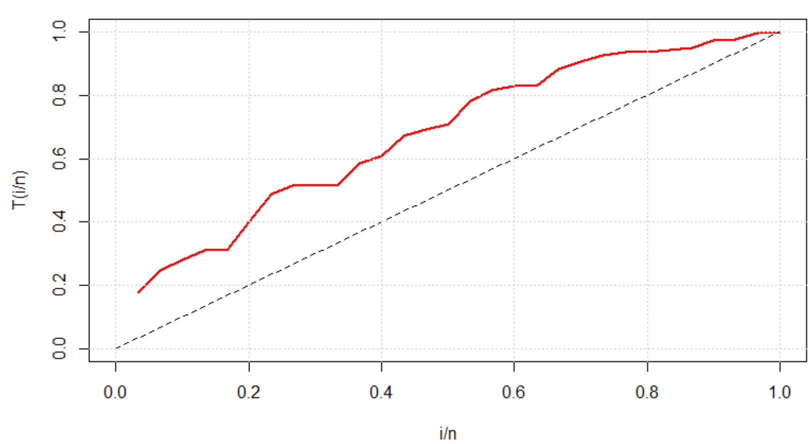

Figure 5.29: TTT-transform plot for the new treatment data

Table 5.23 presents the MLEs of the parameters, standard errors, z-values and the *P*-values of the LLIEW, EEFr, WBXII, WLx and MBIII distributions. All the parameters of LLIEW distributions were significant at 5% significance.

Table 5.23: Maximum likelihood estimates for new treatment data

Model	Estimate	Standard error	z-value	*P*-value
LLIEW	$\alpha = 1.0042$	8.9483×10^{-3}	1.1223×10^{2}	$< 2.2\times10^{-16}$ *
	$\lambda = 51.9041$	1.1184×10^{1}	4.6410×10^{0}	3.468×10^{-6} *
	$\theta = 73.9547$	8.8072×10^{0}	8.3971×10^{0}	$< 2.2\times10^{-16}$ *
	$\gamma = 122.1440$	5.2675×10^{0}	2.3188×10^{1}	$< 2.2\times10^{-16}$ *
EEFr	$\alpha = 6.7559$	1.3282×10^{1}	5.086×10^{-1}	6.110×10^{-1}
	$\beta = 13.2263$	8.3600×10^{1}	1.582×10^{-1}	8.743×10^{-1}
	$\sigma = 102.1598$	5.3650×10^{0}	1.9042×10^{1}	$< 2.2\times10^{-16}$ *
	$\theta = 0.2090$	4.1738×10^{-1}	5.008×10^{-1}	6.165×10^{-1}
WBXII	$\alpha = 1.8978$	1.8583×10^{0}	1.0213×10^{0}	3.071×10^{-1}
	$\beta = 0.8840$	1.0890×10^{0}	8.1180×10^{-1}	4.169×10^{-1}
	$a = 0.0029$	5.3721×10^{-3}	5.341×10^{-1}	5.933×10^{-1}
	$b = 0.7513$	9.6251×10^{-1}	7.805×10^{-1}	4.351×10^{-1}
WLx	$a = 0.0030$	4.9069×10^{-3}	6.1960×10^{-1}	5.355×10^{-1}
	$b = 0.0395$	1.3901×10^{-2}	2.8402×10^{0}	4.509×10^{-3} *
	$\alpha = 36.2046$	1.4301×10^{-5}	2.5315×10^{6}	$< 2.2\times10^{-16}$ *
	$\beta = 1.6803$	1.1109×10^{-5}	1.5125×10^{5}	$< 2.2\times10^{-16}$ *
MBIII	$\alpha = 638.6739$	1.2456×10^{2}	5.1276×10^{0}	2.934×10^{-7} *
	$\beta = 1.5098$	1.8543×10^{-1}	8.1421×10^{0}	3.886×10^{-16} *
	$\lambda = 559.3717$	1.1292×10^{2}	4.9536×10^{0}	7.286×10^{-7} *

Table 5.24 presents the goodness of fit statistics and the log-likelihood values for the new treatment data. The LLIEW distribution has the highest log-likelihood value and lowest information criteria statistics. This implies that, the LLIEW distribution provides a better fit to the data set than the EEFr, MBIII, WBXII and WLx distributions.

Table 5.24: Model selection criteria for the new treatment data

Model	Log-likelihood	AIC	AICc	BIC
LLIEW	-34.0300	76.0513	77.6513	81.6560
EEFr	-34.5300	77.0567	78.6567	82.6615
MBIII	-35.0600	76.1148	77.0379	80.3184
WBXII	-35.2900	78.5791	80.1791	84.1839
WLx	-35.4400	78.8760	80.4760	84.4808

The variance covariance matrix for the estimated parameters is given by

$$v^{-1} = \begin{bmatrix} 8.0072\times10^{-5} & -7.0861\times10^{-2} & -5.5803\times10^{-2} & 3.3375\times10^{-2} \\ -7.0861\times10^{-2} & 1.2508\times10^{2} & 9.8498\times10^{1} & -5.8911\times10^{1} \\ -5.5803\times10^{-2} & 9.8498\times10^{1} & 7.7567\times10^{1} & -4.6392\times10^{1} \\ 3.3375\times10^{-2} & -5.8911\times10^{1} & -4.6392\times10^{1} & 2.7747\times10^{1} \end{bmatrix}.$$

Hence, the estimated 95% CI for the parameters α, λ, θ and γ are $[0.9867, 1.0217]$, $[29.9835, 73.8247]$, $[56.6926, 91.2168]$ and $[111.8197, 132.4683]$ respectively. From the estimates, it can be observed that, none of them contain zero. This implies that, the estimated parameters of LLIEW distribution were all significant at 5% CI.

CHAPTER SIX

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

6.0 Introduction

This chapter presents summary, conclusions and recommendations.

6.1 Summary

In this study, a new statistical family of distributions called T-IE$\{Y\}$ family of distributions was proposed and studied using T-X{Y} frame work. Three sub-families of T-IE$\{Y\}$family distributions were developed. These were T-IE{Weibull}, T-IE{Logistic} and T-IE{Lomax} family of distributions. The three sub-families of distributions were used to develop three special distributions. These distributions were LLIEW, GIEL and WIEL distribution. Both LLIEW distribution and WIEL distribution have four parameters each while GIEL distribution have two parameters.

Plots for the CDFs, PDFs and hazard rate functions of these distributions have been provided. It is evident from Figure 4.2 that the density function of LLIEW can be positively skewed, negatively skewed, J-shape and reverse-J shape and unimodal. Also, the hazard rate function of LLIEW distribution could be increasing, decreasing or inverted bathtub.

Furthermore, the density function of GIEL distribution is unimodal, positively skewed and reverse-J with the hazard rate function being increasing, decreasing and inverted bathtub shape. Also, the density of WIEL distribution is positively skewed and symmetric with the hazard rate function being decreasing and inverted bathtub shape.

The statistical properties of the T-IE$\{Y\}$ family of distributions such as quantile function, mode, moments and Shannon entropy were derived and five methods of parameter estimating procedures were discussed. These methods are Maximum likelihood estimation, Ordinary Least squares estimators, Weighted least square estimators, Percentile based estimators and Cramér-von Mises estimators.

In addition, the results of simulation and real data application of the special distributions were developed. Simulation was done to compare the performance of the estimators using GIEL distribution. The MLE emerged as the best method in estimating the parameters of GIEL distribution.

Applications of the special distributions were illustrated using eleven (11) lifetime data sets of which two (2) data set are censored and one (1) is bimodal. The special distributions were compare with existing distributions. With these data sets, the special distributions performed generally better than the other distributions.

6.2 Conclusions

The T-IE{Y} family of distributions are improvement of the IE distribution. This new family of distributions was used to develop three special cases namely, LLIEW, WIEL and GIEL distributions. The PDFs and the hazard rate functions of these special cases have various shapes. For instance, the density function of LLIEW distribution assume various shapes such as positively skewed, negatively skewed, J-shape and reversed-J shape by different combinations of parameter values. The hazard rate function on the other hand may take both monotonic and non-monotonic shapes namely increasing, decreasing, increasing then decreasing and inverted bathtub shapes.

The density function of WIEL distribution can be approximately symmetric, positively skewed and unimodal with varying degree of kurtosis. The hazard rate function for the WIEL distribution could be decreasing, increasing then decreasing and inverted bathtub. This implies that, WIEL distribution can be used to modal both monotonic and non-monotonic failure rates.

The density function of GIEL distribution can be positively skewed or reversed-J with varying combination of parameter values. The hazard rate function on the other hand could be increasing, decreasing and inverted bathtub. This implies that, GIEL distribution can be used to modal both monotonic and non-monotonic failure rates.

The statistical properties of the T-IE{Y} family of distributions were derived. Five estimation procedures were developed and Monte Carlo simulation performed to examine the performance of the estimators. Among the estimation procedures, the MLE was seen as the best method in estimating the parameters of GIEL distribution.

The GIEL distribution was applied to two (2) complete data sets and one censored data set. In all these application, in can be concluded that, the GIEL distribution can applied to both censored and uncensored data sets. This is because, it performed better in both type of data sets than the competing distributions such as GIE, KIE, BXII and WD.

The WIEL distribution was applied to three complete data set of which one is a bimodal data set. It can be concluded from the application that, the WIEL distribution can be applied to data sets that are unimodal, bimodal, over-dispersed, extremely skewed and data sets with heavy tails.

The LLIEW distribution was applied to five (5) data sets of which one is a censored data set. The distributions of these data sets varied in degree of skewness and kurtosis. From these applications, it can be concluded that, LLIEW distribution can be applied favorably to data sets that are extremely skewed, heavily tailed, over-dispersed and data sets that have bathtub failure rate.

6.2 Recommendations

The PDFs of the special distributions are positively skewed, negatively skewed, reverse j-shape and j-shape. Also, the hazard rate functions are increasing, decreasing, inverted bathtub and bathtub failure rates. Hence, it is recommended that, any data set that exhibit any of the above characteristics can be modeled using the T-IE{Y} family of distributions.

The data sets used in this study were both complete and censored data sets with varying degree of skewness and kurtosis. These data sets were either unimodal or bimodal. However, multimodal samples may arise in different fields of study. For example, the covid-19 cases have had multi-waves with different peaks across the globe. Hence, further demonstration of the application of the developed distributions should consider the use of multimodal data sets.

Events are influenced by a number of different factors. For instance, maternal mortality maybe caused by a number of factors including, age of the expectant mother, environmental factors, and socioeconomic factors. It is necessary to investigate the contribution of each factor to the mortality rate for better policy decision. Thus, parametric regression models for the special distributions can be developed to examine the relationship between the dependent and the independent variables of the distributions.

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APPENDICES

APPENDIX A1

Second derivative of the total log-likelihood function of LLIEW distribution with respect to the various parameters.

$$\frac{\partial^2 \ell}{\partial \gamma^2}=\frac{n}{\gamma^2}+\frac{2\sum_{i=1}^{n}\log(z_i)}{\gamma^3}+(\theta-1)\sum_{i=1}^{n}\left(\frac{2\log[\log(z_i)]\log(z_i)^{\frac{1}{\gamma}}}{\gamma^3}+\frac{\log[\log(z_i)]^2\log(z_i)^{\frac{1}{\gamma}}}{\gamma^4}\right)-$$

$$2\sum_{i=1}^{n}\left(-\frac{\theta^2\log(z_i)^2 z_i^{2/\gamma}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{2\theta-2}}{\alpha^2\gamma^4\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)^2}\right)+\frac{(\theta-1)\theta\log(z_i)^2 z_i^{\frac{2}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-2}}{\alpha^2\gamma^4\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)}$$

$$\frac{2\theta\log(z_i)z_i^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha\gamma^3\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)}+\frac{\theta\log(z_i)^2 z_i^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha\gamma^4\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)}$$

$$\frac{\partial^2 \ell}{\partial \gamma\partial \theta}=\sum_{i=1}^{n}-\frac{\log[\log(z_i)]\log(z_i)^{\frac{1}{\gamma}}}{\gamma^2}-$$

$$2\sum_{i=1}^{n}\left(\frac{\theta\log(z_i)\log\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)(z_i)^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)\Big)^{2\theta-1}}{\alpha\gamma^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)^2}-\frac{\log(z_i)z_i^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha\gamma^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)}-\frac{\theta\log(z_i)\log[\alpha^{-1}(z_i)^{\frac{1}{\gamma}}]z_i^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha\gamma^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)}\right)$$

$$\frac{\partial^2 \ell}{\partial \gamma\partial \alpha}=-2\sum_{i=1}^{n}\left(-\frac{\theta^2\log(z_i)\left(z_i\right)^{2/\gamma}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{2\theta-2}}{\alpha^3\gamma^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)^2}+\frac{(\theta-1)\theta\log(z_i)\left(z_i\right)^{\frac{2}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-2}}{\alpha^3\gamma^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)}+\frac{\theta\log(z_i)\left(z_i\right)^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha^2\gamma^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)}\right)$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{n\theta}{\alpha^2} - 2\sum_{i=1}^{n}\left(-\frac{\theta^2 (z_i)^{2/\gamma}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{2\theta-2}}{\alpha^4\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)^2} + \frac{(\theta-1)\theta(z_i)^{2/\gamma}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-2}}{\alpha^4\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)} + \frac{2\theta(z_i)^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha^3\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)} \right)$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \theta} = -\frac{n}{\alpha} - 2\sum_{i=1}^{n}\left(\frac{\theta \log(\alpha^{-1}(z_i)^{\frac{1}{\gamma}})(z_i)^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{2\theta-1}}{\alpha^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)^2} - \frac{(z_i)^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)} - \frac{\theta \log(\alpha^{-1}(z_i)^{\frac{1}{\gamma}})(z_i)^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)} \right)$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \gamma} = -2\sum_{i=1}^{n}\left(\frac{\theta^2 \log[z_i]\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right) z_i^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{2\theta-2}}{\alpha^3\gamma^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)^2} + \frac{(\theta-1)\log(z_i) z_i^{\frac{2}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-2}}{\alpha^3\gamma^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)} + \frac{\theta \log[z_i]\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right) z_i^{\frac{1}{\gamma}}\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta-1}}{\alpha^3\gamma^2\left(1+\left(\alpha^{-1}(z_i)^{\frac{1}{\gamma}}\right)^{\theta}\right)} \right)$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2} + \sum_{i=1}^{n}\left(\frac{\exp(-\frac{2\lambda}{x_i})}{\left(1-\exp(-\frac{\lambda}{x_i})\right)^2 x_i^2} + \frac{(\exp(-\frac{\lambda}{x_i})}{\left(1-\exp(-\frac{\lambda}{x_i})\right) x_i^2} \right)$$

APPENDIX A2

Second derivative of the total log-likelihood function of GIEL distribution with respect to the various parameters.

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2} + \left(\frac{1}{\alpha} - 1\right)\sum_{i=1}^{n}\left(-\frac{\exp(\frac{2\lambda}{x_i})}{\left(-1+\exp(\frac{\lambda}{x_i})\right)^2 x_i^2} + \frac{\exp(\frac{\lambda}{x_i})}{\left(-1+\exp(\frac{\lambda}{x_i})x_i^2\right)} \right) -$$

$$\sum_{i=1}^{n}\left(\frac{\exp(\frac{\lambda}{x_i})\left(\exp(\frac{\lambda}{x_i})-1\right)^{-1+\frac{1}{\alpha}}}{\alpha x_i^2} + \frac{\left(\frac{1}{\alpha}-1\right)\exp(\frac{2\lambda}{x_i})\left(\exp(\frac{\lambda}{x_i})-1\right)^{\frac{1}{\alpha}-2}}{\alpha x_i^2} \right)$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = \sum_{i=1}^{n} \left(\frac{\exp(\frac{\lambda}{x_i})\left(\exp(\frac{\lambda}{x_i}) - 1\right)^{-1+\frac{1}{\alpha}}}{\alpha^2 x_i} + \frac{\exp(\frac{\lambda}{x_i})\left(\exp(\frac{\lambda}{x_i}) - 1\right)^{-1+\frac{1}{\alpha}} \log[\exp(\frac{\lambda}{x_i}) - 1]}{\alpha^3 x_i} \right) - \frac{\sum_{i=1}^{n} \frac{\exp(\frac{\lambda}{x_i})}{\left(\exp(\frac{\lambda}{x_i}) - 1\right) x_i}}{\alpha^2}$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{n}{\alpha^2} - \sum_{i=1}^{n} \left(\frac{2\left(\exp(\frac{\lambda}{x_i}) - 1\right)^{\frac{1}{\alpha}} \log[\exp(\frac{\lambda}{x_i}) - 1]}{\alpha^3} + \frac{\left(\exp(\frac{\lambda}{x_i}) - 1\right)^{\frac{1}{\alpha}} \log[\exp(\frac{\lambda}{x_i}) - 1]^2}{\alpha^4} \right)$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n} \left(\frac{\exp(\frac{\lambda}{x_i})\left(\exp(\frac{\lambda}{x_i}) - 1\right)^{\frac{1}{\alpha}-1}}{\alpha^2 x_i} + \frac{\exp(\frac{\lambda}{x_i})\left(\exp(\frac{\lambda}{x_i}) - 1\right)^{\frac{1}{\alpha}-1} \log[\exp(\frac{\lambda}{x_i}) - 1]}{\alpha^3 x_i} \right)$$

APPENDIX A3

Second derivative of the total log-likelihood function of WIEL distribution with respect to the various parameters.

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n}{\beta^2} + \sum_{i=1}^{n} \log[-1 - \alpha(z_i)^{-1/\gamma}]^2 \left(-1 - \alpha(z_i)^{-1/\gamma}\right)^{\beta}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \alpha} = \sum_{i=1}^{n} \left(-z_i^{-1/\gamma} \left(-1 - \alpha(z_i)^{-1/\gamma}\right)^{\beta-1} - \beta \log[-1 - \alpha(z_i)^{-1/\gamma}] z_i^{-1/\gamma} \left(-1 - \alpha(z_i)^{-1/\gamma}\right)^{\beta-1} \right)$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \gamma} = \sum_{i=1}^{n} \frac{\log[z_i](z_i)^{-1/\gamma}}{\gamma^2\left(-1 + (z_i)^{-1/\gamma}\right)} +$$

$$\sum_{i=1}^{n} \left(-\frac{\alpha \log[z_i](z_i)^{-1/\gamma}\left(-1 - \alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma^2} - \frac{\alpha\beta \log[z_i] \log[-1 - \alpha(z_i)^{-1/\gamma}](z_i)^{-1/\gamma}\left(-1 - \alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma^2} \right)$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n} (\beta - 1)\beta (z_i)^{-2/\gamma} \left(-1 - \alpha(z_i)^{-1/\gamma}\right)^{\beta-2}$$

$$\frac{\partial^2 \ell}{\partial\alpha\partial\beta}=\sum_{i=1}^{n}\left(-\left(z_i\right)^{-1/\gamma}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}-\beta log[-1-\alpha(z_i)^{-1/\gamma}](z_i)^{-1/\gamma}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}\right)$$

$$\frac{\partial^2 \ell}{\partial\alpha\partial\gamma}=\sum_{i=1}^{n}\left(\frac{\alpha\left(\beta-1\right)\beta\log[z_i](z_i)^{-2/\gamma}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-2}}{\gamma^2}-\frac{\beta\log[z_i](z_i)^{-1/\gamma}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma^2}\right)$$

$$\frac{\partial^2 \ell}{\partial\lambda^2}=-\frac{n}{\lambda^2}+\left(1-\frac{1}{\gamma}\right)\sum_{i=1}^{n}-\frac{\exp(-\frac{\lambda}{x_i})}{x_i^2 z_i}+\left(\beta-1\right)\sum_{i=1}^{n}\frac{\exp(-\frac{\lambda}{x_i})\left(z_i\right)^{-1-\frac{1}{\gamma}}}{\gamma x_i^2\left(-1-\alpha(z_i)^{-1/\gamma}\right)}+\sum_{i=1}^{n}-\frac{\exp(-\frac{\lambda}{x_i})\alpha\beta\left(z_i\right)^{-1-\frac{1}{\gamma}}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma x_i^2}$$

$$\frac{\partial^2 \ell}{\partial\lambda\partial\alpha}=\sum_{i=1}^{n}\left(-\frac{\exp(-\frac{\lambda}{x_i})\alpha\left(\beta-1\right)\beta\left(z_i\right)^{-1-\frac{1}{\gamma}}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-2}}{\gamma x_i}+\frac{\exp(-\frac{\lambda}{x_i})\beta\left(z_i\right)^{-1-\frac{1}{\gamma}}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma x_i}\right)$$

$$\frac{\partial^2 \ell}{\partial\lambda\partial\beta}=\sum_{i=1}^{n}-\frac{\exp(-\frac{\lambda}{x_i})\left(z_i\right)^{-1-\frac{1}{\gamma}}}{\gamma x_i\left(-1+(z_i)^{-1/\gamma}\right)}+$$

$$\sum_{i=1}^{n}\left(\frac{\exp(-\frac{\lambda}{x_i})\alpha\left(z_i\right)^{-1-\frac{1}{\gamma}}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma x_i}+\frac{\exp(-\frac{\lambda}{x_i})\alpha\beta\log[-1-\alpha(z_i)^{-1/\gamma}]z_i^{-1-\frac{1}{\gamma}}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma x_i}\right)$$

$$\frac{\partial^2 \ell}{\partial\lambda\partial\gamma}=\frac{\sum_{i=1}^{n}\frac{\exp(-\frac{\lambda}{x_i})}{x_i z_i}}{\gamma^2}+\left(\beta-1\right)\sum_{i=1}^{n}\left(\frac{\exp(-\frac{\lambda}{x_i})\log[z_i]\left(z_i\right)^{-1-\frac{2}{\gamma}}}{\gamma^3 x_i\left(-1+(z_i)^{-1/\gamma}\right)^2}+\frac{\exp(-\frac{\lambda}{x_i})\left(z_i\right)^{-1-\frac{1}{\gamma}}}{\gamma^2 x_i\left(-1+(z_i)^{-1/\gamma}\right)}-\frac{\exp(-\frac{\lambda}{x_i})\log[z_i]\left(z_i\right)^{-1-\frac{1}{\gamma}}}{\gamma^3 x_i\left(-1+(z_i)^{-1/\gamma}\right)}\right)+$$

$$\sum_{i=1}^{n}\left(-\frac{\exp(-\frac{\lambda}{x})\alpha^2\left(\beta-1\right)\beta\log[z_i]\left(z_i\right)^{-1-\frac{2}{\gamma}}\left(-1+(z_i)^{-1/\gamma}\right)^{\beta-2}}{\gamma^3 x_i}\right)-$$

$$\sum_{i=1}^{n}\left(\frac{\exp(-\frac{\lambda}{x_i})\alpha\beta\left(z_i\right)^{-1-\frac{1}{\gamma}}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma^2 x_i}+\frac{\exp(-\frac{\lambda}{x_i})\alpha\beta\log[z_i]\left(z_i\right)^{-1-\frac{1}{\gamma}}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma^3 x_i}\right)$$

$$\frac{\partial^2 \ell}{\partial \gamma^2} = -\frac{2\sum_{i=1}^{n}\log[z_i]}{\gamma^3} + (\beta-1)\sum_{i=1}^{n}\left(-\frac{\log[z_i]^2 (z_i)^{-2/\gamma}}{\gamma^4\left(-1+(z_i)^{-1/\gamma}\right)^2} - \frac{2\log[z_i](z_i)^{-1/\gamma}}{\gamma^3\left(-1+(z_i)^{-1/\gamma}\right)} + \frac{\log[z_i]^2 (z_i)^{-1/\gamma}}{\gamma^4\left(-1+(z_i)^{-1/\gamma}\right)}\right) +$$

$$\sum_{i=1}^{n}\left(\frac{\alpha^2(\beta-1)\beta\log[z_i]^2 (z_i)^{-2/\gamma}\left(-1+(z_i)^{-1/\gamma}\right)^{\beta-2}}{\gamma^4}\right) +$$

$$\sum_{i=1}^{n}\left(\frac{2\alpha\beta\log[z_i]^2 (z_i)^{-1/\gamma}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma^3} - \frac{\alpha\beta\log[z_i]^2 (z_i)^{-1/\gamma}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma^4}\right) - n\log''(\gamma)$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \alpha} = \sum_{i=1}^{n}\left(\frac{\alpha(\beta-1)\beta\log[z_i](z_i)^{-2/\gamma}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-2}}{\gamma^2} - \frac{\beta\log[z_i](z_i)^{-1/\gamma}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma^2}\right)$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \beta} = \sum_{i=1}^{n}\frac{\log[z_i](z_i)^{-1/\gamma}}{\gamma^2\left(-1+(z_i)^{-1/\gamma}\right)} +$$

$$\sum_{i=1}^{n}\left(-\frac{\alpha\log[z_i](z_i)^{-1/\gamma}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma^2} - \frac{\alpha\beta\log[z_i]\log[-1-\alpha(z_i)^{-1/\gamma}](z_i)^{-1/\gamma}\left(-1-\alpha(z_i)^{-1/\gamma}\right)^{\beta-1}}{\gamma^2}\right)$$

APPENDIX B

DATA

The following are the data used to test the performance of the new family of distributions developed.

1: Relief times of patients receiving Analgesic

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3	1.7	2.3	1.6	2

2: Roofing sheet data

36.8	47.2	35.6	36.7	55.8	58.7	42.3	37.8	55.4	45.2
31.8	48.3	45.3	48.5	52.8	45.4	49.8	48.2	54.5	50.1
48.4	44.2	41.2	47.2	39.1	40.7	40.3	41.2	30.4	42.8
38.9	34	33.2	56.8	52.6	40.5	40.6	45.8	58.9	28.7
37.3	36.8	40.2	58.2	59.2	42.8	46.3	61.2	58.4	38.5
34.2	41.3	42.6	43.1	42.3	54.2	44.9	42.8	47.1	38.9
42.8	29.4	32.7	40.1	33.2	31.6	36.2	33.6	32.9	34.5
33.7	39.9								

3: Dialysis

16	13*	28	318	12	245	9	30	196
154	333	8*	38	70*	25*	4*	177	114
459*	108*	562	24*	66	46*	40	201	156
30	25	26	58	43	30	5*	8	16*
78	8*							

* = means censored

4: Waiting time of bank customers

23	261	87	7	120	14	62	47	225	71
246	21	42	20	5	12	120	11	3	14
71	11	14	11	16	90	1	16	52	95

5: Skin folds

28	98	89	68.9	69.9	109	52.3	52.8	46.7	82.7
42.3	109.1	96.8	98.3	103.6	110.2	98.1	57	43.1	71.1
29.7	96.3	102.8	80.3	122.1	71.3	200.8	80.6	65.3	78
65.9	38.9	56.5	104.6	74.9	90.4	54.6	131.9	68.3	52
40.8	34.3	44.8	105.7	126.4	83	106.9	88.2	33.8	47.6
42.7	41.5	34.6	30.9	100.7	80.3	91	156.6	95.4	43.5
61.9	35.2	50.9	31.8	44	56.8	75.2	76.2	101.1	47.5
46.2	38.2	49.2	49.6	34.5	37.5	75.9	87.2	52.6	126.4
55.6	73.9	43.5	61.8	88.9	31	37.6	52.8	97.9	111.1
114	62.9	36.8	56.8	46.5	48.3	32.6	31.7	47.8	75.1
110.7	70	52.5	67	41.6	34.8	61.8	31.5	36.6	76
65.1	74.7	77	62.6	41.1	58.9	60.2	43	32.6	48
61.2	171.1	113.5	148.9	49.9	59.4	44.5	48.1	61.1	31
41.9	75.6	76.8	99.8	80.1	57.9	48.4	41.8	44.5	43.8
33.7	30.9	43.3	117.8	80.3	156.6	109.6	50	33.7	54
54.2	30.3	52.8	49.5	90.2	109.5	115.9	98.5	54.6	50.9
44.7	41.8	38	43.2	70	97.2	123.6	181.7	146.3	42.3
40.5	64.9	34.1	55.7	113.5	75.7	99.9	91.2	71.6	103.6
46.1	51.2	43.8	30.5	37.5	96.9	57.7	125.9	49	143.5
102.8	46.3	54.4	58.3	34	112.5	49.3	67.2	56.5	47.6
60.4	34.9								

6: Height data

55.00	63.25	64.75	65.25	66.75	67.75	70.00	60.00	63.25	65.00
65.50	66.75	67.75	70.25	60.25	63.25	65.00	65.75	66.75	68.13
70.38	61.00	63.25	65.13	66.00	67.00	68.75	71.00	61.75	63.38
65.13	66.00	67.13	69.00	71.00	62.25	64.00	65.17	66.25	67.25
69.00	71.25	62.25	64.25	65.25	66.25	67.38	69.25	71.75	62.63
64.25	65.25	66.25	67.50	69.50	62.75	64.50	65.25	66.50	67.50
69.88	63.00	64.75	65.25	66.75	67.75	70.00	65.50	68.50	70.75
72.00	72.50	75.00	65.75	68.75	70.75	72.00	72.88	75.00	66.00
69.13	71.00	72.00	73.00	75.50	66.25	69.25	71.00	72.00	73.00
76.00	66.75	69.25	71.50	72.00	73.00	77.00	67.00	69.38	71.50
72.25	73.25	77.13	67.13	69.75	71.63	72.25	73.25	77.50	67.5
70.00	71.75	72.25	73.63	77.50	68.00	70.13	71.75	72.38	74.00
78.50	68.13	70.50	72.00	72.500	74.63				

7: Failure times of Kevlar 373/epoxy strands data set

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.5650	0.5671	0.6566
0.6748	0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113
0.9120	0.9836	1.0483	1.0596	1.0773	1.1733	1.2570	1.2766	1.2985	1.3211
1.3503	1.3551	1.4595	1.4880	1.5728	1.5733	1.7083	1.7263	1.7460	1.7630
1.7746	1.8275	1.8375	1.8503	1.8808	1.8878	1.8881	1.9316	1.9558	2.0048
2.0408	2.0903	2.1093	2.1330	2.2100	2.2460	2.2878	2.3203	2.3470	2.3513
2.4951	2.5260	2.9911	3.0256	3.2678	3.4045	3.4846	3.7433	3.7455	3.9143
4.8073	5.4005	5.4435	5.5295	6.5541	9.0960				

8: Failure times of air conditioning system data

23	261	87	7	120	14	62	47	225	71
246	21	42	20	5	12	120	11	3	14
71	11	14	11	16	90	1	16	52	95

9: endurance deep-groove ball bearing data

17.88	45.6	54.12	68.88	105.84	28.92	48.4	55.56
84.12	127.92	33	51.84	67.8	93.12	128.04	41.52
51.96	68.64	98.64	173.4	42.12	54.12	68.64	105.12

10: 6061-T6 Aluminum data

70	90	96	97	99	100	103	104	104	105
107	108	108	108	109	109	112	112	113	114
114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130
130	130	131	131	131	131	131	132	132	132
133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142
142	142	144	144	145	146	148	148	149	151
151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196
212									

11: New treatment data

5*	7*	8*	9	9*	12	15	16	16*	16*
19*	20*	23*	24*	25*	29*	31*	32	32*	36
38*	40*	41*	41*	42*	43*	48*	49*	58*	59*

*: means censored

Table 12: Summary of relief times of analgesic data

N	Min.	Max.	Med.	Mean	Var.	Skewness	Kurtosis
20	1.100	1.475	1.700	1.900	0.496	1.592	2.347

Table 13: summary of TCS procedure for roofing sheet data

N	Min.	Max.	Med.	Mean	Var.	Skewness	Kurtosis
72	28.7000	61.2000	42.3000	43.0900	68.0859	0.4150	-0.6827

Table 14: Summary of Bank data

N	Min.	Max.	Med.	Mean	Var.	Skewness	kurtosis
100	0.8000	38.5000	4.6750	13.0200	52.3741	1.4728	2.4300

Table 15: Summary of skinfolds data

N	Min.	Max.	Med.	Mean	Var.	Skewness	Kurtosis
202	28.0000	200.0000	58.6000	69.0700	1067.0000	1.1720	1.3228

Table 16: Summary of height data

N	Min.	Max.	Med.	Mean	Var.	Skewness	Kurtosis
126	55.0000	78.5000	68.4400	68.5500	17.2870	-0.0491	0.0045

Table 17: Summary of life of fatigue fracture of Kevlar 373/epoxy data

N	Min.	Max.	Med.	Mean	Var.	Skewness	kurtosis

76	0.0251	9.0960	1.7360	1.9590	2.4774	1.9401	4.94643

Table 18: Summary of air condition data

N	Min.	Max.	Med.	Mean	Var.	Skewness	Kurtosis
30	1.0	261.0	22.0	59.6	5,167.4	1.609	4.967

Table 19: Summary of endurance deep-groove ball bearing data

N	Min.	Max.	Med.	Mean	Var.	Skewness	Kurtosis
24	17.8800	173.4000	61.6800	71.4700	1358.1150	0.9424	0.3514

Table 20: Summary of fatigue life of 6061-T6 aluminum data

N	Min.	Max.	Med.	Mean	Var.	Skewness	Kurtosis
101	70.000	212.000	133.000	133.700	499.778	0.3256	0.9730

APPENDIX C

R CODES

```
#### LLIEW Distribution PDF ####

LLIEW_PDF<-function(x, alpha, lambda, theta, gamma){

  A<-lambda*theta*exp(-(lambda/x))

  B<-(alpha^theta)*gamma*(x^2)*(1-exp(-(lambda/x)))

  C<-(-log(1-exp(-(lambda/x))))^((1/gamma)-1)

  D<-((-log(1-exp(-(lambda/x))))^(1/gamma))^(theta-1)

  E<-(1+(((-log(1-exp(-lambda/x)))^(1/gamma))/alpha)^theta)^2

  PDF<-(A/B)*((C*D)/E)

  return(PDF)}

#### LLIEW Distribution Hazard rate function ####

LLIEW_HRF<-function(x,alpha,lambda,theta,gamma){

  A<-lambda*theta*exp(-(lambda/x))

  B<-(alpha^theta)*gamma*(x^2)*(1-exp(-(lambda/x)))

  C<-(-log(1-exp(-(lambda/x))))^((1/gamma)-1)

  D<-((-log(1-exp(-(lambda/x))))^(1/gamma))^(theta-1)

  E<-1+((-log(1-exp(-(lambda/x))))/alpha)^(theta/gamma)

  HRF<-(A/B)*((C*D)/E)

  return(HRF)
```

```
}

#### LLIEW Distribution Log-likelihood ####

LL_LLIEW<-function(alpha,lambda,theta,gamma){
  A<-(-(lambda/x))
  B<-(-log(1-exp(A)))
  C<-(B)^(1/gamma)
  D<-((alpha^(-1))*C)^theta
  E<-(lambda*theta*exp(A)*(B^((1/gamma)-1))*((B)^(1/gamma))^(theta-1))
  G<-((alpha^theta)*gamma*(x^2)*(1-exp(A))*(1+((alpha^(-1))*C)^theta)^2)
  PDF<-E/G
  LL<--sum(log(PDF))
  return(LL)
}

#### LLIEW Distribution quantile function ####

quantile<-function(alpha,lambda,u){
  A<-(-log(u))^alpha
  quant<-lambda/log(A+1)
  return(quant)
```

```
}

#### LLIEW Distribution Optimization ####

Library(bbmle) ## calling R package bbmle###

Fit<-mle2(LL_LLIEW, start=list(alpha=alpha, lambda=lambda, theta=theta, gamma=gamma),
data=list(x), method=”BFGS”)

Summary(fit) #### summary of results ####


#### GIEL Distribution PDF ####

GIEL_PDF<-function(x,alpha,lambda){

  A<-lambda*(exp(lambda/x))

  B<-alpha*(x)^(2)

  C<-((exp(lambda/x))-1)^((1/alpha)-1)

  D<-exp(-(exp((lambda/x))-1)^(1/alpha))

  PDF<-(A/B)*(C)*(D)

  return(PDF)

}

#### GIEL Distribution Hazard rate function ####

GIEL_HF<-function(x, alpha, lambda){

  A<-exp(lambda/x)
```

```
  B<-alpha*(x^2)

  C<-((lambda*A)/B)*((A-1)^((1/alpha)-1))*exp(-(A-1)^(1/alpha))

  D<-1-exp(-(A-1)^(1/alpha))

  HF<-(C/D)

  return(HF)

}

#### GIEL Distribution Log-likelihood ####

LL_GIEL<-function(alpha,lambda){

  A<-lambda*(exp(lambda/x))

  B<-alpha*(x)^(2)

  C<-((exp(lambda/x))-1)^((1/alpha)-1)

  D<-exp(-(exp((lambda/x))-1)^(-1/alpha))

  PDF<-(A/B)*(C)*(D)

  LL<--sum(log(PDF))

  return(LL)

}

#### GIEL Distribution quantile function ####

quantile<-function(alpha, lambda, u){
```

```
  A<-(-log(u))^alpha
  quant<-lambda/log(A+1)
  return(quant)
}
#### GIEL Distribution Optimization ####
Library(bbmle) ## calling R package bbmle###
Fit<-mle2(LL_LLIEW, start=list(alpha=alpha, lambda=lambda), data=list(x), method="BFGS")
Summary(fit) #### summary of results ####


#### WIEL Distribution PDF ####
wl_pdf<-function(x,alpha,beta,lambda,gamma){
  A<-exp(-(lambda/x))
  B<-alpha*beta*lambda*A
  C<-gamma*(x^2)*(1-A)^((1/gamma)-1)
  D<-(((1-A)^(-1/gamma))-1)^(beta-1)
  E<-exp(((-alpha*(1-A)^(-1/gamma)))-1)^(beta)
  pdf<-(B/C)*D*E
  return(pdf)
```

```
}}

#### WIEL Distribution Hazard rate function ####

wl_HR<-function(x,alpha,beta,lambda,gamma){

  A<-exp(-(lambda/x))

  B<-alpha*beta*lambda*A

  C<-gamma*(x^2)*(1-A)^((-1/gamma)-1)

  D<-(((1-A)^(1/gamma))-1)^(beta-1)

  HR<-(B/C)*D

  return(HR)

}

#### WIEL Distribution Log-likelihood ####

LL_WIEL<-function(alpha,lambda,beta,gamma){

  A<-exp(-lambda/x)

  B<-(alpha*beta*lambda)*(A)

  C<-(gamma*(x^2))

  D<-((1/gamma)-1)

  E<-(1-A)

  H<-(-1/gamma)
```

```
 G<-((E)^(H))

 I<-(G-1)^(beta-1)

 J<-E^D

 K<-(alpha*(G-1)^beta)

 PDF<-(B/(C*J))*I*exp(-K)

 LL<--sum(log(PDF))

 return(LL)

}

#### WIEL Distribution Optimization ####

Library(bbmle) ## calling R package bbmle###

Fit<-mle2(LL_LLIEW, start=list(alpha=alpha, lambda=lambda, theta=theta, gamma=gamma),
data=list(x), method=”BFGS”)

Summary(fit) #### summary of results ####
```

APPENDIX D

The PDFs of the various distributions used to compare with the special cases of the T-IE{Y} family of distributions:

1. Generalized inverse exponential distribution

$$g_X(x) = a\tfrac{\theta}{x^2}\exp(-\tfrac{\theta}{x})(1-\exp(-\tfrac{\theta}{x}))^{a-1}, x>0, \theta>0, a>0.$$

2. Kumaraswamy inverse exponential distribution

$$g_X(x) = ab\tfrac{\lambda}{x^2}\exp(-\tfrac{\lambda}{x})\{\exp(-\tfrac{\lambda}{x})\}^{a-1}[1-\{\exp(-\tfrac{\lambda}{x})\}^{a}]^{b-1}$$

3. Burr type XII distribution

$$g_X(x) = \beta\alpha x^{\alpha-1}[1+x^{\alpha}]^{-\beta-1}$$

4. Weibull distribution

$$g_X(x) = \tfrac{\beta}{\theta}(\tfrac{x}{\theta})^{\beta-1}\exp(-\tfrac{x}{\theta})$$

5. Weibull Burr XII distribution

$$g_X(x) = \tfrac{\alpha\beta ckx^{c-1}}{1+x^{c}}\{k\log(1+x^{c})\}^{\beta-1}$$

6. Exponentiated inverse Rayleigh distribution

$$g_X(x) = \tfrac{2\alpha\sigma^2}{x^3}\exp(-\tfrac{\sigma}{x})^2[1-\exp(-\tfrac{\sigma}{x})^2]^{\alpha-1}$$

7. Inverse Rayleigh distribution

$$g_X(x) = \tfrac{2\sigma^2}{x^3}\exp(-\tfrac{\sigma}{x})^2$$

8. Weibull Lomax distribution

$$\tfrac{ab\alpha}{\beta}(1+\tfrac{x}{\beta})^{b\alpha-1}\{1-[1+\tfrac{x}{\beta}]^{-\alpha}\}^{b-1}\exp\{-a\{[1+\tfrac{x}{\beta}]^{\alpha}-1\}^{b}\}$$

9. Modified Burr III

$$g_X(x) = \alpha\beta x^{-(\beta+1)}[1+\gamma x^{-\beta}]^{-(\frac{\alpha}{\gamma}+1)}$$

10. Inverse Weibull

$$g_X(x) = \beta\alpha^{\beta}x^{-(\beta+1)}\exp(-\tfrac{\alpha}{x})^{\beta}$$

11. Exponentiated exponential Fréchet distribution

$$g_X(x) = \alpha\beta\theta\sigma^{\theta}x^{-(\theta+1)}\exp(-\tfrac{\sigma}{x})^{\theta}[1-\exp(-\tfrac{\sigma}{x})^{\theta}]^{\alpha-1}[1-\{1-\exp(-\tfrac{\sigma}{x})^{\theta}\}^{\alpha}]^{\beta-1}$$

12. Lomax-Weibull {Log-logistic} distribution

$$g_X(x) = \frac{kc}{\beta\gamma}(\tfrac{x}{\gamma})^{c-1}\exp(\tfrac{x}{\gamma})^c[\exp(\tfrac{x}{\gamma})^c - 1]^{\frac{1}{\beta}-1}[1+(\exp(\tfrac{x}{\gamma})^c - 1)^{\frac{1}{\beta}}]^{-k-1}$$