UNIVERSITY FOR DEVELOPMENT STUDIES

COMPLEMENTARY POWER SERIES EXPONENTIATED NADARAJAH-HAGHIGHI DISTRIBUTION

ISSAHAKU ADAM

2019

UNIVERSITY FOR DEVELOPMENT STUDIES

COMPLEMENTARY POWER SERIES EXPONENTIATED NADARAJAH-HAGHIGHI DISTRIBUTION

ISSAHAKU ADAM (DIPLOMA IN EDUCATION, B.ED BASIC, PGD STATISTICS)

(UDS/MAS/0001/2017)

THESIS SUBMITTED TO THE DEPARTMENT OF STATISTICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY FOR DEVELOPMENT STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF MASTER OF PHILOSOPHY DEGREE IN APPLIED STATISTICS

NOVEMBER, 2019

DECLARATION

STUDENT

I hereby declare that this thesis is the result of my own original work and that no part of it has been presented for another degree in this University or elsewhere:

Candidate's Signature: ………...…………… Date: ……………..………

Name: ISSAHAKU ADAM

SUPERVISOR

I hereby declare that the preparation and presentation of the thesis was supervised in accordance with the guidelines on supervision of thesis laid down by the University for Development Studies.

Supervisor's Signature: ………………… Date: …………………….

Name: DR. SULEMAN NASIRU

ABSTRACT

A new class of distributions, called complementary power series exponentiated Nadarajah-Haghighi distribution was developed in this study by compounding the exponentiated Nadarajah-Haghighi distribution with zero truncated power series distributions. The new class of distributions were developed using the concepts of latent complementary risk scenario, in which the lifetime associated with a particular risk is not observable; rather we observe only the maximum lifetime value among all risks. The statistical properties such as quantile, moments, moment generating function, stochastic ordering property and order statistics for the new class of distributions were derived. In order to estimate the parameters of the new class of distributions, the maximum likelihood method was employed to develop estimators for the parameters. Special sub-distributions namely, complementary Poisson exponentiated Nadarajah-Haghighi, complementary geometric Nadarajah-Haghighi, complementary binomial Nadarajah-Haghighi and complementary logarithmic Nadarajah-Haghighi distributions were developed from the new class of distributions. A study of the failure rate of the special sub-distributions revealed that they exhibit different kinds of non-monotonic failure rates including bathtub and upside-down bathtub. Monte Carlo simulations were performed to examine the behavior of the estimators and the results showed that the estimators were able to estimate the parameters well. The applications of the specials distributions were illustrated using two lifetime datasets and the results revealed that the special sub-distributions perform better than the exponentiated Nadarajah-Haghighi distribution.

ACKNOWLEDGEMENTS

I owe my deepest gratitude to my thesis supervisor, Dr. Suleman Nasiru for his unwavering support and guidance in the preparation and completion of this research. I equally express my sincere gratitude to the entire lecturers, of the Statistics Department, for their time, patience, valuable suggestions and meaningful contributions to the success of this research. I must offer my special thanks and immense gratitude to my dearest friend, Godwin Dzakpasu for his support and encouragement. Finally, I am grateful to my family for their unwavering support and encouragement.

DEDICATION

This thesis is dedicated to Almighty God for all he has done for me and also to my family.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ACRONYMS

AB	Average Bias
AIC	Akaike Information Criterion
AICc	Corrected Akaike Information Criterion
BIC	Bayesian Information Criterion
CBENH	Complementary Binomial Exponentiated Nadarajah-Haghighi
CDF	Cumulative Distribution Function
CGENH	Complementary Geometric Exponentiated Nadarajah-Haghighi
CLENH	Complementary Logarithmic Exponentiated Nadarajah-Haghighi
CP	Coverage Probability
CPENH	Complementary Poisson Exponentiated Nadarajah-Haghighi
CPSENH	Complementary Power Series Exponentited Nadarajah-Haghighi
ENH	Exponentiated Nadarajah-Haghighi
ME	Mean Estimate
MGF	Moment Generating Function

MLE	Maximum Likelihood Estimation
MSE	Mean Square Error
NH	Nadarajah – Haghighi
PDF	Probability Density Function
RMSE	Root Mean Square Error
SO	Stochastic Orders

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Barrage of modified versions of existing models have been proposed in literature by researchers in recent time with the primary objective of making them more flexible in providing good parametric fit to a given data. The exponential distribution which is popularly known in literature because of its constant hazard rate and memory-less property is not suitable for lifetime and reliability analysis where the need for a distribution with monotone (increasing and decreasing) and non-monotonic failure rates behaviors are required.

Owing to these drawbacks of the exponential distribution, extended forms of the distribution have been proposed by quite a number of researchers in recent time to make it capable of providing rational parametric fit to specified data set. Some of these generalizations include: exponentiated exponential distribution (Gupta and Kundu, 1999; 2001); beta-exponential distribution (Nadarajah and Kotz, 2006); and extended exponential distribution (Gómez *et al*., 2014). Another extension of the exponential distribution which has attracted the attention of researchers recently is the Nadarajah-Haghighi (NH) distribution developed by Nadarajah and Haghighi (2011). The NH distribution was proved to have increasing, decreasing and constant failure rates; and also capable of modeling lifetime datasets which has its mode fixed at zero.

However, the NH model is not suitable for modeling data that exhibit non-monotonic failure rates. Thus, researchers in distribution theory are proposing new modifications or generalizations of the NH distribution to make it more flexible. Some of these extensions include: inverted NH distribution (Tahir *et al*., 2018); Weibull NH distribution (Peña-Ramírez, 2018); transmuted NH distribution (Kumar and Kumar, 2018); exponentiated NH distribution (Abdul-Moniem, 2015) and Topp-Leone NH distribution (Yousof and Korkmaz, 2017). In line with the goal of developing more flexible distributions by generalizing existing classical distributions, this study develops another extension of the NH distribution called the complementary power series exponentiated NH (CPSENH) distribution.

1.2 Problem Statement

Although the NH distribution possess certain characteristics which makes it good for modeling some types of lifetime dataset, it has some limitations when it comes to analyzing data that exhibit non-monotonic failure rates which includes upside-down bathtub, bathtub, modified bathtub and modified upside-down bathtub failure rates. Thus, the need to introduce current modifications of the NH model to address some of these shortcomings is vital. This study therefore develops, study the statistical properties and demonstrate the applications of the new generalizations of the NH distribution called the CPSENH distribution.

1.3 General Objective

To develop, study the statistical properties and demonstrate the applications of the CPSENH distribution.

1.4 Specific Objectives

i. To develop the CPSENH distribution.
ii. To derive the statistical properties of the CPSENH distribution.
iii. To develop estimators for the parameters of the CPSENH distribution using maximum likelihood method.
iv. Perform Monte Carlo simulations to assess the properties of the estimators.
v. To assess the flexibility of the new distribution by means of goodness-of-fit and illustration using real data sets.

1.5 Significance of the Study

Statistical distributions play a major role in parametric statistical modeling and inference. This implies that identifying appropriate distributions for modeling real datasets improve the power and efficiency of the statistical test associated with the datasets. Thus, developing new generalizations of existing distributions to improve their goodness-of-fit is imperative.

1.6 Thesis Outline

The thesis is organized into five chapters including this one. Chapter two presents literature on modified distributions. Chapter three presents the methodology of the study. Chapter four presents the results and discussion. Finally, chapter five presents the summary, conclusion and recommendations of the study.

.

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

In this chapter, literature on generalized distributions developed by several researchers and some properties of these generalized distributions are presented.

2.1 Reviews on Modifications and Generalizations of Nadarajah-Haghighi Distribution

In recent times, more researchers have worked extensively and some are still working to come up with generators aimed at compounding some well- known classical distributions to make them more flexible and provide a better fit.

One of the latest is the current group of models known as the exponentiated generalized power series family pioneered by Nasiru *et al*. (2018). They investigated the finite sample properties of the estimators of the special distributions by simulation and the result obtained revealed that the parameters of the distributions were good with regards to the simulations method and this was demonstrated using real datasets.

Tahir *et al*. (2018) introduced a new model by name inverted NH distribution. They estimated the model parameters and studied its characteristics. After further studies the new model was deemed to be a better fit to the real datasets that was provided. The model can be a good substitute to existing ones in literature.

Alizadeh *et al*. (2018) developed a new model with four parameters known as the extended exponentiated NH model by modifying the NH model. Certain characteristics of this model were derived; one of such is the incomplete moment. Its importance and flexibility was demonstrated using real datasets.

Yousof and Kurmaz (2017) introduced a recent model which is known as the Topp-Leone NH distribution. This model can be used to analyze different forms of data. Some characteristics were developed. The study also revealed that the model was better than some existing distributions.

Guerra *et al*. (2018) came up with a new distribution by name the logistic NH distribution. Some characteristics of this model were expressed. They illustrated the potency of their distribution using two real datasets. In both cases it was revealed that the model fitted better than existing ones in literature.

Peña – Ramirez *et al*. (2018) introduced the Weibull NH distribution. Its density function is more flexible and exhibits various shapes. The study revealed that in empirical applications, the more tractable than some NH models. It can therefore be used as an effective substitute to the previously developed models.

Kumar and Kumar (2018) introduced a model called the transmuted extended exponential model. This was obtained by modifying the extended exponential model. The characteristics of this new model were expressed and the performance of the model was ascertained. It was shown that the model is quite good and out-performed some existing ones.

Abdul-Monien (2015) introduced a new distribution called exponentiated NH exponential model which is a modified version of the NH exponential model. They demonstrated its usefulness by application to real data and it was evident that the model fits better than certain existing ones.

Khan *et al*. (2018) introduced a new model by name weighted NH distribution. They illustrated its usefulness by applying to four datasets and it showed that the weighted NH model provided better fit to the datasets. Their model was seen to be a substitute to older ones. Finally, they concluded that the weighted NH distribution gave more flexibility to different kind of datasets.

Dias *et al*. (2018) develop a model by name beta NH distribution which generalizes the NH. They demonstrated the usefulness by application to different datasets. Both applications showed that the beta NH is an alternative to the exponentiated Weibull, beta Weibull, Weibull, generalized exponential, extended exponential distributions and exponentiated NH distributions.

Saboor *et al*. (2017) develop a lifetime distribution known as beta exponentiated NH model. The developed distribution can be applied to model different datasets that exhibit different kinds of shapes. The extended distribution proved to be better than the various existing models.

Chesneau *et al.* (2018) introduced a new distribution called a weighted transmuted exponential distribution. This was done by re-parameterization technique. Certain characteristics of the new distribution were determined and further studies revealed the usefulness of the new distribution.

Yousof *et al*. (2017) worked on a new model for analysis of lifetime data referred to as the odd Lindley NH distribution. The strength of the distribution was demonstrated to be good. Certain characteristics were also developed and the model was shown to exhibit various shapes.

Also, Vatto *et al*. (2016) pioneered a new distribution called the exponentiated generalized NH model. They studied and derived some of its characteristics exhibited the shapes of its hazard function. They also provided a maximum likelihood procedure for estimating the exponentiated generalized NH distribution parameters.

Anwar and Bibi (2018) introduced a model named half-logistic generalized Weibull distribution. This new distribution has sub-models and certain characteristics were expressed. Its usefulness and potentiality was demonstrated on two datasets. Their study revealed that the recent one out-classed the models it was compared with.

Elbatal *et al*. (2018) defined and studied a modern class of distributions referred to as the generalized Burr XII power series distribution. The study elaborated and explained expressions for some of its characteristics and behaviors. They performed further studies and the importance of the developed model was also determined.

Bera (2015) proposed a modern family of distributions known as the Kumaraswamy inverse Weibull Poisson model. The zero truncated Poisson distribution was compounded with the Kumaraswamy inverse Weibull model. He studied its characteristics and presented vivid expressions in the work.

Okasha (2017) studied and defined a recent model referred to as the Topp-Leone (*J*-shaped) geometric distribution. The model developed was based on a compounding process. Further studies showed the derivation of some of its characteristics and the usefulness of the model was illustrated using real datasets.

Nasiru *et al*. (2018) introduced the Poisson exponentiated Erlang-truncated exponential distribution and studied its statistical properties. The results of the simulation study revealed the stability of the parameters and the importance of the new distribution was demonstrated by applying it to real dataset. It was shown that the model out-performed similar models.

Shafiei *et al*. (2015) pioneered a new model mainly referred to as the inverse Weibull power series distribution. They employed some techniques under favorable constraints and characterized the distribution by employing certain concepts. The strength of the model was exhibited using two real datasets.

Asgharzadeh *et al*. (2013) introduced a modern model called Pareto Poisson-Lindley distribution. The new model was achieved by a compounding method.

They established certain characteristics of the distribution and also showcased some of its behaviors. The results of the simulation displayed showed good performance.

Alkarni (2016) proposed a modern class of distributions known as generalized extended Weibull power series class of models. He followed the same procedure adopted by Adamidis and Loukas (1998). He also worked on some sub-models and established a number of characteristics of the derived model.

Bourguignon *et al*. (2015) introduced a recent model popularly known as the gamma-NH distribution. They demonstrated the various shapes exhibited by the distribution functions. It is worth mentioning that after careful examination, the model was deemed to be a good fit for real datasets.

Muhammad (2017) introduced another model known as the generalized half-logistic Poisson model which exhibits favorable behaviors. The practical importance, applicability and tractability were demonstrated using real data and this showed that the generalized half-logistic Poisson distribution out-performed certain models.

Alizadeh *et al*. (2018) worked on a modern family of continuous models by name the complementary generalized transmuted Poisson –G family, an extension of the transmuted family pioneered by Shaw and Buckley (2007). Special models were provided and some general characteristics were expressed. The new model outranked other ones mentioned in literature.

Aryal and Yousof (2017) engineered and investigated a current family of distribution by name the exponentiated generalized-G Poisson class of models. They derived certain mathematical characteristics of the current distribution and certain behaviors were displayed. The model was shown to be very acceptable in terms of performance.

Hassan *et al*. (2016) introduced a current class of models named the complementary exponentiated inverted Weibull power series family of distributions. This distribution is a generalization of some lifetime distributions. Its performance exceeded the models mentioned in the analysis.

Muhammad (2017) worked on a recent distribution by name the complementary exponentiated Burr XII Poisson Model. Several mathematical and shape characteristics were provided. Further studies revealed the potency of the model. It was clearly evident that the model outmatches other distributions in literature.

Cordeiro and Silva (2014) engineered a new distribution referred to as the complementary extended Weibull power series family of distributions. The characteristics of the new family were investigated. The study also highlighted various shapes of the distribution and the applicability was also displayed.

Alizadeh *et al*. (2017) pioneered a current model known as the exponentiated power Lindley series model. Special cases were developed and studied. The distribution shows a number of shapes. Subsequent examination showed that the new model out-classed several models in terms of performance.

Nasir *et al*. (2019) engineered a recent model called the Burr XII power series class of distributions. The new distribution shows good strength and has some sub-models. The characteristics of this new distribution derived in the study were explicitly expressed. From the results, it is evident that the model can out-class existing ones.

Louzada *et al*. (2013) developed a recent model referred to as the complementary exponentiated exponential geometric distribution, and it gave good fit to real sets of data. This family of distributions can be applied to varied forms of data existing in literature.

Louzada *et al*. (2014) introduced a recent distribution known as the long-term exponentiated complementary exponential geometric distribution. The new distribution is obtained from the exponentiated complementary exponential geometric and accommodates a number of shapes. Like all other distributions, certain characteristics were derived.

Alkarni (2013) proposed a current family of models referred to as a class of truncated binomial lifetime distribution. The theory behind this new model was expressed and vividly explained in the work. The importance of the new distribution was illustrated with the help of real datasets.

Rashid *et al*. (2017) introduced a modern distribution with the aim of adding an extra dimension to the Lindley power series. The new model is known as the complementary compound Lindley power series distribution. This new model contains special cases of several lifetime distributions.

Flores *et al*. (2013) introduced a new distribution by name the complementary exponential power series distribution. Certain characteristics of the new distribution were expressed in the work. Its legitimacy was also demonstrated and further analysis shows that the model fits real datasets very well.

A current model named as the complementary Burr III Poisson distribution was developed by Hassan *et al*. (2015). The distribution was achieved by mixing two distributions. Certain characteristics of the new model were derived and an intensive simulation study displayed its performance.

Hassan *et al*. (2012) developed a distribution by name an exponentiated exponential binomial distribution. The characteristics of this model along with certain behaviors exhibited by the distribution were highlighted in the work with great precision. The model out-performed the existing models noted in this research work.

Louzada *et al*. (2014) engineered a current class of models known as the exponentiated exponential geometric distribution. The new model exhibited a number of shapes and this can be considered as improvement upon the existing one which was generalized. The characteristics of the new model were studied and its performance was examined.

Fatima and Roohi (2015) introduced a recent modification to exponentiated Pareto-I distribution known as the transmuted exponentiated Pareto-I model. This new model exhibits a number of shapes. The distribution's suitability in modeling breaking strength of materials was highlighted with real datasets. Furthermore, some characteristics of the model were expressed and its potency was provided through application to two datasets.

Rahmouni and Orabi (2018) introduced a current distribution by name the exponential-generalized truncated geometric distribution. Some characteristics of this new model and certain shapes were exhibited. The potency of the new model was shown to be very good and it fits well to real datasets.

Cancho *et al*. (2011) developed a new model referred to as the Poisson-exponential lifetime distribution which exhibits a variety of shapes. Certain characteristics were expressed and the Fisher information matrix was derived and the model's capacity to fit real datasets was demonstrated in the study.

Peña-Ramirez *et al*. (2018) worked on a current model known as the exponentiated power generalized Weibull distribution. The suitability to model certain functions and the flexibility of the distribution was provided. Some characteristics of the distribution were vividly expressed in the study.

Tahmasebi and Jafari (2015) introduced a well-known family of models referred to as the exponentiated extended Weibull-power series class of distributions. The sub-models of this distribution were derived and clearly defined. Various characteristics of this family of models were stated and expressed in the study.

Warahana-Liyanage and Pararai (2015) advanced a family of distributions known as the Lindley power series distribution. Various statistical characteristics of the distribution were depicted in the study. Special cases of the Lindley power series were developed. They simulated several times to test how well its parameters perform.

Elgarhy *et al*. (2018) developed and examined the exponential generalized Kumaraswamy model. In this study, some mathematical characteristics such as moments were expressed. The model parameter estimation was done and the applicability of this model was exhibited with real datasets and the results revealed were positive.

Rodrigues *et al*. (2016) pioneered a new model referred to as the exponentiated Kumaraswamy inverse Weibull model. Some mathematical characteristics of the model were discussed. The model parameters were examined using maximum likelihood estimation method. The importance of the model was illustrated with real datasets.

Elbatal *et al*. (2017) developed a new model which is referred to as the exponential Pareto power series models. Special cases of this class were developed and some of its characteristics were worked upon. The legitimacy of this modified model was tested and its usefulness was displayed using real datasets.

Tahmasebi and Jafari (2015) introduced the generalized Gompertz-power series family of models a compound of the power series and generalized Gompertz models. The distribution comprised of sub-models which were duly discussed in the study to great effect. Advanced studies revealed the potentiality and usefulness of the model.

CHAPTER THREE

METHODOLOGY

3.0 Introduction

The techniques of developing new distribution using the exponentiated Nadarajah-Haghighi (ENH) distribution, power series class of distributions and further highlighting the method of estimating the model parameters and model selection criteria are presented in this chapter.

3.1 **Data and Source**

The study employed two lifetime secondary data to demonstrate the applications of the developed distributions. The first dataset consists of the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. The dataset presented in Table 3.1 can be found in Bjerkedal (1960) and Nasiru *et al*. (2018).

Table 3.1: Survival times of guinea pigs

0.1	0.93	1.08	1.22	1.53	1.83	2.3	2.93
0.33	0.96	1.08	1.22	1.59	1.95	2.31	3.27
0.44	1	1.09	1.24	1.6	1.96	2.4	3.42
0.56	1	1.12	1.3	1.63	1.97	2.45	3.47
0.59	1.02	1.13	1.34	1.63	2.02	2.51	3.61
0.72	1.05	1.15	1.36	1.68	2.13	2.53	4.02
0.74	1.07	1.16	1.39	1.71	2.15	2.54	4.32
0.77	1.07	1.2	1.44	1.72	2.16	2.54	4.58
0.92	1.08	1.21	1.46	1.76	2.22	2.78	5.55

The second dataset consists of 101 observations on the failure time (in hours) of Kevlar 49/epoxy strands with pressure at 90%. The dataset was first presented in Barlow *et al*. (1984) and can also be found in Nasiru *et al*. (2018). Table 3.2 displays the failure times in hours of Kevlar 49/epoxy strands.

Table 3.2: Failure times of Kevlar 49/epoxy strands

0.01	0.07	0.13	0.36	0.63	0.8	1.02	1.31	1.54	2.02	7.89
0.01	0.07	0.18	0.38	0.65	0.8	1.03	1.33	1.54	2.05	
0.02	0.08	0.19	0.4	0.67	0.83	1.05	1.34	1.55	2.14	
0.02	0.09	0.2	0.42	0.68	0.85	1.1	1.4	1.58	2.17	
0.02	0.09	0.23	0.43	0.72	0.9	1.1	1.43	1.6	2.33	
0.03	0.1	0.24	0.52	0.72	0.92	1.11	1.45	1.63	3.03	
0.03	0.1	0.24	0.54	0.72	0.95	1.15	1.5	1.64	3.03	
0.04	0.11	0.29	0.56	0.73	0.99	1.18	1.51	1.8	3.34	
0.05	0.11	0.34	0.6	0.79	1	1.2	1.52	1.8	4.2	
0.06	0.12	0.35	0.6	0.79	1.01	1.29	1.53	1.81	4.69	

3.2 Exponentiated Nadarajah-Haghighi Distribution

A random variable X is said to have exponentiated NH distribution (Abdul-Moniem, 2015) if its cumulative density function (CDF) is given as

$$G(x)=\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}, x>0, \beta>0, \alpha>0, \theta>0 \qquad (3.1)$$

and its corresponding probability density function (PDF) and hazard rate function takes the form:

$$g(x)=\frac{\alpha\beta\theta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^{\beta}}}{\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{1-\theta}}, x>0, \alpha>0, \beta>0, \theta>0 \qquad (3.2)$$

and

$$\tau(x)=\frac{\alpha\beta\theta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}}{1-\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}}, x>0 \qquad (3.3)$$

respectively. Here, α is the scale parameter and both β,θ are shape parameters.

3.3 Power Series Class of Distribution

Suppose N is a discrete random variable from a power series distribution (truncated at zero) and whose PDF is given by

$$P(N=n)=\frac{a_n\lambda^n}{C(\lambda)}, n=1,2......., \qquad (3.4)$$

where $C(\lambda)=\sum_{n=1}^{\infty}a_n\lambda^n$ and a_n depends on n, $\lambda\in(0,S)$, S can be ∞ and $\lambda>0$. $C(\lambda)$ is finite and its first, second and third derivative with respect to λ are defined and given by $C^{'}(.), C^{''}(.)$, and $C^{'''}(.)$, respectively. The power series family of distribution consists of Poisson, binomial, geometric and logarithmic distribution. Table 3.2 displays some useful quantities of the zero truncated power series distribution.

Table 3.3: Useful quantities of some power series distributions

Distribution	$C(\lambda)$	$C^{'}(\lambda)$	$C^{-1}(\lambda)$	a_n	s
Poisson	$e^{\lambda}-1$	e^{λ}	$\log(1+\lambda)$	$(n!)^{-1}$	$(0,\infty)$
Geometric	$\lambda(1-\lambda)^{-1}$	$(1-\lambda)^{-2}$	$\lambda(1+\lambda)^{-1}$	1	$(0,1)$
Logarithmic	$-\log(1-\lambda)$	$(1-\lambda)^{-1}$	$1-e^{-\lambda}$	n^{-1}	$(0,1)$
Binomial	$(1+\lambda)^m-1$	$\frac{m}{(1+\lambda)^{1-m}}$	$(\lambda+1)^{\frac{1}{m}}-1$	$\binom{m}{n}$	$(0,\infty)$

3.4 Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) is the most common technique employed for estimating parameters of a statistical model by choosing a set of values of the model parameters which maximizes the likelihood function. Let $X = (x_1, x_2, ..., x_n)^T$ be a vector of random variables in one class of distribution on R^n and indexed by a $k-$ dimensional parameters $\boldsymbol{\theta} = (\theta_1, \theta_2 ..., \theta_k)^T$ where $\boldsymbol{\theta} \in \Omega \subset R^k$ and $k \leq n$. Let $F(X / \boldsymbol{\theta})$ be the distribution function of X and that the joint density function $f(x_1, x_2, ..., x_n / \boldsymbol{\theta})$ exist. Then the likelihood of $\boldsymbol{\theta}$ is the function

$$L(\boldsymbol{\theta}) = f(x_1, x_2, ..., x_n / \boldsymbol{\theta}), \qquad (3.5)$$

which is the probability of observing the given data as a function of $\boldsymbol{\theta}$. The maximum likelihood estimates of $\boldsymbol{\theta}$ are those values of $\boldsymbol{\theta}$ which maximizes the likelihood function. If $X = (x_1, x_2, ..., x_n)$ are independent and identically distribut

ed, then the likelihood is given by

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i / \boldsymbol{\theta}). \qquad (3.6)$$

Practically, it is often favorable to handle the logarithms of the likelihood function, the log-likelihood function, is given by

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log f(x_i / \boldsymbol{\theta}). \qquad (3.7)$$

Because logarithm is a monotone function when the likelihood function is maximized, the log-likelihood function is also maximized and vice versa. The likelihood equations are attained by setting the first partial derivatives of the log-likelihood function with respect to $\theta_1, \theta_2, ..., \theta_k$ to zero; that are $\frac{\partial \ell(\boldsymbol{\theta} / x_1, x_2, ..., x_n)}{\partial \theta_i} = 0, i = 1, 2,$ and solving the system of likelihood equations.

3.5 Methods of Evaluating Maximum Likelihood Estimators

Suppose $x_1, x_2, ..., x_n$ represent a random sample of size n from the sampling model $f(X / \theta)$, where θ is an unknown parameter. An estimator of θ obtained by techniques such as method of moments and maximum likelihood estimation, is a function of the sample, that is a statistic $\theta = T(x_1, x_2, ..., x_n)$. The mean square error and bias (equivalently root mean square error and average bias) is used to study the behavior of an estimator or asymptotic properties of the estimator.

3.5.1 Mean Square Error of an Estimator

Let $X = (x_1, x_2, ..., x_n)$ be a random sample and θ be the estimator of the unknown parameter θ from the random sample. Then obviously the deviation of $\hat{\theta}$ from the true value of θ, $|\hat{\theta} - \theta|$ measures the quality of the estimator. That is, the mean square error (MSE) of an estimator $\hat{\theta}$ of a parameter θ is the function of θ and is expressed as

$$MSE_{\hat{\theta}} = E\left(\hat{\theta} - \theta\right)^2 = \operatorname{var}\left(\hat{\theta}\right) + \left(E\left(\hat{\theta}\right) - \theta\right)^2 = \operatorname{var}\left(\hat{\theta}\right) + \left(Bias\left(\hat{\theta}\right)\right)^2. \qquad (3.8)$$

The expectation in (3.8) corresponds to the random variables $x_1, x_2, ..., x_n$ since they are the only random components in the expression. The sequence of estimators $\{\hat{\Theta}_n\}$ is weakly consistent or equivalently MSE consistent if $\hat{\Theta}_n \to \theta$ in probability as $n \to \infty$. That is, $\forall \in > 0$, if $n \to \infty$

$$P\left(|\hat{\Theta}_n - \theta| > \in\right) \to 0. \qquad (3.9)$$

3.5.2 Bias of an Estimator

The bias of an estimator is defined as:

$$Bias\left(\hat{\theta}\right) = E\left(\hat{\theta}\right) - \theta, \qquad (3.10)$$

this is the difference between the expected value of $\hat{\theta}$ and θ, where $\hat{\theta}$ is an estimator of θ, an unknown population parameter. If $E\left(\hat{\theta}\right) \neq \theta$ then the estimator has either a positive or negative bias. That is, on the average the estimator tends to over (or under) estimate the population parameter.

An estimator is unbiased if $E\left(\hat{\theta}\right)=\theta, \forall \theta$. For an unbiased estimator $\hat{\theta}$,

$$MSE_{\theta} = E\left(\hat{\theta}-\theta\right)^2 = \operatorname{var}\left(\hat{\theta}\right). \tag{3.11}$$

If an estimator is unbiased, its MSE is equal to its variance. The sequence of estimators $\{\hat{\Theta}_n\}$ is asymptotically unbiased if $E\left(\hat{\Theta}_n\right) \to \theta$ as $n \to \infty$.

3.6 Model Comparison and Model Selection Criteria

To illustrate the relevance and flexibility of our proposed model in modeling lifetime real dataset, we compare its performance with other existing competing models with regards to information lost. The lesser the information lost, the higher the model's quality. Essentially, a comparison of different model-selection techniques' ability to detect a true model involves a trade-off between goodness-of-fit and models parsimony. So, we employed information criteria methods and goodness-of-fit statistics that penalize model for complexity, to keep the model from over fitting, to assess the best model from a number of alternative models which may have different number of parameters. The most commonly used information criteria are Akaike information criterion (AIC), corrected Akaike information criterion (AICc) and the Bayesian information criterion (BIC). The information criterion chooses models with smaller values of AIC, AICc, and BIC for a given set of candidate models and specified dataset.

The AIC (Akaike, 1974) measure the quality of statistical models for given dataset. It quantifies information lost when the data generating procedure is represented by a statistical model by attaining equilibrium in the trade-off between goodness-of-fit of the model and its complexity. Assume we have a statistical model of some data y . Let k be the number of estimated parameters in the model and $\hat{L}$ the maximum value of the model's likelihood function. Then the AIC is given by

$$AIC = 2k - 2\log\left(\hat{L}\right). \qquad (3.12)$$

The AIC rewards goodness of fit, but it also includes a penalty (to minimize over fitting) that is an increasing function of the number of estimated parameters.

AICc (Hurvich and Tsai, 1989) is AIC with correction for finite sample sizes defined as:

$$AICc = -2\log\left(\hat{L}\right) + 2k\left(\frac{n}{n-k-1}\right), \qquad (3.13)$$

where, n is the number of observations, and k is the number of estimated parameters. That is, AICc is essentially AIC with bigger penalty for more parameters. It is convenient to use AICc if the sample size is small or when the model has numerous parameters (Anderson, 2002).

The BIC (Schwarz, 1978) is another criterion for model selection among finite sets of models. In fitting models, it is possible to increase the likelihood by adding parameters, but with trade-off for over fitting. Both BIC and AIC try to resolve this problem by adding a penalty term for the number of parameters in the model; the penalty term is bigger in BIC than AIC. The BIC is express by:

$$BIC = \log(n)k - 2\log\left(\hat{L}\right), \qquad (3.14)$$

where $\hat{L}$ is the maximized value of the model likelihood function, n is the sample size, k is the number of parameters to be estimated. However, the smaller values of the model selection criteria, the better the fit of the given model.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.0 Introduction

The chapter is divided into six sections, namely: complementary power series ENH distribution, statistical properties, estimation of parameters, special distributions, simulations and applications.

4.1 Complementary Power Series Eponentiated Nadarajah-Haghighi Distribution

Given that the random variable N represents the number of failure causes, $n = 1, 2, ...,$ and the underlying distribution of N is the zero truncated power series distribution. Suppose that $X_1, X_2, ..., X_N$ is a sequence of independent and identically distributed, continuous random variables independent of N that follows ENH distribution with CDF $G(x)$ and, parameters α, β and θ. These random variables denote the lifetimes associated with the failure causes. Usually the number of causes N and the lifetime X_i associated with a particular cause are not observable in a latent complementary risk scenario, but only the maximum lifetime $X_{(n)}$ among all the independent causes is often observed. Thus, we only observe the random variable

$$X_{(n)} = \max\{X_1, X_2, ..., X_N\}. \tag{4.1}$$

The conditional CDF of $X_{(n)} \mid N = n$ is given by

$$\begin{aligned} F(x \mid N = n) &= \prod_{i=1}^{n} G(x) \\ &= [G(x)]^n \\ &= \left(1 - e^{1-(1+\alpha x)^\beta}\right)^{n\theta}. \end{aligned} \tag{4.2}$$

Hence, the marginal CDF of $X_{(n)}$ is

$$\begin{aligned} F(x) &= \sum_{i=1}^{n} \frac{a_n \lambda^n}{C(\lambda)} [G(x)]^n \\ &= \sum_{i=1}^{n} \frac{a_n \lambda^n}{C(\lambda)} \left[\left(1 - e^{1-(1+\alpha x)^\beta}\right)^{\theta}\right]^n \\ &= \frac{C\left[\lambda\left(1 - e^{1-(1+\alpha x)^\beta}\right)^{\theta}\right]}{C(\lambda)}, x > 0, \alpha > 0, \beta > 0, \theta > 0, \lambda > 0. \end{aligned} \tag{4.3}$$

The CPSENH distribution has a number of sub-distributions. These include: the complementary Poisson ENH (CPENH), complementary geometric ENH (CGENH), complementary binomial ENH (CBNH) and complementary logarithmic ENH (CLENH) distribution. The proposed distribution can be referred to as a CPSENH class of distributions since it contains a number of sub-models. The newly developed model has some important applications in areas such as medical, industrial and finance where complementary risk problems arise.

The corresponding PDF of the CPSENH distribution is obtained by differentiating the marginal CDF in equation (4.3) and is given by

$$f(x)=\lambda\alpha\beta\theta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}\frac{C^{'}\left[\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]}{C(\lambda)}, x>0, \tag{4.4}$$

where $\alpha>0,\lambda>0$ are scale parameters and $\beta>0,\theta>0$ are shape parameters. Henceforth, we represent a random variable X that follows the CPSENH distribution as $X\sim CPSENH(x;\alpha,\beta,\lambda,\theta)$. It is worth mentioning that when the parameters $\beta=1$ and $\theta=1$ the CPSENH distribution reduces to the complementary exponential power series distribution developed by Flores *et al*. (2013).

The survival function plays a critical role in both engineering and biological studies. For instance, in the engineering sciences, it is used to estimate the reliability of a system. In the biological sciences and other related fields, the survival functions can be used to study the average time to the occurrence of events. The survival function of the CPSENH is

$$\begin{aligned}S(x)&=1-F(x)\\&=1-\frac{C\left[\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]}{C(\lambda)}, x>0.\end{aligned} \tag{4.5}$$

The hazard rate function of a random variable is useful when investigating the failure rate of a component. It is the instantaneous rate at which events happen given no previous events (instantaneous failure rate). The hazard rate function of the CPSENH random variable is defined as

$$\tau(x)=\frac{f(x)}{S(x)}$$

$$=\lambda\alpha\beta\theta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}\frac{C'\left[\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]}{C(\lambda)-C\left[\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]}, x>0. \qquad (4.6)$$

Proposition 4.1. The ENH converges to the CPSENH when $\lambda \to 0^{+}$.

Proof. Since $C(\lambda)=\sum_{n=1}^{\infty}a_n\lambda^n$, we have

$$F(x)=\frac{\sum_{n=1}^{\infty}a_n\lambda^n\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{n\theta}}{\sum_{n=1}^{\infty}a_n\lambda^n}.$$

Considering $\lambda \to 0^{+}$, we obtain

$$\lim_{\lambda\to 0^{+}}F(x)=\lim_{\lambda\to 0^{+}}\frac{\sum_{n=1}^{\infty}a_n\lambda^n\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{n\theta}}{\sum_{n=1}^{\infty}a_n\lambda^n}.$$

By applying L'Hôpital's rule, we obtain

$$\lim_{\lambda\to 0^+} F(x) = \lim_{\lambda\to 0^+} \frac{\sum_{n=1}^{\infty} na_n\lambda^{n-1}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{n\theta}}{\sum_{n=1}^{\infty} na_n\lambda^{n-1}}$$

$$= \lim_{\lambda\to 0^+} \frac{a_1\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta} + \sum_{n=2}^{\infty} na_n\lambda^{n-1}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{n\theta}}{a_1 + \sum_{n=2}^{\infty} na_n\lambda^{n-1}}$$

$$= \lim_{\lambda\to 0^+} \frac{a_1\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}}{a_1 + \sum_{n=2}^{\infty} na_n\lambda^{n-1}} + \lim_{\lambda\to 0^+} \frac{\sum_{n=2}^{\infty} na_n\lambda^{n-1}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{n\theta}}{a_1 + \sum_{n=2}^{\infty} na_n\lambda^{n-1}}$$

$$= \left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}.$$

This completes the proof.

Proposition 4.2. The CPSENH can be written as an infinite mixture of the density of the largest order statistic of the ENH with parameters α, β and $n\theta$.

Proof. Using $C^{'}(\lambda) = \sum_{n=1}^{\infty} na_n\lambda^{n-1}$, the PDF of the CPSENH distribution can be written as

$$f(x) = \frac{\lambda\theta\alpha\beta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}}{C(\lambda)} \sum_{n=1}^{\infty} na_n\left[\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{na_n\lambda^n}{C(\lambda)}\theta\alpha\beta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{n\theta-1}$$

$$= \sum_{n=1}^{\infty} \mathrm{P}(N=n)g_{(n)}(x;\alpha,\beta,n\theta),$$

where $\mathrm{P}(N=n)=\dfrac{a_n\lambda^n}{C(\lambda)}$ and

$$g_{(n)}(x;\alpha,\beta,n\theta)=n\theta\alpha\beta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{n\theta-1}$$

is the density function of the largest order statistic of the ENH.

4.2 Statistical Properties

It is often imperative to derive the statistical properties when new distributions are developed. This section presents statistical properties such as the quantile function, moments, moment generating function (MGF), stochastic ordering property and order statistics.

4.2.1 Quantile Function

The quantile function or the inverse CDF of a random variable is very useful when generating random numbers from a given probability distribution. It can also be used to describe some properties of a distribution such as the median, kurtosis and skewness.

Proposition 4.3. The CPSENH quantile is given by

$$x_u = \frac{1}{\alpha}\left[\left[1-\log\left(1-\left(\frac{C^{-1}(uC(\lambda))}{\lambda}\right)^{\frac{1}{\theta}}\right)\right]^{\frac{1}{\beta}} - 1\right], 0 \le u \le 1, \tag{4.7}$$

where C^{-1} is the inverse of C.

Proof. By definition, the quantile function is given by

$$F^{-1}(u) = \inf\{x_u : F(x_u) > u\}, 0 \le u \le 1. \tag{4.8}$$

For a continuous increasing function F, $F^{-1}(u)$ is the unique real number x_u such that $F(x_u) = u$. Thus, equating the CDF of the CPSENH distribution to u and solving for x_u yields

$$x_u = \frac{1}{\alpha}\left[\left[1-\log\left(1-\left(\frac{C^{-1}(uC(\lambda))}{\lambda}\right)^{\frac{1}{\theta}}\right)\right]^{\frac{1}{\beta}} - 1\right], 0 \le u \le 1.$$

This completes the proof.

It can be seen that the quantile function of the CPSENH class of distributions is tractable and can be used for generating random numbers from the distributions. Sometimes the data may contain outliers or extreme values and the median may be required as the most appropriate central tendency measure instead of the mean. The median of the CPSENH class of distributions is obtained by substituting $u = 0.5$ into equation (4.7). Hence, the median is given by

$$\text{Median} = \frac{1}{\alpha}\left[\left[1-\log\left(1-\left(\frac{C^{-1}(0.5C(\lambda))}{\lambda}\right)^{\frac{1}{\theta}}\right)\right]^{\frac{1}{\beta}}-1\right]. \tag{4.9}$$

4.2.2 Moments

The moments of a random variable are vital in statistical inference. They are employed to study important characteristics of a distribution such as the measures of central tendency, measures of dispersion and measures of shapes. The r^{th} non-central moment of the CPSENH random variable is derived here.

Proposition 4.4. If $X \sim CPSENH(x;\alpha,\beta,\lambda,\theta)$, then the r^{th} non-central moment of X is given by

$$\mu_r^{'} = \sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{r}\frac{(-1)^{i+r-j}\alpha^{-r}n\theta a_n\lambda^n e^{i+1}}{(i+1)^{\frac{j}{\beta}+1}C(\lambda)}\binom{r}{j}\binom{n\theta-1}{i}\Gamma(\tfrac{j}{\beta}+1, i+1),\ r=1,2,..., \tag{4.10}$$

where $\Gamma(a,x)=\int_x^{\infty} t^{a-1}e^{-t}dt$ is the upper incomplete gamma function.

Proof. By definition, the r^{th} non-central moment of a continuous random variable X with the support $(0,\infty)$ is given by

$$\mu_r^{'} = E(X) = \int_0^{\infty} x^r f(x)dx.$$

From Proposition 4.2, we have

$$\mu_r^{'}=\int_0^{\infty}x^r\sum_{n=1}^{\infty}P(N=n)g_{(n)}(x;\alpha,\beta,n\theta)dx$$
$$=\sum_{n=1}^{\infty}P(N=n)\int_0^{\infty}x^r g_{(n)}(x;\alpha,\beta,n\theta)dx$$
$$=\sum_{n=1}^{\infty}\frac{a_n\lambda^n}{C(\lambda)}n\theta\alpha\beta\int_0^{\infty}x^r(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{n\theta-1}dx.$$

Employing the expansion $(1-z)^{\eta-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\eta-1}{i}z^i, |z|<1$ and that fact that

$0<1-e^{1-(1+\alpha x)^{\beta}}<1$, we have

$$\mu_r^{'}=\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i n\theta\alpha\beta a_n\lambda^n e^{i+1}}{C(\lambda)}\binom{n\theta-1}{i}\int_0^{\infty}x^r(1+\alpha x)^{\beta-1}e^{-(i+1)(1+\alpha x)^{\beta}}dx.$$

Let $y=(i+1)(1+\alpha x)^{\beta}$, by change of subject $x=\alpha^{-1}\left[\left(\frac{y}{i+1}\right)^{\frac{1}{\beta}}-1\right]$. As $x\to 0, y\to i+1$ and as $x\to\infty, y\to\infty$. Also, $\frac{dy}{i+1}=\alpha\beta(1+\alpha x)^{\beta-1}dx$. Hence,

$$\mu_r^{'}=\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^{-r}n\theta a_n\lambda^n e^{i+1}}{C(\lambda)}\binom{n\theta-1}{i}\int_{i+1}^{\infty}\left[\left(\frac{y}{i+1}\right)^{\frac{1}{\beta}}-1\right]^r e^{-y}\frac{dy}{i+1}.$$

The binomial theorem as stated in Graham (1994) is the power series identity $(x+b)^v=\sum_{j=0}^{\infty}\binom{v}{j}x^j b^{v-j}$. Adopting the binomial theorem, we have

$$\mu_r' = \sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{r}\frac{(-1)^{i+r-j}\alpha^{-r}n\theta a_n\lambda^n e^{i+1}}{(i+1)^{\frac{j}{\beta}+1}C(\lambda)}\binom{r}{j}\binom{n\theta-1}{i}\int_{i+1}^{\infty} y^{\frac{j}{\beta}}e^{-y}dy$$

$$=\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{r}\frac{(-1)^{i+r-j}\alpha^{-r}n\theta a_n\lambda^n e^{i+1}}{(i+1)^{\frac{j}{\beta}+1}C(\lambda)}\binom{r}{j}\binom{n\theta-1}{i}\Gamma(\tfrac{j}{\beta}+1, i+1), r=1,2,....$$

This completes the proof for the r^{th} non-central moment.

4.2.3 Moment Generating Function

The MGFs are special functions employed to establish the moments if they exist for a random variable and functions of moments such as mean and variance, kurtosis and skewness in a much simpler way.

Proposition 4.5. If $X \sim CPSENH(x;\alpha,\beta,\lambda,\theta)$, then the MGF is

$$M_X(t) = \sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{r}\frac{(-1)^{i+r-j}t^r\alpha^{-r}n\theta a_n\lambda^n e^{i+1}}{r!(i+1)^{\frac{j}{\beta}+1}C(\lambda)}\binom{r}{j}\binom{n\theta-1}{i}\Gamma(\tfrac{j}{\beta}+1, i+1), \quad (4.11)$$

Proof. If the MGF exist, then

$$M_X(t) = E\left(e^{tX}\right) = \int_0^{\infty} e^{tx} f(x)dx.$$

Employing Taylor series expansion, the MGF can be expressed as

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r',$$

where $\mu_r^{'}$ is the r^{th} non-central moment. Thus, substituting the r^{th} non-central moment gives the MGF as

$$M_X(t)=\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{r}\frac{(-1)^{i+r-j}t^r\alpha^{-r}n\theta a_n\lambda^n e^{i+1}}{r!(i+1)^{\frac{j}{\beta}+1}C(\lambda)}\binom{r}{j}\binom{n\theta-1}{i}\Gamma(\tfrac{j}{\beta}+1,i+1),$$

This completes the proof.

4.2.4 Stochastic Ordering Property

Stochastic orders (SO) are very useful in many areas of applied probability and statistics. In the fields of reliability and maintainability theory, SO have relevant applications in, for instance, defining notions of positive and negative aging, bounding system reliabilities and availability, and comparing maintenance policies (Ohnishi, 2002). Stochastic ordering is the common way of showing ordering mechanism in lifetime distributions. A random variable X_1 is said to be greater than a random X_2 in likelihood ratio order if $f_{X_1}(x)/f_{X_2}(x)$ is an increasing function of x.

Proposition 4.6. Let $X_1 \sim CPSENH(x;\alpha,\beta,\lambda,\theta)$ and $X_2 \sim ENH(x;\alpha,\beta,\theta)$, then X_1 is greater than X_2 in likelihood ratio order $(X_2 \leq_{lr} X_1)$ provided $\lambda > 0$.

Proof. The ratio of the densities of X_1 and X_2 is

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \frac{\lambda C^{'}\left[\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]}{C(\lambda)}.$$

Differentiating the ratio of the two densities with respect to x, we have

$$\frac{d}{dx}\frac{f_{X_1}(x)}{f_{X_2}(x)} = \lambda^2\theta\alpha\beta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}\frac{C^{''}\left[\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]}{C(\lambda)}.$$

Hence, for $\lambda > 0$, $\dfrac{d}{dx}\dfrac{f_{X_1}(x)}{f_{X_2}(x)} > 0$ for all x. This completes the proof.

4.2.5 Order Statistics

Order statistics are essential tools in non-parametric statistics and inference. They are obtained from transformation that involves the ordering of an entire set of observations on a random variable. Since order statistics have myriad of applications in several areas of statistics, it is important to derive some common order statistics for the CPSENH class of distributions. Suppose $X_1, X_2, ..., X_n$ are independent identically distributed random sample of size n from CPSENH class of distributions with CDF $F(x)$ and PDF $f(x)$. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ represent the order statistics obtained from the sample. The PDF of the k^{th} order statistic, for $k = 1, 2, ..., n$, is given by

$$f_{X_{k:n}}(x) = \frac{1}{B(k, n-k+1)} \left[F(x)\right]^{k-1} \left[1 - F(x)\right]^{n-k} f(x), \tag{4.12}$$

where $B(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}dy$ is the beta function. Using the fact that $0 < F(x) < 1$ for $x > 0$, we have

$$\left[1 - F(x)\right]^{n-k} = \sum_{m=0}^{n-k} (-1)^m \binom{n-k}{m} \left[F(x)\right]^m. \tag{4.13}$$

Hence, substituting equation (4.13) into equation (4.12), we obtain

$$f_{X_{k:n}}(x) = \frac{1}{B(k, n-k+1)} \sum_{m=0}^{n-k} (-1)^m \binom{n-k}{m} \left[F(x)\right]^{m+k-1} f(x). \tag{4.14}$$

Finally substituting the CDF and PDF of the CPSENH class of distributions into equation (4.14) yields the PDF of the k^{th} order statistic for the CPSENH class of distributions as

$$f_{X_{k:n}}(x) = \frac{\lambda\alpha\beta\theta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^{\beta}} \left(1 - e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}}{B(k, n-k+1)} \times$$

$$\sum_{m=0}^{n-k} (-1)^m \binom{n-k}{m} \left[\frac{C\left[\lambda\left(1 - e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]}{C(\lambda)} \right]^{m+k-1} \frac{C'\left[\lambda\left(1 - e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]}{C(\lambda)}. \tag{4.15}$$

4.3 Estimation of Model Parameters

In order to illustrate the applications of the developed distribution with regards to modeling real datasets, it is essential to develop estimators for estimating the parameters of the distribution. In this section, estimators are developed for estimating the parameters of the CPSENH class of distributions. Suppose $x_1, x_2, ..., x_n$ are possible outcomes of a random sample of size n from the CPSENH and $z_i = e^{1-(1+\alpha x_i)^{\beta}}$, then the log-likelihood function is

$$\ell = n\log(\lambda\theta\alpha\beta) + (\beta-1)\sum_{i=1}^{n}\log(1+\alpha x_i) + \sum_{i=1}^{n}(1-(1+\alpha x_i)^{\beta}) + (\theta-1)\sum_{i=1}^{n}\log(1-z_i) - n\log C(\lambda) + \sum_{i=1}^{n}\log C^{'}\left[\lambda\left(1-z_i\right)^{\theta}\right]. \quad (4.16)$$

By differentiating the total log-likelihood function with respect to the parameters λ, θ, β and α, the score functions are obtained as:

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \frac{nC^{'}(\lambda)}{C(\lambda)} + \sum_{i=1}^{n}\frac{(1-z_i)^{\theta}C^{''}\left[\lambda(1-z_i)^{\theta}\right]}{C^{'}\left[\lambda(1-z_i)^{\theta}\right]}, \quad (4.17)$$

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log(1-z_i) + \sum_{i=1}^{n}\frac{\lambda(1-z_i)^{\theta}C^{''}\left[\lambda(1-z_i)^{\theta}\right]\log(1-z_i)}{C^{'}\left[\lambda(1-z_i)^{\theta}\right]}, \quad (4.18)$$

$$\frac{\partial \ell}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log(1+\alpha x_i)-\sum_{i=1}^{n}(1+\alpha x_i)^{\beta}\log(1+\alpha x_i)+(\theta-1)\sum_{i=1}^{n}\frac{z_i(1+\alpha x_i)^{\beta}\log(1+\alpha x_i)}{1-z_i}$$
$$+\lambda\theta\sum_{i=1}^{n}\frac{(1-z_i)^{\theta-1}z_i C''\left[\lambda\left(1-z_i\right)^{\theta}\right](1+\alpha x_i)^{\beta}\log(1+\alpha x_i)}{C'\left[\lambda\left(1-z_i\right)^{\theta}\right]}, \tag{4.19}$$

and

$$\frac{\partial \ell}{\partial \alpha}=\frac{n}{\alpha}+(\beta-1)\sum_{i=1}^{n}\frac{x_i}{1+\alpha x_i}-\beta\sum_{i=1}^{n}x_i(1+\alpha x_i)^{\beta-1}+\beta(\theta-1)\sum_{i=1}^{n}\frac{x_i(1+\alpha x_i)^{\beta-1}z_i}{1-z_i}+$$
$$\lambda\theta\beta\sum_{i=1}^{n}\frac{x_i(1+\alpha x_i)^{\beta-1}z_i(1-z_i)^{\theta-1}C''\left[\lambda\left(1-z_i\right)^{\theta}\right]}{C'\left[\lambda\left(1-z_i\right)^{\theta}\right]}. \tag{4.20}$$

The normal equations that need to be solved simultaneously to obtain the maximum likelihood estimates of the parameters are obtained by equating equations (4.17), (4.18), (4.19) and (4.20) to zero. That is, $\frac{\partial \ell}{\partial \lambda}=0, \frac{\partial \ell}{\partial \theta}=0, \frac{\partial \ell}{\partial \beta}=0$ and $\frac{\partial \ell}{\partial \alpha}=0$. The resulting normal equations do not have closed form and so the maximum likelihood estimates are obtained by solving the equations using numerical methods.

4.4 Special Distributions

In this section, we present the CDF, PDF and the hazard functions of the special sub-models of the CPSENH class of distributions. These are: the CPENH, CGENH, CBENH and CLENH distributions.

4.4.1 Complementary Poisson Exponentiated Nadarajah-Haghighi Distribution

The zero truncated Poisson distribution is a special case of the power series distribution with $a_n = \frac{1}{n!}$ and $C(\lambda) = e^{\lambda} - 1, (\lambda > 0)$. From equation (4.3), the CDF of the CPENH distribution is

$$F(x) = \frac{e^{\lambda(1-e^{1-(1+\alpha x)^{\beta}})^{\theta}} - 1}{e^{\lambda} - 1}, x > 0, \tag{4.21}$$

where $\alpha > 0, \lambda > 0$ are scale parameters and $\beta > 0, \theta > 0$ are shape parameters. When $\beta = 1$, the CPENH distributions reduces to the complementary Poisson exponentiated exponential distribution. When $\beta = 1$ and $\theta = 1$, the CPENH distribution reduces to the complementary exponential Poisson distribution developed by Flores *et al*. (2013). Figure 4.1 displays the plot of the CDF of the CPENH distribution for some selected parameter values.

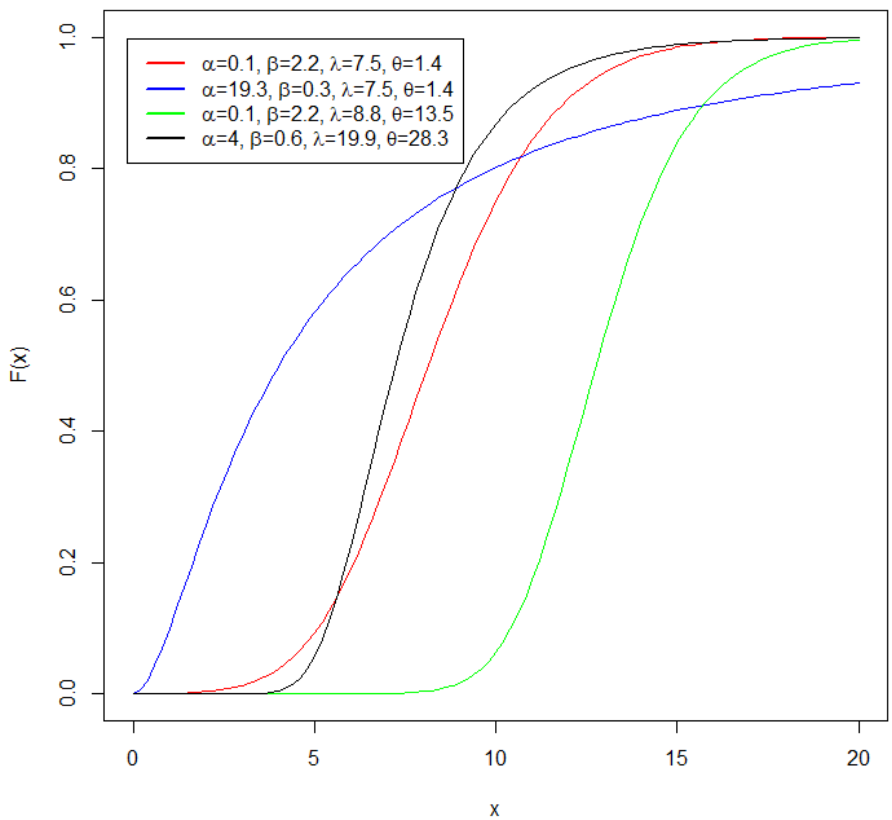

Figure 4.1: Plot of CDF of CPENH

The CPENH distribution PDF is given by

$$f(x) = \lambda\alpha\beta\theta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^{\beta}} \left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1} \frac{e^{\lambda(1-e^{1-(1+\alpha x)^{\beta}})^{\theta}}}{e^{\lambda}-1}, x>0. \qquad (4.22)$$

Figure 4.2 shows the PDF of the CPENH distribution for some selected parameter values. It can be seen that the PDF for some chosen parameter values can be approximately symmetric, left skewed and right skewed.

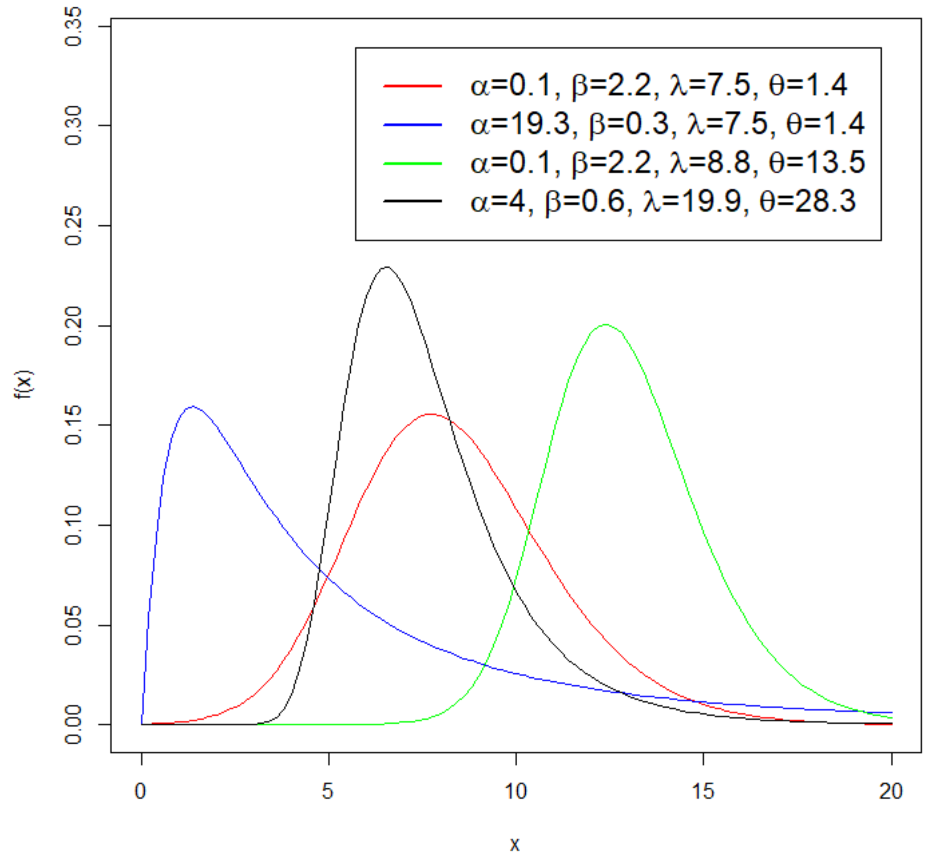

Figure 4.2: Plot of PDF of CPENH

The CPENH distribution hazard function is given by

$$\tau(x)=\lambda\alpha\beta\theta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}\frac{e^{\lambda(1-e^{1-(1+\alpha x)^{\beta}})^{\theta}}}{e^{\lambda}-e^{\lambda(1-e^{1-(1+\alpha x)^{\beta}})^{\theta}}},x>0. \qquad (4.23)$$

The plots of the CPENH hazard function is given in Figure 4.3. From Figure 4.3, it can be seen that the hazard function exhibit different kinds of non-monotonic shapes such as the bathtub, upside-down bathtub and decreasing failure rate.

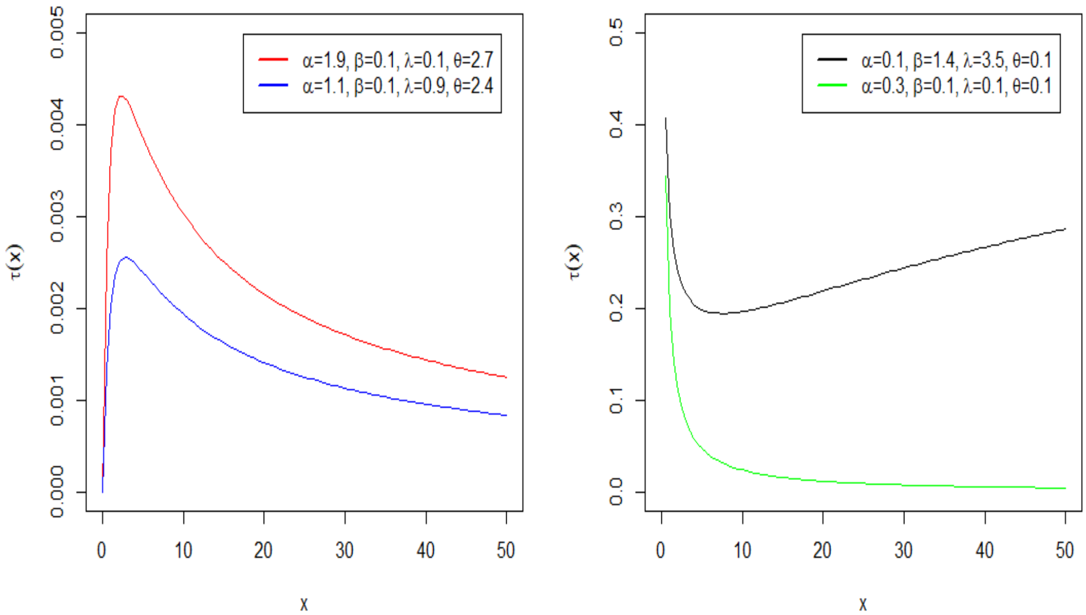

Figure 4.3: Plots of hazard rate function for CPENH

4.4.2 Complementary geometric Exponentiated Nadarajah-Haghighi Distribution

The zero truncated geometric distribution is a special case of the power series distribution with $a_n = 1$ and $C(\lambda) = \frac{\lambda}{1-\lambda}, (0 < \lambda < 1)$. From equation (4.3), the CDF of the CGENH distribution is given by

$$F(x) = \frac{(1-\lambda)\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}}{1-\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}}, x > 0, \tag{4.24}$$

where $\alpha > 0, 0 < \lambda < 1$ are scale parameters and $\beta > 0, \theta > 0$ are shape parameters. It is worth mentioning that λ is also valid for $(-\infty, 1)$. When $\beta = 1$, the CGENH reduces to the complementary geometric exponentiated exponential distribution. When $\beta = 1$ and $\theta = 1$, the CGENH reduces to the complementary exponential geometric distribution developed by Flores *et al*. (2013). Figure 4.4 shows the CDF of the CGENH distribution for some parameter values.

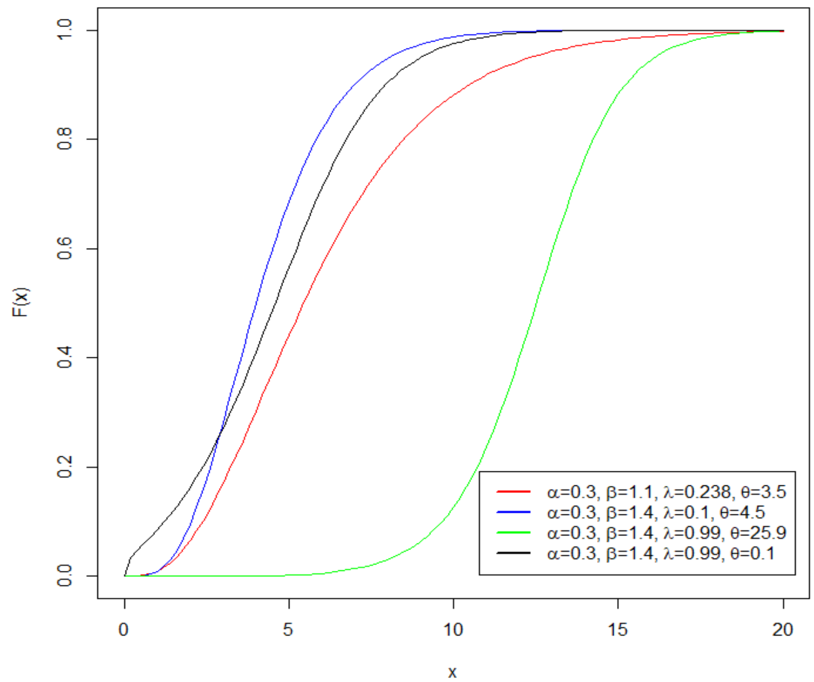

Figure 4.4: Plot of CDF of CGENH

The CGENH distribution PDF is

$$f(x)=\frac{(1-\lambda)\alpha\beta\theta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}}{\left[1-\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]^{2}}, x>0. \tag{4.25}$$

Figure 4.5 is the plot of the CGENH distribution PDF for some chosen values. The density of the CGENH distribution can exhibit both left skewed and right skewed shapes.

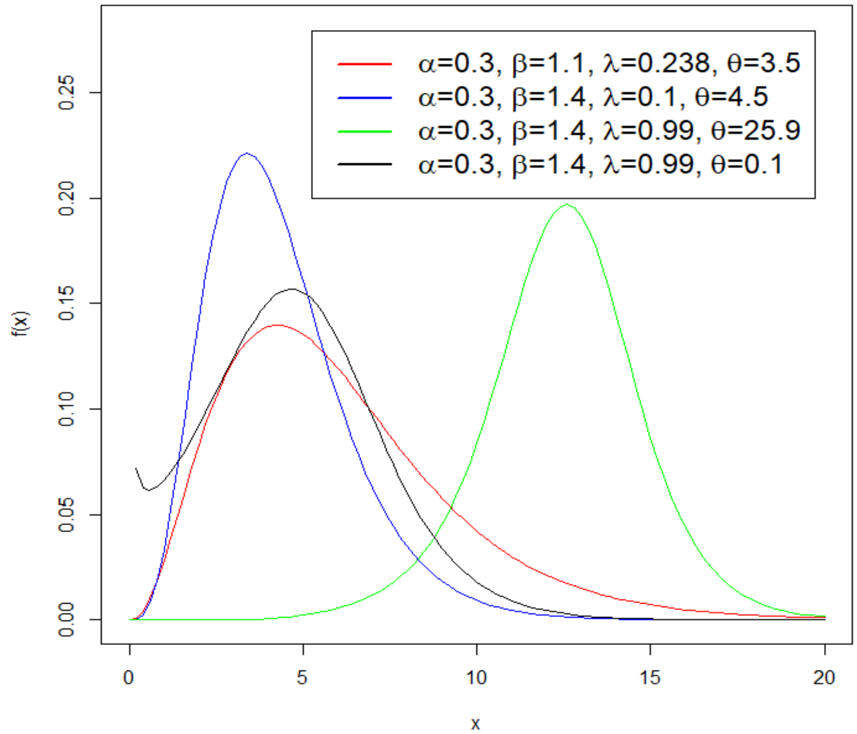

Figure 4.5: Plot of PDF of CGENH

The CGENH hazard function is

$$\tau(x)=\frac{(1-\lambda)\alpha\beta\theta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}}{\left[1-\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]\left[1-\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]},x>0. \tag{4.26}$$

The plots of the CGENH hazard function are presented in Figure 4.6. The CGENH hazard function can exhibit both upside-down bathtub and bathtub shapes as shown in Figure 4.6.

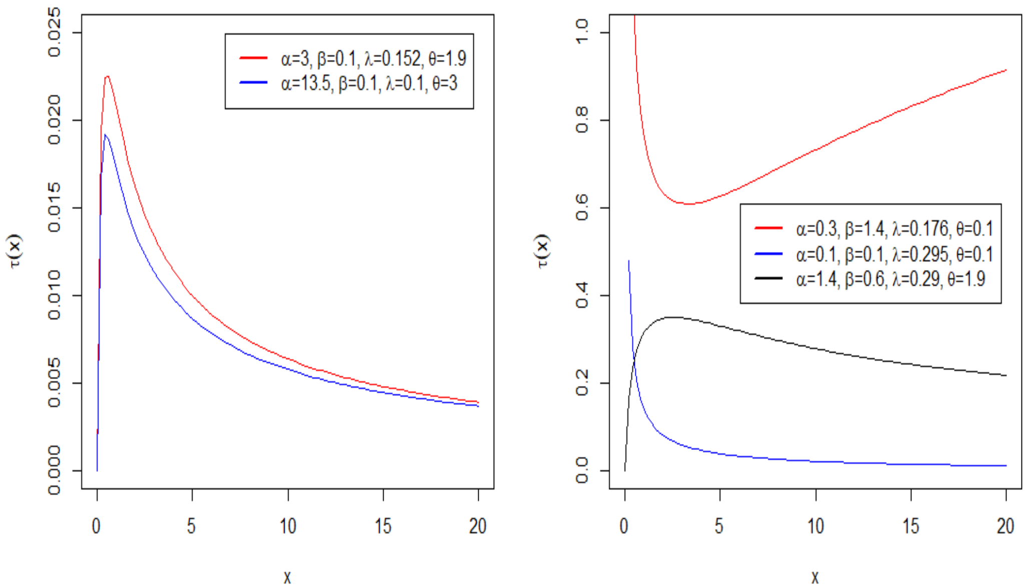

Figure 4.6: Plots of the hazard rate function of the CGENH

4.4.3 Complementary Binomial Exponentiated Nadarajah-Haghighi Distribution

The zero truncated binomial distribution is a special case of the power series distribution with $a_n = \binom{m}{n}$, and $C(\lambda) = (1+\lambda)^m - 1, (\lambda > 0)$, where $m (n \leq m)$ is the number of replicas and is a positive integer. Using equation (4.3), the CDF of the CBENH is

$$F(x) = \frac{\left[1 + \lambda\left(1 - e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]^{m} - 1}{(1+\lambda)^{m} - 1}, x > 0, \qquad (4.27)$$

where $\alpha > 0, \lambda > 0$ are scale parameters and $\beta > 0, \theta > 0$ are shape parameters. When $\beta = 1$, the CBENH reduces to the complementary binomial exponentiated exponential distribution. When $\beta = 1$ and $\theta = 1$, the CBENH distribution reduces to the complementary exponential binomial distribution developed by Flores *et al*. (2013). Figure 4.7 shows the CDF of the CBENH for some selected parameter values and $m = 5$.

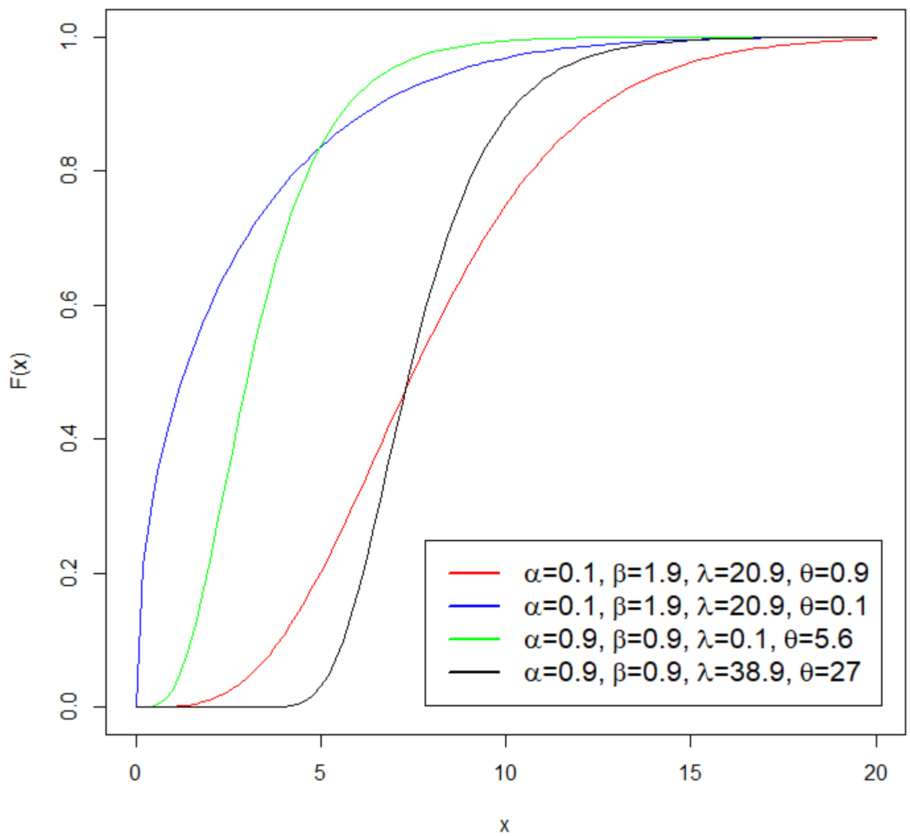

Figure 4.7: Plot of CDF of the CBENH

The CBENH PDF is

$$f(x)=m\lambda\alpha\beta\theta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}\frac{\left[1+\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]^{m-1}}{(1+\lambda)^{m}-1},x>0. \qquad (4.28)$$

The PDF of the CBENH exhibit right skewed shapes with different degrees of kurtosis for some chosen values as displayed in Figure 4.8.

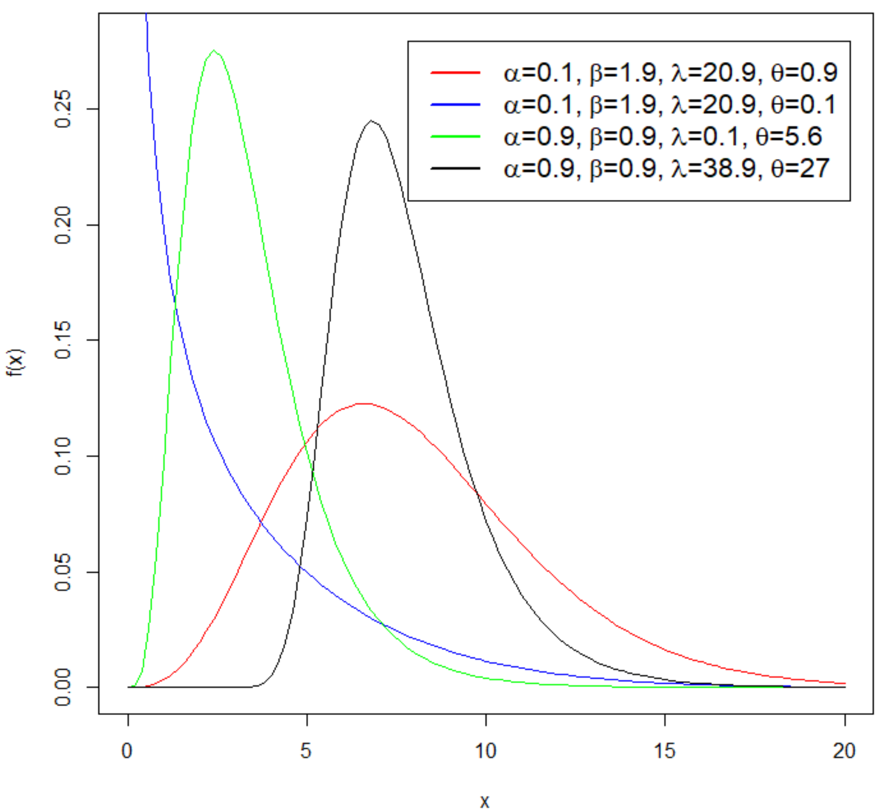

Figure 4.8: Plot of PDF of the CBENH

The CBENH hazard function is

$$\tau(x)=m\lambda\alpha\beta\theta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}\frac{\left[1+\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]^{m-1}}{(1+\lambda)^{m}-\left[1+\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]^{m}},x>0. \quad (4.29)$$

The CBENH hazard function plot for $m=5$ are shown in Figure 4.9. It exhibits decreasing, bathtub and upside-down bathtub failure rates.

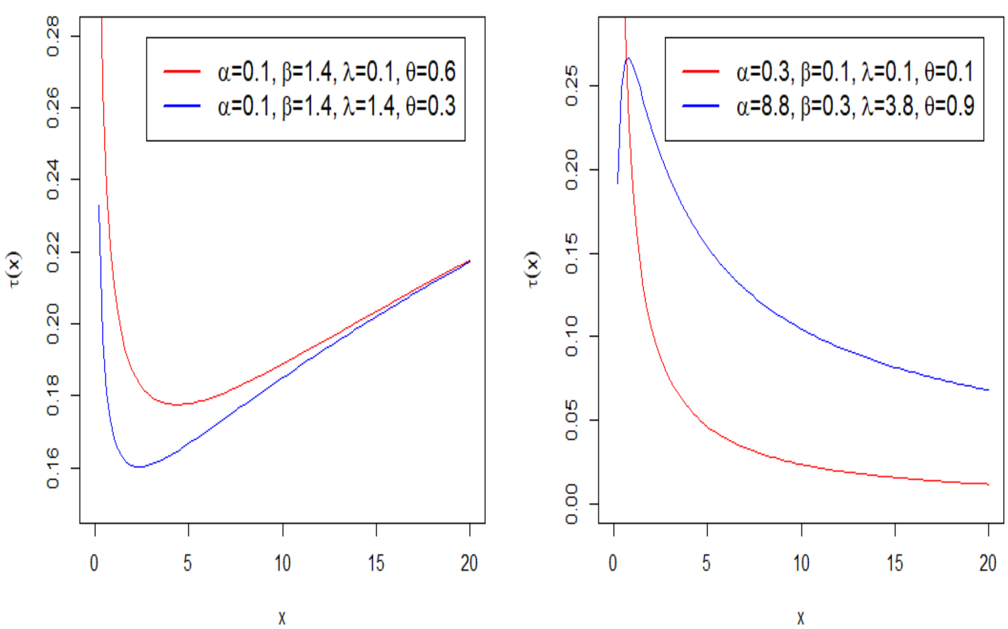

Figure 4.9: Plots of hazard rate function of the CBENH

4.4.4 Complementary Logarithmic Exponentiated Nadarajah-Haghighi Distribution

The zero truncated logarithmic distribution is a special case of the power series distribution with $a_n = \frac{1}{n}$ and $C(\lambda) = -\log(1-\lambda), (0 < \lambda < 1)$. Using equation (4.3) the CDF of the CLENH distribution is defined as

$$F(x) = \frac{\log\left[1-\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]}{\log(1-\lambda)}, x > 0, \tag{4.30}$$

where $\alpha > 0, 0 < \lambda < 1$ are scale parameters and $\beta > 0, \theta > 0$ are shape parameters. It is imperative to note λ is also valid for $(-\infty, 1)$. When $\beta = 1$, the CLENH reduces to the complementary logarithmic exponentiated exponential distribution. When $\beta = 1$ and $\theta = 1$, the CLENH distribution reduces to the complementary exponential logarithmic distribution developed by Flores *et al*. (2013). Figure 4.10 shows the CDF of the CLENH for some chosen values.

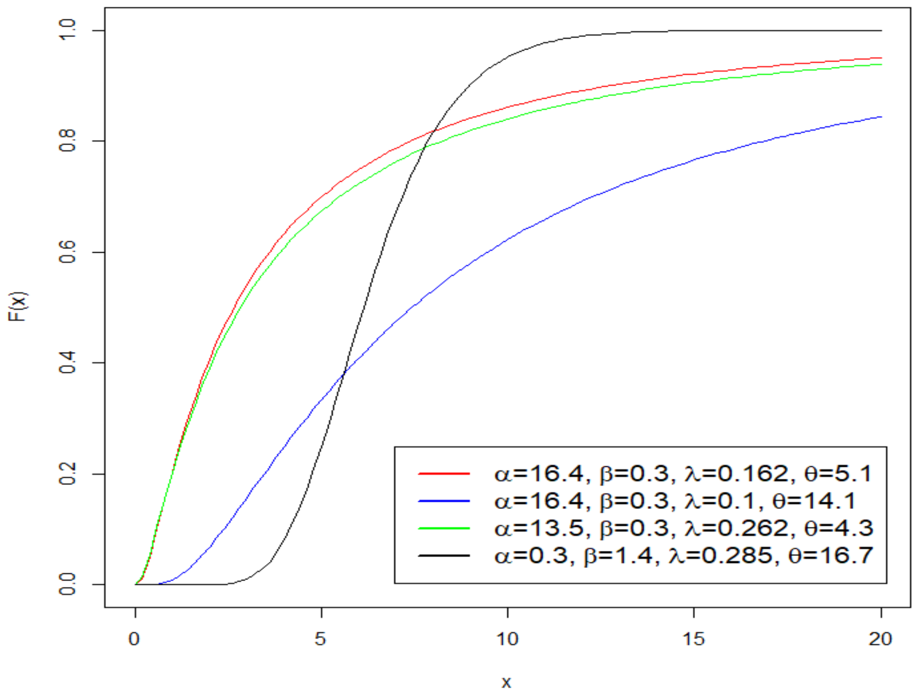

Figure 4.10: Plot of CDF of the CLENH

The CLENH PDF is given by

$$f(x)=\frac{\lambda\alpha\beta\theta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}}{\log(1-\lambda)\left[\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}-1\right]},x>0 \tag{4.31}$$

Figure 4.11 displays the density function of the CLENH distribution. From Figure 4.11, it can be seen that the PDF of the CLENH distribution exhibit right skewed shapes for some given parameter values.

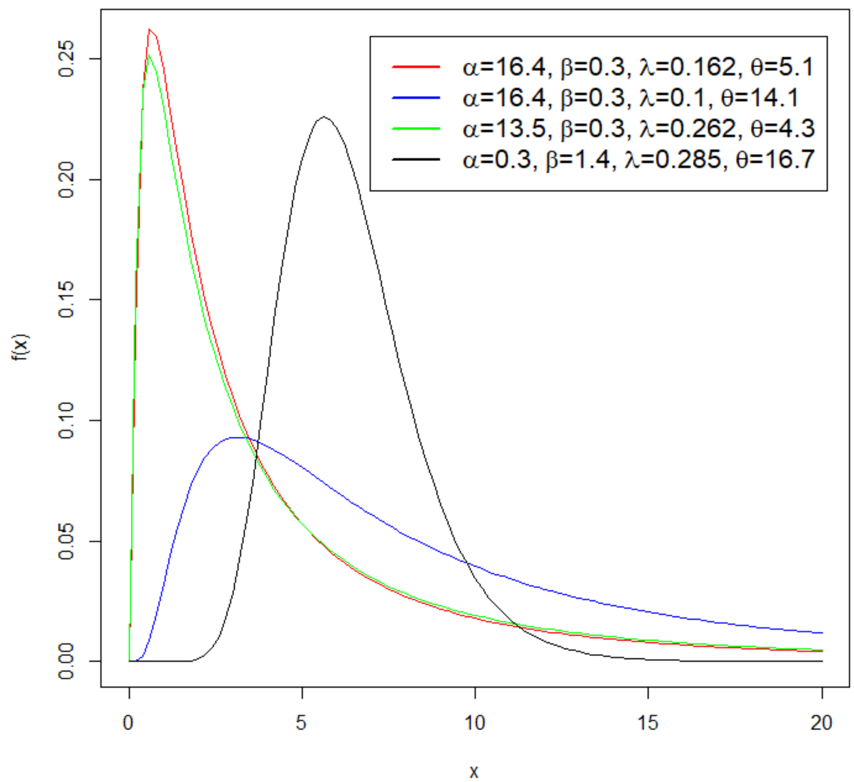

Figure 4.11: Plot of PDF of the CLENH

The CLENH hazard function is

$$\tau(x)=\frac{\lambda\alpha\beta\theta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta-1}}{\left[1-\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right]\log\left[\frac{\left(1-\lambda\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\theta}\right)}{(1-\lambda)}\right]},x>0. \tag{4.32}$$

The plot of the CLENH hazard function is shown in Figure 4.12. The CLENH hazard function can exhibit decreasing and upside-down bathtub failure rates as shown in Figure 4.12.

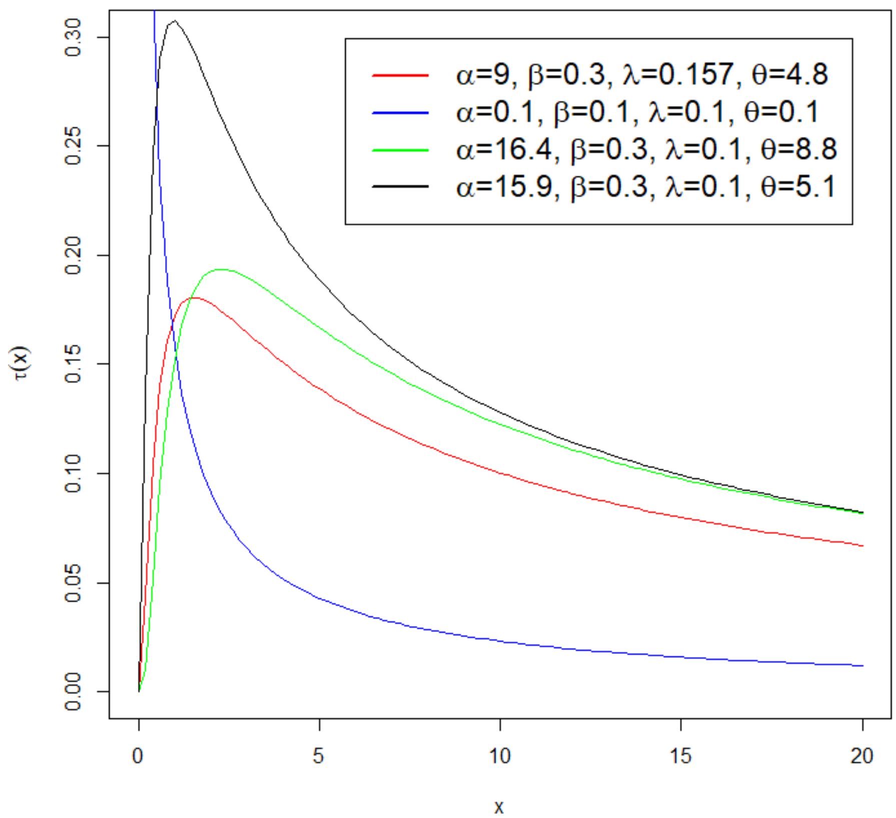

Figure 4.12: Plot of hazard rate function of the CLENH

4.5 Monte Carlo Simulation

The estimators for the parameters of the CPSENH were assessed using Monte Carlo simulations. For the purpose of illustration the CPENH was used for the experiment. The simulation experiment was carried using sample sizes $n = 30, 70, 100, 150, 250, 350$ and 500. For each sample size, the experiment was replicated $1,000$ times. The following sets of parameter values $\text{I}: \alpha = 0.8, \beta = 0.6, \lambda = 0.1, \theta = 0.2$ and $\text{II}: \alpha = 2.5, \beta = 1.5, \lambda = 0.8, \theta = 0.5$ were used to obtain random samples from the CPENH model. Table 4.1 displays the mean estimate (ME), average bias (AB), root mean square error (RMSE) and coverage probability (CP) for the estimators. As shown in Table 4.1, the ME vary with respect to the sample sizes and get more closer to the actual parameter values as the sample sizes increases. The ABs for α, β and λ were generally positive whiles that of θ was negative. However, looking at the absolute values of the ABs, it can be seen that it decays towards zero as the sample sizes increases. The RMSEs of the estimators of the parameters decreases as the sample size increases. This is an indication that the consistency property of the maximum likelihood estimators will be achieved as $n \to \infty$. The CPs for the 95% interval also vary as the sample size increases with most of the CPs close to the nominal value of 0.95. Thus, the estimators estimates the parameters well.

Table 4.1: Monte Carlo simulation results

		I				II			
Parameter	n	**ME**	**AB**	**RMSE**	**CP**	**ME**	**AB**	**RMSE**	**CP**
α	30	33.0427	32.2427	581.964	0.8570	4.5322	2.0322	8.5066	0.9970
	70	6.9063	6.1063	176.32	0.8110	3.5966	1.0966	3.0157	0.9910
	100	1.1614	0.3614	1.0489	0.8070	3.3584	0.8584	2.6670	0.9820
	150	1.0050	0.2050	0.6303	0.8260	3.1758	0.6758	2.2831	0.9800
	250	0.8905	0.0905	0.4214	0.8490	2.9509	0.4509	1.8909	0.9710
	350	0.8401	0.0401	0.2335	0.8800	2.8938	0.3938	1.5985	0.9700
	500	0.8215	0.0215	0.1314	0.9000	2.7433	0.2433	1.3823	0.9540
β	30	0.7199	0.1199	0.5169	0.8250	2.3875	0.8875	3.1290	0.9500
	70	0.6628	0.0628	0.3655	0.8380	2.0995	0.5995	2.3098	0.9140
	100	0.6341	0.0341	0.1822	0.8300	2.1643	0.6643	2.5023	0.9110
	150	0.6211	0.0211	0.1240	0.8670	1.9935	0.4935	2.2711	0.9110
	250	0.6172	0.0172	0.0826	0.8610	1.9008	0.4008	1.7366	0.9050
	350	0.6156	0.0156	0.0640	0.8830	1.7256	0.2256	1.1896	0.9180
	500	0.6142	0.0142	0.0501	0.8990	1.6983	0.1983	0.8000	0.9170
λ	30	1.3070	1.2070	1.7117	0.9920	2.1288	1.3288	1.9047	0.9770
	70	0.8254	0.7254	1.1165	0.9890	1.9527	1.1527	1.9431	0.9820
	100	0.7353	0.6353	0.9918	0.9830	1.7101	0.9101	1.7900	0.9860
	150	0.5585	0.4585	0.7170	0.9740	1.5188	0.7188	1.5721	0.9930
	250	0.4023	0.3023	0.4786	0.9510	1.2816	0.4816	1.2383	0.9910
	350	0.3330	0.2330	0.3786	0.9460	1.1142	0.3142	0.9609	0.9900
	500	0.2760	0.1760	0.2953	0.9530	0.9508	0.1508	0.6630	0.9870
θ	30	0.1950	-0.0050	0.1320	0.9720	0.4195	-0.0805	0.2408	0.9810
	70	0.1851	-0.0149	0.0523	0.9820	0.4156	-0.0844	0.1601	0.9480
	100	0.1840	-0.0160	0.0380	0.9780	0.4319	-0.0681	0.1435	0.9500
	150	0.1874	-0.0126	0.0305	0.9860	0.4444	-0.0559	0.1268	0.9560
	250	0.1899	-0.0101	0.0219	0.9860	0.4577	-0.0423	0.1030	0.9650
	350	0.1921	-0.0079	0.0184	0.9840	0.4757	-0.0243	0.0827	0.9770
	500	0.1945	-0.0055	0.0153	0.9860	0.4860	-0.0140	0.0646	0.9820

4.6 Applications to Lifetime Data

The applications of the special distributions were demonstrated in this section using two lifetime datasets. The performance of the special distributions was compared using the AIC, AICc and BIC. The ENH distribution was also fitted to the datasets and its performance was compared to that of the special distributions.

4.6.1 Guinea Pigs Survival Times Data

Table 4.2 presents the descriptive statistics of the guinea pigs survival time data. From Table 4.2, the minimum and maximum survival times of the guinea pigs were 0.10 and 5.55 days respectively. The average survival time was 1.768 days. The coefficient of skewness of 1.37 and excess kurtosis value of 2.20 revealed that the survival times was right skewed and more peaked than the normal curve.

Table 4.2: Descriptive statistics of survival times (in days) of guinea pigs

Minimum	Maximum	Mean	Skewness	Excess Kurtosis
0.1000	5.5500	1.7680	1.3700	2.2000

The survival times were modeled using the CPENH, CGENH, CBENH, CLENH and ENH distributions. It is important to mention that $m=5$ replicas were used to fit the CBENH distribution to the dataset. The estimates for the parameters are shown in Table 4.3. For the CPENH, the parameters β and λ were significant at the 5% significance level. The parameters of the CGENH and ENH were all significant at the 5% significance level. For the CBENH only the parameter β was significant and for the CLENH only the parameter θ was significant.

Table 4.3: Maximum likelihood estimates for the guinea pigs data

Model	Estimate	Standard error	z-value	*P*-value
CPENH	$\hat{\alpha} = 3.4171$	3.4898	0.9792	0.3275
	$\hat{\beta} = 0.7056$	0.2252	3.1333	0.0017^{*}
	$\hat{\lambda} = 5.3599$	2.6241	2.0425	0.0411^{*}
	$\hat{\theta} = 1.9241$	1.8721	1.0278	0.3041
CGENH	$\hat{\alpha} = 0.011$	0.0011	9.3973	$< 2.2000 \times 10^{-16}$ *
	$\hat{\beta} = 28.8790$	0.0006	48243.4513	$< 2.2000 \times 10^{-16}$ *
	$\hat{\lambda} = -7.4780$	0.0118	-634.4898	$< 2.2000 \times 10^{-16}$ *
	$\hat{\theta} = 2.8333$	0.3114	9.0980	$< 2.2000 \times 10^{-16}$ *
CBENH	$\hat{\alpha} = 2.1747$	1.9103	1.1384	0.2549
	$\hat{\beta} = 0.8067$	0.2595	3.1081	0.0019^{*}
	$\hat{\lambda} = 1.7851$	1.3931	1.2814	0.2001
	$\hat{\theta} = 2.1203$	1.4868	1.4261	0.1538
CLENH	$\hat{\alpha} = 0.3864$	0.5828	0.6630	0.5073
	$\hat{\beta} = 1.6910$	1.4880	1.1364	0.2558
	$\hat{\lambda} = -2.8430$	8.6290	-0.3295	0.7418
	$\hat{\theta} = 3.1306$	0.8219	3.8090	0.0001^{*}
ENH	$\hat{\alpha} = 5.8242 \times 10^{-3}$	4.0393×10^{-4}	14.4190	$< 2.2000 \times 10^{-16}$ *
	$\hat{\beta} = 70.4950$	5.8150×10^{-5}	1.2123×10^{6}	$< 2.2000 \times 10^{-16}$ *
	$\hat{\theta} = 1.5948$	0.2300	6.9360	4.0340×10^{-12} *

*: means significant at the 5% significance level

Table 4.4 presents the goodness-of-fit statistics for the fitted models. It can be seen from Table 4.4 that the CPENH was the best model for the data since it has the highest log-likelihood value and the least values for the AIC, AICc and BIC.

Table 4.4: Goodness-of-fit statistics for guinea pigs data

Model	log-likelihood	AIC	AICc	BIC
CPENH	-92.8300*	193.6503*	194.2473*	202.7570*
CGENH	-93.1200	194.2475	194.8445	203.3542
CBENH	-92.9700	193.9487	194.5457	203.0554
CLENH	-93.9300	195.8527	196.4497	204.9593
ENH	-98.5600	203.1276	203.4805	209.9576

*: means best based on the goodness-of-fit statistic

Figure 4.13 shows the histogram of the guinea pigs data and the densities of the fitted distributions on the left, and the empirical CDF of the guinea pigs data and the fitted CDFs on the right. It can be seen that the fitted distributions mimic the empirical density and CDF of the guinea pigs data.

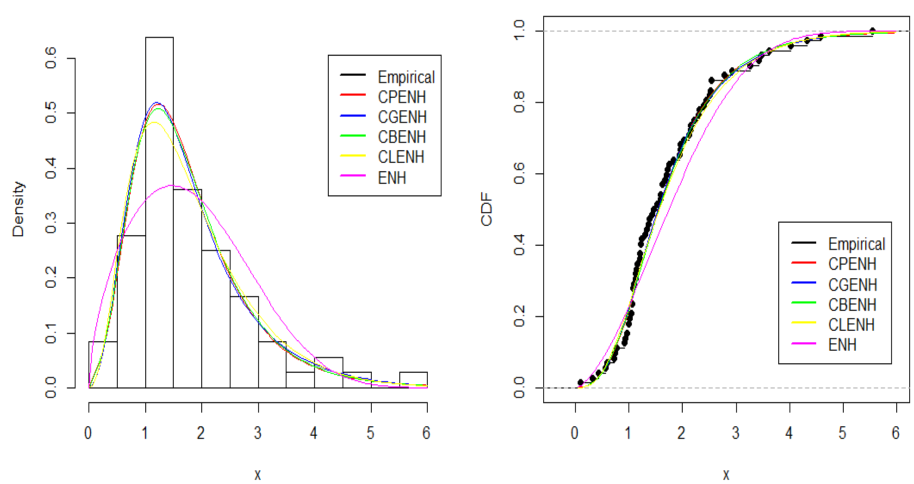

Figure 4.13: Plots of fitted densities and CDFs for guinea pigs data

The probability-probability plots of the fitted distributions were plotted to examine how well the distributions fits the given dataset. From Figure 4.14, the special distributions provided better fit to the dataset than the ENH distribution as the plot of their observed probability against the expected cluster along the diagonal as compared to that of the ENH distribution.

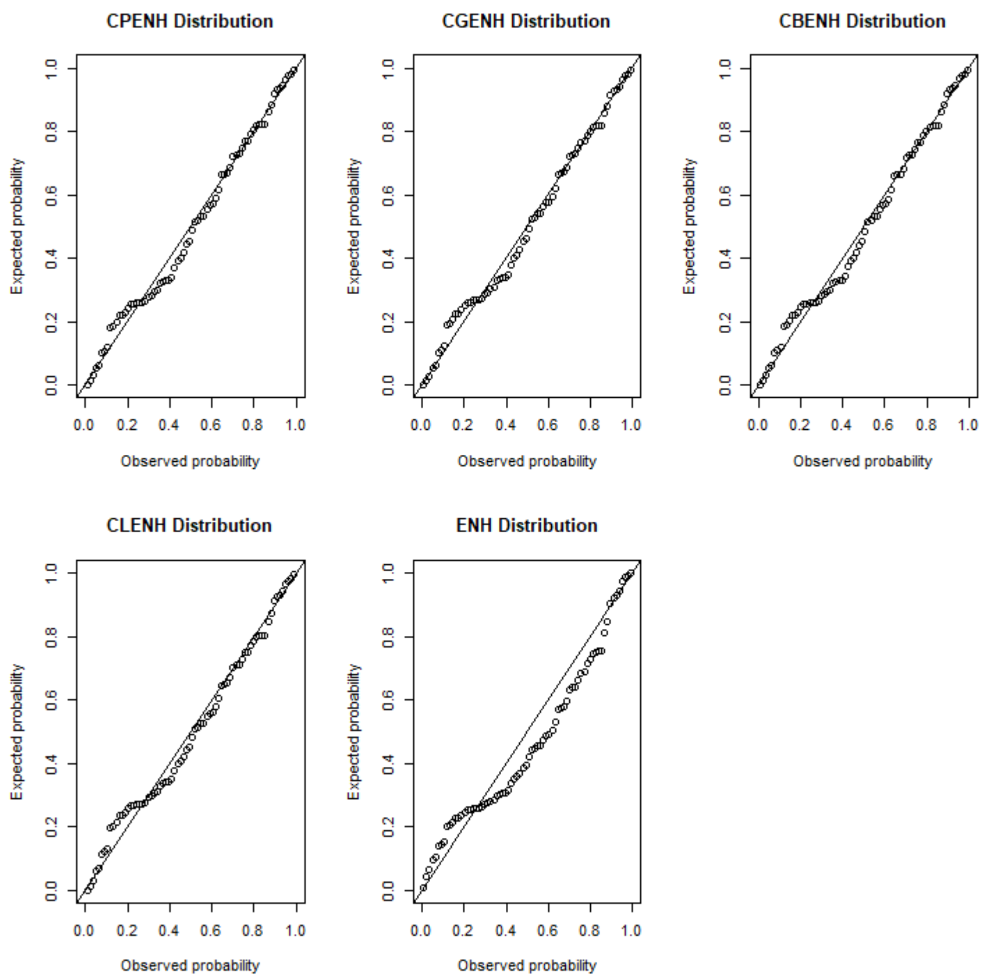

Figure 4.14: Probability-probability plots of fitted distributions for guinea pigs data

4.6.2 Kevlar 49/Epoxy Strands Failure Times Data

Table 4.5 displays the descriptive statistics of the Kevlar 49/Epoxy strands failure times data. From Table 4.5, the minimum and maximum failure times were 0.01 and 7.89 hours respectively. The mean failure time was 1.025 hours. The skewness value of 3.05 implies that the failure times are right skewed. The excess kutosis value of 14.47 implies that the distribution of the failure times is more peaked than the normal curve and the observations are closely distributed around their average value.

Table 4.5: Descriptive statistics of failure times of Kevlar 49/Epoxy strands

Minimum	Maximum	Mean	Skewness	Excess Kurtosis
0.0100	7.8900	1.0250	3.0500	14.4700

The estimates for the parameters of the CPENH, CGENH, CBENH, CLENH and ENH are shown in Table 4.6. It is worth stating that the CBENH distribution was fitted to the dataset using $m=5$ replicas. The parameters β and θ for the CPENH and CBENH distributions were significant at the 5% significance level. All the parameters for the CGENH distribution were significant with the exception of α. The parameters λ and θ were significant for the CLENH. The parameters for the ENH were all significant at the 5% significance level as shown in Table 4.6.

Table 4.6: Maximum likelihood estimates for the Kevlar/Epoxy strands data

Model	Estimate	Standard error	z-value	*P*-value
CPENH	$\hat{\alpha}=1.3412$	0.9016	1.4875	0.1369
	$\hat{\beta}=0.8757$	0.2586	3.3862	0.0007^{*}
	$\hat{\lambda}=1.1919$	1.1833	1.0073	0.3138
	$\hat{\theta}=0.7369$	0.1994	3.6957	0.0002^{*}
CGENH	$\hat{\alpha}=2.8075$	3.5242	0.7966	0.4257
	$\hat{\beta}=0.6956$	0.2759	2.5216	0.0117^{*}
	$\hat{\lambda}=0.7088$	0.3071	2.3080	0.0210^{*}
	$\hat{\theta}=0.6784$	0.2231	3.0409	0.0024^{*}
CBENH	$\hat{\alpha}=1.1961$	0.7440	1.6077	0.1079
	$\hat{\beta}=0.9118$	0.2634	3.4615	0.0005^{*}
	$\hat{\lambda}=0.2730$	0.3324	0.8212	0.4115
	$\hat{\theta}=0.7538$	0.1917	3.9331	$8.3860\times10^{-5\,*}$
CLENH	$\hat{\alpha}=0.0855$	0.1019	0.8385	0.4017
	$\hat{\beta}=3.7972$	3.7968	1.0001	0.3173
	$\hat{\lambda}=-150.3863$	0.0265	-5667.2575	$<2.0000\times10^{-16\,*}$
	$\hat{\theta}=1.4450$	0.1295	11.1558	$<2.0000\times10^{-16\,*}$
ENH	$\hat{\alpha}=120.0200$	5.7146×10^{-3}	21002.0536	$<2.0000\times10^{-16\,*}$
	$\hat{\beta}=0.2534$	1.1294×10^{-2}	22.4385	$<2.0000\times10^{-16\,*}$
	$\hat{\theta}=3.8001$	5.0750×10^{-1}	7.4879	$7.0000\times10^{-16\,*}$

*: means significant at the 5% significance level

Comparative analysis of the fitted distributions was performed and the results as shown in Table 4.7 implies that the CGENH provided the best fit to the dataset since it has the highest value of the log-likelihood and the least values for the AIC, AICc and BIC.

Table 4.7: Goodness-of-fit statistics for the Kevlar/Epoxy strands failure data

Model	log-likelihood	AIC	AICc	BIC
CPENH	-102.3400	212.6781	213.0948	223.1386
CGENH	-102.0000*	211.9907*	212.4074*	222.4512*
CBENH	-102.4200	212.8330	213.2497	223.2934
CLENH	-108.5900	225.1750	225.5917	235.6355
ENH	-116.4200	238.8321	239.0795	246.6775

*: means best based on the goodness-of-fit statistic

Figure 4.15 displays the histogram and the fitted densities on the left, and the empirical CDF and the fitted CDFs on the right. From figure 4.15, the fitted distributions mimic the empirical density and CDF of the Kevlar/Epoxy strands failure data.

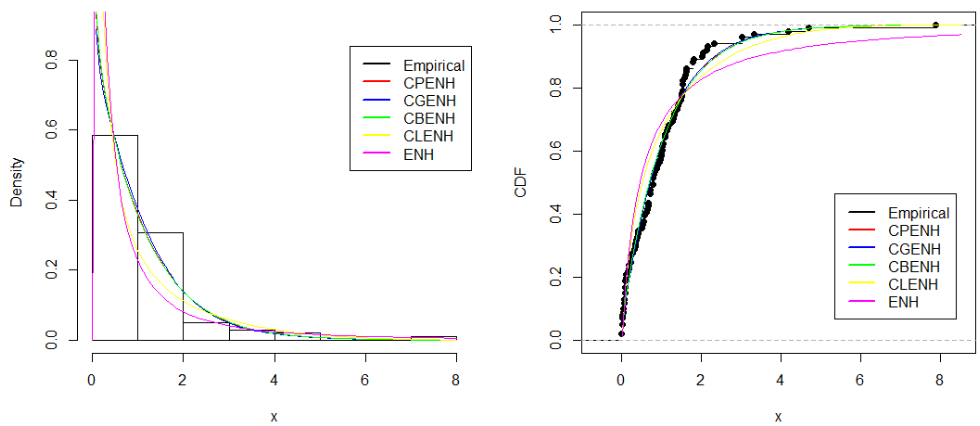

Figure 4.15: Plots of fitted densities and CDFs for Kevlar/Epoxy strands data

The probability-probability plots of the fitted distributions were used to assess how well the distributions fit the data. From Figure 4.16, it can be seen that the CGENH, CPENH and CBENH distributions provided better fit to the dataset as compared to the CLENH and ENH distributions.

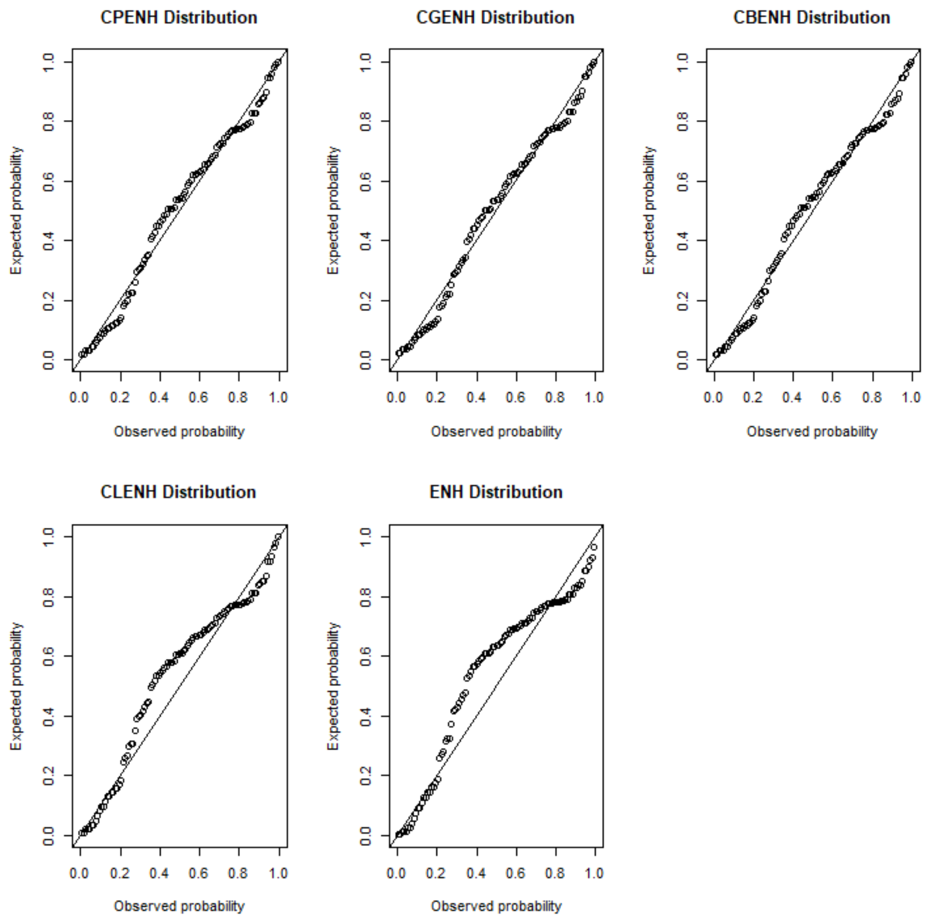

Figure 4.16: Probability-probability plots of fitted distributions for Kevlar strands data

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.0 Introduction

This chapter presents the summary of the study, conclusions and recommendations.

5.1 Summary

Statistical probability distributions play significant role in parametric modeling and inferences. Many statistical modeling or inferences make assumptions about the underlying statistical distribution of the data generating process. However, most of the dataset may not follow the existing classical distributions. Thus, the need to develop new or generalize the existing probability distributions for modeling datasets is imperative.

This study adopted the concept of complementary risk scenario to generalize the ENH distribution. The new generalization of the ENH distribution was named the CPSENH distribution. The limiting distribution of the CPSENH distribution was shown to be the ENH distribution. It is important to mention that the CPSENH distribution generalizes the complementary exponential power series distribution developed by Flores *et al*. (2013). The CDF, PDF, survival and hazard rate functions of the CPSENH distribution were derived. The PDF of the CPSENH distribution was expressed as an infinite mixture of the density of the largest order statistic of the ENH distribution to make it easy to derive the statistical properties of the distribution. Statistical properties such as the quantile, moments, moment generating function, stochastic ordering property and order statistics of the CPSENH distribution were derived.

The CPSENH distribution contains a number of sub-distributions. These include: CPENH, CGENH, CBENH and CLENH distribution. The CDF, PDF and hazard rate function of the sub-distributions were derived. The plots of the PDF and hazard rate function revealed that the sub-distributions can exhibit different kinds of shapes that are suitable for modeling lifetime datasets. The PDFs can exhibit reversed-J, unimodal, approximately symmetric, right skewed and left skewed shapes with different degrees of kurtosis. The hazard rate functions exhibit both monotonic and non-monotonic failure rates.

In order to estimate the parameters of the new distribution, the maximum likelihood method was employed to develop estimators for the parameters of the CPSENH distribution. Monte Carlo simulation experiments were performed to assess the properties of the estimators and the results revealed that the estimators were able estimate the parameters well.

Applications of the sub-distributions were illustrated using two lifetime datasets. The first dataset consists of the survival times of guinea pigs and the second dataset comprises the failure times of Kevlar/Epoxy strands. Exploratory analyses of the datasets revealed that both datasets were right skewed and more peaked than the normal distribution. The performances of the distributions fitted to the datasets were compared using the log-likelihood, AIC, AICc and BIC. For the guinea pigs data, the CPENH distribution was selected as the best distribution whiles for the Kevlar/Epoxy strands data, the CGENH distribution was selected as the best. The histogram of the datasets and the densities of the fitted distributions were used to examine how well the sub-distributions fit the datasets. Similarly, the empirical CDFs of the datasets and the fitted CDFs were plotted to explore whether the distributions fits the given datasets well. Also, the P-P plots of the fitted distributions were also plotted to assess the performance of the distributions. The sub-distributions were compared to the ENH distribution and the results revealed that all the sub-distributions performed better than the ENH distribution.

5.2 Conclusions

This study developed the CPSENH distribution. The properties of the distribution were established and sub-distribution such as the CPENH, CGENH, CBENH and CLENH distributions were studied. The findings of the study revealed that the sub-distributions exhibit different kinds of failure rates such as decreasing, bathtub and upside-down bathtub shapes. This makes them suitable for modeling lifetime datasets with such kinds of failure rates.

Estimators were developed for estimating the parameters of the distribution and Monte Carlo simulation experiments were carried out to examine the behaviour of the estimators. The simulations indicated the estimators were able to estimate the parameters well.

Applications of the sub-distributions were illustrated using two lifetime datasets and the results revealed that each of the sub-distributions performs better than the ENH distribution. However, for the guinea pigs data, the CPENH distribution was the best model and for the Kevlar/Epoxy strands data the CGENH distribution was the best model.

5.3 Recommendations for Further Studies

i. In this study, the applications of the developed distributions were shown using complete samples of survival datasets in the estimation of the model parameters. However, there are situations were censored observations may be encountered. Thus, subsequent further research should consider application of the developed distributions using censored survival datasets.

ii. This study used the stochastic representation $X_{(n)} = \max(X_1, X_2, ..., X_N)$ to develop the CPSENH distribution. By considering the stochastic represent ation $X_{(1)} = \min(X_1, X_2, ..., X_N)$, a new distribution that is useful in modeling lifetime data from series system can be developed.

REFERENCES

Abdul-Moniem, I., B. (2015). Exponentiated Nadarajah and Haghighi's exponential distribution. *International Journal of Mathematical Analysis and Applications*, **2**(5): 68-73.

Adamidis, k., and Lukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics Probability letters,* **39**(1): 35 – 42.

Adamidis, K., Dimitrakopoullou, T. and Loukas, S. (2005). On an extension of the exponential-geometric distribution. *Statistics and Probability Letter*, **73**(3): 259 – 269.

Akaike, H. (1974). A new look at the statistical model. *IEEE transactions on Automatic Control*, **19(**6): 716 – 723.

Alizadeh, M., Bagheri, S., F., Samani, E., B., Ghobadi, S., and Nadarajah, S. (2017). Exponentiated power Lindley power series class of distribution: Theory and applications. *Communication in Statistics-Simulation and Computation*, **47**(9): 2499 – 2531.

Alizadeh, M., Rasekhi, M., Yousof, H., M., Ramires, T., G. (2018). Extended exponentiated Nadarajah-Haghighi model: mathematical properties characterizations and applications. *Studia Scientiarum Mathematicarum Hungarica*, **55**(4): 421 -558.

Alizadeh, M., Yousof, H., M., Afifiy, A., Z., and Mansoor M. (2018). The complementary generalized transmuted Poisson –G family of distributions. *Austrian Journal of Statistics*, **47**(4):51 – 71.

Alkarni, S., H. (2013). A class of truncated binomial lifetime distribution. *Open Journal of Statistics*, **3**(5): 305 – 311.

Alkarni, S., H. (2016). Generalised extended Weibull power series family of distribution. *Journal of Data Science*, **14**(3): 415 – 440.

Anderson, D., R. (2002). Model selection and multi-model inference. *A practical Information-Theoretical Approach. Springer*

Anwar, M., and Bibi, A. (2018). The half-Logistic generalised Weibull Distribution. Journal of probability and statistics volume 2018, htt://doi.org/10.1155/2018/8767826.

Aryal, G., R. and Yousof, H., M. (2017). The exponentiated generalized – G Poisson family of distribution. *Stochastic and Quality Control*, **32**(1): 7 - 23

Asgharzadeh, A., Bakouch, H., S., and Esmaeili, L. (2013). Pareto Poisson – Lindley distribution with appliations. *Journal of Applied Statistics*, **40(**8): 1717 – 1734.

Barlow, R., E. and Doksum, K., A. (1972). Isotonic tests for convex orderings. *In Proceedings of 6th Berkeley Symposium*, **1**: 293-323.

Bera, W., T. (2015). The Kumaraswamy inverse Weibull Poisson distribution with applications. *Thesis and Dissertations (All),* Indiana University of Pennsylvania.

Bjerkedal, T. (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *American Journal of Hygiene*, **72**: 130-148.

Bourguignon, M., Lima, M., C., S., Leao, J., Nascimento, A., D., C., Pinho, L., G., B. and Cordeiro, G., M. (2015). A new generalised gamma distribution with application. *American Journal of Mathematical and Management Sciences*, **34**(4): 309 – 342.

Bourguignon, M., Silva, R., B. and Cordeiro, G., M. (2014). The Weibull-G family of probability distribution. Journal of Data Science, **12**(1): 53 – 68.

Cancho, V., G., Barriga, G., D., C., and Louzada, F. (2011). The Poisson – exponential lifetime distributions. *Computational Statistics and Data Analysis*, **55**(1): 677 – 686.

Chahkandi, M. and Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate. *Computational Statistics and Data Analysis*, **53**(12): 4433 – 4440.

Chesneau, C., Bakouch, H. and Khan, M. (2018). A weighted transmutted exponential distributions with environmental applications. Available at:

https://hal.archives-ouvertes.fr/hal-01508880v accessed on 21st November, 2018.

Cooner, F., Carlin, B., Banerjee, S. and Sinha, D. (2007). Flexible cure rate modeling under latent activation schames. *Journal of American Statistical Association*, **102**(478): 560 – 572.

Cordeiro, G., M., and Silva, R., B. (2014). The complementary extended Weibull power series classof distributions. Ciencia e Natura, Santa Maria, **36,** 1 – 13.

Dias C., R., B., Alizadeh M. and Cordeiro, G. (2018). Beta Nadarajah – Haghighi distribution. *Hacettepe Journal of Mathematics and Statistic*, **47**(5): 1302 – 1320.

Elbatal, I., Altun, E., Afify, A., Z. and Ozel, G. (2018). The generalised Burr XII power series distributions with properties and applications. *Annals of Data Science*. Doi: 10.1007/s40745-018-0171-2

Elbatal, I., Zayed, M., Rasekhi, M. and Butt, N., S. (2017). The exponential Pareto power series distribution: Theory and Applications, *Pakistan Journal of Statistics and Operation Research*, **13(**3): 603 - 615.

Elgarhy, M., Ahsan ul Haq, M. and Qurat ul Ain, (2018). Exponential generalized Kumaswamy distribution with applications. *Annals of Data Science*, **5**(2): 273 – 292.

Fatima, A. and Roohi, A. (2015). Transmuted exponential Pareto-I distribution. *Pakistan, Journal, Statistics*, **32**(1): 63 – 80.

Flores, J., Borges, P., Cancho, V., G. and Louzada, F. (2013). The complementary power series distribution. *Brazilian Journal of Probability and Statistics*, **27**(4): 565-584.

Gómez, Y., M., Bolfarine, H. and Gómez, H., W. (2014). A new extension of the exponential distribution. *Revista Colombiana de Estadistica*, **37**(1): 25-34.

Graham, R. (1994). *Concrete mathematics*: A Foundation for Computer Science. Addison-Wesley, Boston.

Guerra, R., R., Pena-Ramirez, F., A. and Cordeiro, G., M. (2018). A new Nadarajah-Haghighi generalization: simulation andapplication. Available at: http://www.est.ufmg.br/~enricoc/pdf/medicina/Grupo18/4bfX5en0VL GQxIOgOxk0ep8IJij2.pdf . [Accessed on 8th December, 2018].

Gupta, R., D. and Kundu, D. (1999). Generalized exponential distributions. *Australian and New Zealand Journal of Statistics*, **41** (2): 173-188.

Gupta, R., D. and Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull distribution. *Biometrical Journal*, **43**(1): 117-130.

Hassan, A., Assar, S. and Ali, K. (2015). The complementary Poisson Lindley class of distribution. *International Journal of Advanced Statistics and Probability*, **3(**2): 146 – 160.

Hassan, A., S., Abd-Elfattah, A., H., and Mokhtar, A., H. (2016). The complementary exponentiated inverted Weibull power series family of distributions and its applications. *British Journal of Mathematics and Computer Science*, **13**(2): 1 – 20.

Hassan, A., S., Abd-Elfattah, A., M. and Mokhtar, A., H. (2015). The complementary Burr III Poisson distribution. *Australian Journal of Basic and Applied Sciences*, **9(**11): 219 – 228.

Hassan, B., S., Miroslav R., M., Asgharzadeh, A., Esmaily, L. and Bander, A., M. (2012). An exponentiated exponential binomial distribution with application . *Statistics and Probability Letters,Elsevier*, **82(**6): 1067 – 1081.

Hurvich, C., M. and Tsai, C. (1989). Regression and time series model selection in small samples. *Biometrika*, **76**(2): 292 – 307.

Khan M., N., Saboor, A., Cordeiro, G., M., Nazir, M. and Pescim, R., R. (2018). Weighted Nadarajah – Haghighi distribution. *U.P.B Scientific Bulletin, Series A: Applied Mathematics and Physics*, **80**(4): 133 – 140.

Kumar, D. and Kumar, M. (2018). A new generalization of the extended exponential distribution with an application. *Annals of Data Science*. Doi: 10.1007/s40745-018-0181-0.

Louzada, F., Marchi, V., and Carpenter, J. (2013). The complementary exponentiated exponential geometric lifetime distribution. *Journal of Probability and Statistics*, http://dx.doi.org/10.1155/2013/502159

Louzada, F., Marcho, V., and Roman, M. (2014). The exponentiated exponential-geometric distribution: a distribution with decreasing, increasing and unimodal failure rate. *A Journal of Theoretical and Applied Statistics,* **48**(1): 167 – 181.

Louzada, F., Roman, M. and Cancho, V., G. (2011). The complementary exponential geometric distribution: Model properties and comparison with its counterpart. *Computational Statistics and Data Analysis,* **55**(8): 2516 – 2524.

Louzada, F., Yamachi, C., Y., Marchi, V., A., A. and Franco, M., A., P. (2014). The long-term exponentiated complementary exponential geometric distribution under a latent complementary causes framework. *Tendências em Matemática Aplicada e Computacional,* **15**(1): 19 – 35.

Mahmoudi, E., and Jafari, A., A. (2012). Generalized exponential – power series distributions. *Computational Statistics and Data Analysis,* 56(12): 4047 – 4066.

Muhammad, M. (2017). Generalized half-logistic Poisson distributions. *Communications for Statistical Applications and Methods*, 24(4): 353 – 356.

Muhammad, M. (2017). The complementary exponentiated Burr XII Poisson distribution: model, properties and application. *Journal of Statistics Applicationand Probability*, **6**(1): 33 – 48

Nadarajah, S. and Haghighi, F. (2011). An extension of the exponential distribution. *Statistics: A Journal of Theoretical and Applied Statistics*, **45**(6): 543-558.

Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution. *Reliability Engineering and System Safety*, 91(6): 689-697.

Nasir, A., Yousof, H., M., Jamal, F. and Korkmaz, M., C. (2019). The exponentiated Burr XII power series distribution: properties and applications. *Stats*, **2**(1): 15 – 31.

Nasiru, S., Atem, B., A., M. and Nantomah, K. (2018). Poisson exponentiated erlang-truncated exponential distribution. *Journal of Statistics Applications and probability*, **7**(2): 1 – 17.

Nasiru, S., Mwita P., N., Ngesa, O. (2018). The exponentiated generalized power series family of distribution. *Annals of Data Science*, htt://doi.org/10.1007/s40745-018-0170-3.

Ohnishi, M. (2002). Stochastic order in reliability theory. In S. Osaki, (eds.). *Stochastic models in reliability and maintenance* (pp. 31 - 63). Springer-Verlage Berlin Heidelberg.

Okasha, H., M. (2017). A new family of Topp and Leone geometric distribution with reliability applications. *Journal of Failure Analysis and Prevention*, **17**(3): 477 - 489

Peña-Ramírez, F., A., Guerra, R., M., Marinho, P., R., D. (2018). The exponentiated power generalized Weibull: properties and application. *Mathematical Science*. Doi: 10.1590/0001-3765201820170423.

Peña-Ramírez, F., A., Guerra, R., R. and Cordeiro, G., M. (2018). A new generalization of the Nadarajah-Haghighi distribution. *XXVIII Simposio Internacional de Estadística*, **23**(27):1-12.

Rahmouni, M., and Orabi, A. (2018). The exponential - generalized truncated geometric distribution. A new lifetime distribution. *International Journal of Statistics and Probability*, **7**(1): 1 – 20.

Rashid, A., Ahmad, Z. and Jan, T., R. (2017). Complementary compound Lindley power series distribution with application. *Journal of Reliability and Statistical Studies*, **10(**2): 143 – 158.

Rodrigues, J., A., Silva, A., P., C., M. and Hamedani, G., G. (2016). The exponentiated Kumaraswamy inverse Weibull distribution with application in survival analysis. *Journal of Statistical Theory and Applications*, **15**(1): 8 – 24.

Saboor, A., Elbatal, I., Khan, M., N., Cordeiro, G., Pescim, R. (2017). The beta exponentiated Nadarajah-Haghighi distribution: theory and application. https://hal.archives-ouvertes.fr/hal-01570564/document. accessed on 17th November, 2018.

Shafiei, S., Darijani, S. and Saboori, H. (2015). Inverses Weibull power series distribution: properties and applications. Journal of Statistical Computation and Simulation, **86**(6): 1 – 26.

Schwarz (1978). Estimating the dimension of a model. *The Annals of Statistics*, **6**(2): 461 – 464.

Shaw, W., T. and Buckley, I., R., C. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions and skew-kurtotic-normal distribution from a rank transmutation map. Research report.

Silva, R., B., Bourguignon, M., Dias, C., R., B. and Codeiro, G., M. (2013). The compound class of extended Weibull power series of distribution. *Computational Statistics and Data Analysis*, **58**: 352 – 367

Tahir, H. M., Cordeiro, G. M., Ali, S., Dey, S. and Manzoor, A. (2018). Inverted Nadarajah-Haghighi distribution: estimation methods and applications.

Journal of Statistical Computation and Simulation. Doi: 10.1080/00949655.2018.1487441.

Tahmasebi, S. and Jafari, A., A. (2015). Exponentiated extended Weibull – power series class of distributions. *Ciencia e Natura, Santa Maria*, **37(**2): 183 – 193.

Tahmasebi, S. and Jafari, A., A. (2015). Generalized Gompertz-power series distributions. https://arxiv.org/abs/1508.07634v1

Vatto, V., T., Nascimento, A., D., C., Miranda, F., W., R., Lima, M., C., S., Pinho, L., G., B. and Cordeiro, G., M. (2016). Exponentiated generalized Nadarajah-Haghighi distribution. *Chilean Journal of Statistics.* Available at: https://arxiv.org/abs/1610.08876. accessed on 11th December, 2018.

Warahena-Liyange, A. and Pararai, M. (2015). The lindley power series class of distributions: model, properties and applications. *Journal of Computations and Modeling*, **5**(3): 35 – 80.

Yousof, H. M. and Korkmaz, M. Ç. (2017). Topp-Leone Nadarajah-Haghighi distribution. *Journal of Statisticians: Statistics and Actuarial Sciences*, **10(**2**)**: 199-128.

Yousof, H., M., Korkmaz, M., C. and Hamedani, G., G. (2017). The odd Lindley Nadarajah-Haghighi distribution. *Journal of Mathematical Computed Science*, **7(**5): 864 – 882.