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# A PROOF OF JENSEN'S INEQUALITY THROUGH A NEW STEFFENSEN'S INEQUALITY

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**Abstract.** In this paper, we present more proofs of the new Steffensen's inequality for convex functions. First, we provide separate proofs for continuous functions followed by a general proof for all $L^1([0,1])$ functions. The last part is dedicated to the proof of the well known Jensen's inequality using the new inequality.

**Keywords:** Steffensen; Jensen; inequality; continuous function.

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## 1. Introduction

The following Steffensen's inequality (1) was unpopular in the research environment for some decades since its discovery in 1918:

$$\int_{b-\lambda}^{b} g(x)dx \leq \int_{a}^{b} g(x)f(x)dx \leq \int_{a}^{a+\lambda} g(x)dx, \tag{1}$$

where $f$ and $g$ are integrable functions defined on $(a,b)$, $g$ is decreasing and for each $x \in (a,b)$, $0 \leq f(x) \leq 1$ and $\lambda = \int_a^b f(x)dx$. This inequality (1) is known in literature as one of the useful inequalities in mathematical analysis (See [2]) and ([4], pp. 311-312). The second of its

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appearance was in [9]. Then in 1959, the inequality attracted the attention of R. Bellman [2] who gave the following generalization

$$\left(\int_a^b f(x)g(x)dx\right)^p \leq \int_a^{a+\lambda} g(x)^p dx \tag{2}$$

where $\lambda = \left(a + \int_a^b f(x)dx\right)^p$, $g \in L^p[a,b]$, $f \geq 0$ and $\int_a^b f^q \leq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ for $p > 1$. This inequality was noted to be invalid and was thus corrected by J.E. Pecaric [7] with the result

$$\left(\int_0^1 f(x)g(x)dx\right)^p \leq \int_0^{\lambda} g(x)^p dx \tag{3}$$

where $g : [0,1] \longrightarrow \Re$ is a nonnegative and nonincreasing function, $f : [0,1] \longrightarrow \Re$ is an integrable function such that $0 \leq f(x) \leq 1$ ($\forall x \in [0,1]$), $p \geq 1$ and $\lambda = \left(\int_0^1 f(x)dx\right)^p$ (see also [5] and [8]).

Now the task in this paper is to give some proofs of (1) with involvement of convex functions and further deduce the Jensen's inequality from the results.

## 2. Preliminaries

Let us present some definitions of convex functions here. The study of convex functions were first defined and systematically studied by J. L. W. V. Jensen who gave the study of (algebraic) inequalities as principal object of his investigation of convex functions (see [1], [6]).

**Definition 1.1.** Let $I$ be an interval in $\Re$. A function $\phi : I \longrightarrow \Re$ is said to be convex if for all $x_1, x_2 \in I$ and for all positive $a_1$ and $a_2$ satisfying $a_1 + a_2 = 1$,

$$\phi(a_1x_1 + a_2x_2) \leq a_1\phi(x_1) + a_2\phi(x_2). \tag{4}$$

A convex function necessarily is continuous for $x_1, x_2 \in I$.

**Definition 1.2.** A function $\phi$ is said to be concave if $-\phi$ is convex (i.e. if the inequality (4) is reversed).

For example, $|x|$, $x^k$ for $k > 1$, $e^x$ etc. are convex functions whiles $x^k$ for $0 < k < 1$, $\log x$, $\sqrt{x}$ for $x \geq 0$ etc. are concave functions.

## 3. Main results

We begin as follows:

**Theorem 3.1.** [3] *Let the function* $f : [0,1] \to \Re$ *be continuous such that* $0 \leq f(x) \leq 1$. *If* $\phi : [0,1] \to \Re$ *is a convex, differentiable function with* $\phi(0) = 0$. *Then*

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx \tag{5}$$

*for all* $x \in [0,1]$.

Next, we give two proofs.

**Proof 1.** The function $\phi(x)$ is convex and differentiable on the interval $[0,1]$ and it's differential, $\phi'(x)$ is nondecreasing for all $x \in [0,1]$. Then $-\phi'(x)$ is decreasing, and making the substitution of $g(x) = -\phi'(x)$, $a = 0$ and $b = 1$ into inequality (1) gives

$$\int_0^\lambda \phi'(x)dx \leq \int_0^1 f(x)\phi'(x)dx \leq \int_{1-\lambda} \phi'(x)dx.$$

This simplifies to

$$\phi(\lambda) - \phi(0) \leq \int_0^1 f(x)\phi'(x)dx \leq \phi(1) - \phi(1-\lambda).$$

Since $\lambda = \int_0^1 f(x)dx$ and $\phi(0) = 0$, thus

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx.$$

**Proof 2.** Let

$$F(x) = \int_0^x f(t)dt \leq x \tag{6}$$

and

$$G(x) = \phi(F(x)) = \phi\left(\int_0^x f(t)dt\right).$$

Since $f$ is continuous and the differential of $F(x)$ denoted $F'(x) = f(x)$, then by the chain rule of differentiation we obtain

$$G'(x) = F'(x)\phi'(F(x)) \leq f(x)\phi'(x).$$

Thus

$$\int_0^1 G'(x)dx \leq \int_0^1 f(x)\phi'(x)dx. \tag{7}$$

Since $\phi(0) = 0$ and

$$\begin{aligned}\int_0^1 G'(x)dx &= G(1) - G(0)\\ &= \phi(F(1)) - \phi(F(0))\\ &= \phi\left(\int_0^1 f(x)dx\right) - \phi(0)\\ &= \phi\left(\int_0^1 f(x)dx\right),\end{aligned}$$

inequality (7) becomes

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx.$$

**Example 3.1.** Consider the convex function $\phi(u) = u^p$ for $1 \leq p < \infty$, then

$$\left(\int_0^1 f(x)dx\right)^p \leq p\int_0^1 f(x)x^{p-1}dx.$$

For a particular case of $p = 2$, we get

$$\left(\int_0^1 f(x)dx\right)^2 \leq 2\int_0^1 xf(x)dx.$$

Let us also prove the new inequality for functions $f \in L^1([0,1])$. We denote by $C_c([0,1])$ the set of all continuous functions on $[0,1]$ with compact support. Thus

$$C_c([0,1]) \subset L^1([0,1]).$$

**Theorem 3.2.** *Let $f \in L^1([0,1])$ such that $0 \leq f(x) \leq 1$ for all $x \in [0,1]$. If $\phi : [0,1] \longrightarrow \Re$ is a convex, differentiable function with $\phi(0) = 0$ then*

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx. \tag{8}$$

**Proof.** Let $f \in L^1([0,1])$ and $\varepsilon = \frac{1}{n} > 0$, there exists sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $C_c([0,1])$ such that $||f - f_n||_1 < \frac{1}{n}$. Since $f_n$ is continuous then by Theorem 3.1, we have

$$\begin{aligned}\phi\left(\int_0^1 f_n(x)dx\right) &\leq \int_0^1 f_n(x)\phi'(x)dx \\ &\leq \int_0^1 f(x)\phi'(x)dx + \int_0^1 [f_n(x) - f(x)]\phi'(x)dx.\end{aligned}$$

Since

$$\left|\int_0^1 f_n(x)dx - \int_0^1 f(x)dx\right| \leq \int_0^1 |f_n(x) - f(x)|dx < \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty$$

Thus

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx. \quad \square$$

**Remark 3.1** Let us remark that $\phi'(x)$ is bounded in $[0,1]$. So

$$\begin{aligned}\int_0^1 |f_n(x) - f(x)|\phi'(x)dx &\leq ||\phi'||_\infty \int_0^1 |f_n(x) - f(x)|dx \\ &\leq \frac{1}{n}||\phi'||_\infty \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

**Example 3.2.** Let

$$f(x) = \begin{cases} a_1 & \text{if} \quad 0 \leq x < x_1 \\ a_2 & \text{if} \quad x_1 \leq x < x_2 \\ \vdots & \\ a_n & \text{if} \quad x_{n-1} \leq x \leq 1 \end{cases}$$

where $0 < a_1 < a_2 < \cdots < a_n < 1$, $x_0 = 0$ and $x_n = 1$. Putting $f$ into (8) and evaluating the result, yields

$$\phi\left(\sum_{i=1}^n a_i(x_i - x_{i-1})\right) \leq \sum_{i=1}^n a_i\{\phi(x_i) - \phi(x_{i-1})\}. \tag{9}$$

**Theorem 3.3.** (Jensen's inequality) *Let* $0 \leq b_1, b_2, \ldots, b_n \leq 1$ *with* $\sum_{j=1}^{n} b_j = 1$ *and let* $0 < x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$. *If* $\phi : [0,1] \longrightarrow \Re$ *is convex and differentiable with* $\phi(0) = 0$, *then*

$$\phi\left(\sum_{j=1}^{n} b_j x_j\right) \leq \sum_{j=1}^{n} b_j \phi(x_j). \tag{10}$$

**Proof.** Expansion of inequality (9) gives

$$\phi\left\{a_1(x_1 - x_0) + a_2(x_2 - x_1) + \cdots + a_n(x_n - x_{n-1})\right\}$$

$$\leq a_1[\phi(x_1) - \phi(x_0)] + a_2[\phi(x_2) - \phi(x_1)] + \cdots + a_n[\phi(x_n) - \phi(x_{n-1})].$$

Since $x_0 = 0$, we have

$$\phi\left(a_n x_n + \sum_{j=1}^{n-1}(a_j - a_{j+1})x_j\right) \leq a_n \phi(x_n) + \sum_{j=1}^{n-1}(a_j - a_{j+1})\phi(x_j). \tag{11}$$

Now, we consider $0 < b_1, b_2, \ldots, b_n < 1$ with $\sum_{j=1}^{n} b_j = 1$. Let

$$a_1 = b_1 + b_2 + \cdots + b_{n-1} + b_n = 1$$

$$a_2 = b_2 + b_3 + \cdots + b_{n-1} + b_n$$

$$a_3 = b_3 + b_4 + \cdots + b_{n-1} + b_n$$

$$\vdots$$

$$a_{n-1} = b_{n-1} + b_n$$

$$a_n = b_n.$$

This shows that

$$a_j - a_{j+1} = b_j, \quad \text{for} \quad j = 1, \cdots, n-1. \tag{12}$$

Putting (12) into (11), yields

$$\phi\left(a_n x_n + \sum_{j=1}^{n-1} b_j x_j\right) \leq a_n \phi(x_n) + \sum_{j=1}^{n-1} b_j \phi(x_j). \tag{13}$$

Since $a_n = b_n$, thus (13) becomes $\phi\left(\sum_{j=1}^{n} b_j x_j\right) \leq \sum_{j=1}^{n} b_j \phi(x_j)$ as required.

## Conclusion

The Steffensen's inequality was discussed for convex and differentiable functions. Simple analytical proofs were provided for the new Steffensen's inequality. The well known Jensen's inequality was also proved from the new inequality.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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