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New Inequalities Involving the Dirichlet Beta and Euler's Gamma Functions

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Abstract: We present some new inequalities involving the Dirichlet Beta and Euler's Gamma functions. The concept of monotonicity of Dirichlet Beta function is also discussed. The generalized forms of the Hölder's and Minkowski's inequalities among other techniques are employed.

Keywords: Dirichlet beta function, Gamma function, Inequality

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1 Introduction

The Dirichlet beta function (also known as the Catalan beta function or the Dirichlet's L-function) is defined for $x > 0$ by [4, p. 56]

$$\beta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t + e^{-t}}\, dt \tag{1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^x}$$

where $\Gamma(x)$ is the classical Euler's Gamma function defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

and satisfying the basic relation

$$\Gamma(x+1) = x\Gamma(x). \tag{2}$$

Let $K(x)$ be defined as

$$K(x) = \beta(x)\Gamma(x) = \int_0^\infty \frac{t^{x-1}}{e^t + e^{-t}}\, dt, \quad x > 0. \tag{3}$$

The Dirichlet beta function, which is closely related to the Riemann zeta function, has important applications in Analytic Number Theory as well as other branches of mathematics. See for instance [2], [3] and the related references therein. In particular, $\beta(1) = \frac{\pi}{4}$ and $\beta(2) = G$, where $G = 0.915965594177\ldots$ is the Catalan constant [5].

For positive integer values of n, the function $\beta(x)$ may be evaluated explicitly by

$$\beta(2n+1) = \frac{(-1)^n E_{2n} \pi^{2n+1}}{4^{n+1}(2n)!}$$

where E_n are the Euler numbers generated by

$$\frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

Also, in terms of the polygamma function $\psi^{(m)}(x)$, the function $\beta(x)$ may be written as [6]

$$\beta(n) = \frac{1}{2^{2n}(n-1)!} \left[\psi^{(n-1)}\left(\frac{1}{4}\right) - \psi^{(n-1)}\left(\frac{3}{4}\right) \right].$$

The main objective of this paper is to establish some inequalities involving the Dirichlet Beta and Euler's Gamma functions. We begin by recalling the following lemmas which shall be required in order to establish our results.

2 Preliminaries

Lemma 1(Generalized Hölder's Inequality). *Let $f_1, f_2, \ldots, f_n$ be functions such that the integrals exist.*

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Then the inequality

$$\int_a^b \left|\prod_{i=1}^n f_i(t)\right| dt \le \prod_{i=1}^n \left(\int_a^b |f_i(t)|^{\alpha_i} dt\right)^{\frac{1}{\alpha_i}} \tag{4}$$

holds for $\alpha_i > 1$ *such that* $\sum_{i=1}^n \frac{1}{\alpha_i} = 1$.

*Proof.*See page 790-791 of [1].

Lemma 2(Generalized Minkowski's Inequality). *Let* $f_1, f_2, \ldots, f_n$ *be functions such that the integrals exist. Then the inequality*

$$\left(\int_a^b \left|\sum_{i=1}^n f_i(t)\right|^u dt\right)^{\frac{1}{u}} \le \sum_{i=1}^n \left(\int_a^b |f_i(t)|^u dt\right)^{\frac{1}{u}} \tag{5}$$

holds for $u \ge 1$.

*Proof.*See page 790-791 of [1].

Lemma 3([9]). *Let* f *and* h *be continuous rapidly decaying positive functions on* $[0,\infty)$. *Further, let* F *and* H *be defined as*

$$F(x) = \int_0^\infty f(t)t^{x-1}\,dt \quad \text{and} \quad H(x) = \int_0^\infty h(t)t^{x-1}\,dt.$$

If $\frac{f(t)}{h(t)}$ *is increasing, then so is* $\frac{F(x)}{H(x)}$.

Lemma 4([7]). *Let* f *and* g *be two nonnegative functions of a real variable and* m, n *be real numbers such that the integrals in* (6) *exist. Then*

$$\int_a^b g(t)\,(f(t))^m\,dt \cdot \int_a^b g(t)\,(f(t))^n\,dt \ge \left(\int_a^b g(t)\,(f(t))^{\frac{m+n}{2}}\,dt\right)^2 \tag{6}$$

Lemma 5([8]). *Let* $f:(0,\infty)\to(0,\infty)$ *be a differentiable, logarithmically convex function. Then the function*

$$g(x) = \frac{(f(x))^\alpha}{f(\alpha x)}$$

is decreasing if $\alpha \ge 1$, *and increasing if* $0 < \alpha \le 1$.

3 Main Results

We present the main findings of the paper in this section.

Theorem 1.*For* $i = 1,2,\ldots,n$, *let* $\alpha_i > 1$ *such that* $\sum_{i=1}^n \frac{1}{\alpha_i} = 1$. *Then the inequality*

$$\frac{\Gamma\left(\sum_{i=1}^n \frac{x_i}{\alpha_i}\right)}{\prod_{i=1}^n (\Gamma(x_i))^{\frac{1}{\alpha_i}}} \le \frac{\prod_{i=1}^n (\beta(x_i))^{\frac{1}{\alpha_i}}}{\beta\left(\sum_{i=1}^n \frac{x_i}{\alpha_i}\right)} \tag{7}$$

is valid for $x_i > 0$.

*Proof.*Let $K(x)$ be defined as in (3). Then by utilizing Lemma 1, we obtain

$$\begin{aligned}
K\left(\sum_{i=1}^n \frac{x_i}{\alpha_i}\right) &= \int_0^\infty \frac{t^{(\sum_{i=1}^n \frac{x_i}{\alpha_i})-1}}{e^t+e^{-t}}\,dt \\
&= \int_0^\infty \frac{t^{\sum_{i=1}^n \frac{x_i-1}{\alpha_i}}}{(e^t+e^{-t})^{\sum_{i=1}^n \frac{1}{\alpha_i}}}\,dt \\
&= \int_0^\infty \prod_{i=1}^n \left(\frac{t^{x_i-1}}{e^t+e^{-t}}\right)^{\frac{1}{\alpha_i}} dt \\
&\le \prod_{i=1}^n \left(\int_0^\infty \frac{t^{x_i-1}}{e^t+e^{-t}}\,dt\right)^{\frac{1}{\alpha_i}} \\
&= \prod_{i=1}^n (K(x_i))^{\frac{1}{\alpha_i}}
\end{aligned}$$

which gives the required result (7).

*Remark.*If $n=2$, $\alpha_1 = a$, $\alpha_2 = b$, $x_1 = x$ and $x_2 = y$, then, we have

$$K\left(\frac{x}{a}+\frac{y}{b}\right) \le (K(x))^{\frac{1}{a}}\,(K(y))^{\frac{1}{b}}$$

which implies that $K(x)$ is logarithmically convex. Also, since every logarithmically convex function is convex, it follows that $K(x)$ is convex.

Corollary 1.*The inequality*

$$\left[\frac{\beta'(x)}{\beta(x)}\right]^2 - \frac{\beta''(x)}{\beta(x)} \le \psi'(x) \tag{8}$$

holds for $x > 0$, *where* $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ *is the Digamma function.*

*Proof.*Since $K(x) = \beta(x)\Gamma(x)$ is logarithmically convex, then $(\ln K(x))'' \ge 0$ which results to (8).

Theorem 2.*Let* $x_i > 0$, $i = 1,2,\ldots,n$ *and* $u \ge 1$. *Then the inequality*

$$\left(\sum_{i=1}^n \beta(x_i)\Gamma(x_i)\right)^{\frac{1}{u}} \le \sum_{i=1}^n (\beta(x_i)\Gamma(x_i))^{\frac{1}{u}} \tag{9}$$

holds.

*Proof.*Let $K(x)$ be defined as in (3). Then by using the fact that $\sum_{i=1}^n a_i^u \le (\sum_{i=1}^n a_i)^u$, for $a_i \ge 0$, $u \ge 1$ in conjunction with Lemma 2, we obtain

$$\begin{aligned}
\left(\sum_{i=1}^{n}K(x_i)\right)^{\frac{1}{u}} &= \left(\sum_{i=1}^{n}\int_0^\infty \frac{t^{x_i-1}}{e^t+e^{-t}}\,dt\right)^{\frac{1}{u}}\\
&= \left(\int_0^\infty \sum_{i=1}^{n}\frac{t^{x_i-1}}{e^t+e^{-t}}\,dt\right)^{\frac{1}{u}}\\
&= \left(\int_0^\infty \sum_{i=1}^{n}\left[\frac{t^{\frac{x_i-1}{u}}}{(e^t+e^{-t})^{\frac{1}{u}}}\right]^u dt\right)^{\frac{1}{u}}\\
&\le \left(\int_0^\infty \left[\sum_{i=1}^{n}\frac{t^{\frac{x_i-1}{u}}}{(e^t+e^{-t})^{\frac{1}{u}}}\right]^u dt\right)^{\frac{1}{u}}\\
&\le \sum_{i=1}^{n}\left(\int_0^\infty \left[\frac{t^{\frac{x_i-1}{u}}}{(e^t+e^{-t})^{\frac{1}{u}}}\right]^u dt\right)^{\frac{1}{u}}\\
&= \sum_{i=1}^{n}(K(x_i))^{\frac{1}{u}}
\end{aligned}$$

which yields the result (9).

Theorem 3. *The function $\beta(x)$ is monotone increasing on $(0,\infty)$. That is, for $0<x\le y$, we have*

$$\beta(x)\le\beta(y). \tag{10}$$

Proof. Let F, H, f and h be defined as

$$F(x)=\int_0^\infty \frac{t^{x-1}}{e^t+e^{-t}}\,dt,\quad H(x)=\int_0^\infty t^{x-1}e^{-t}\,dt=\Gamma(x),$$

$$f(t)=\frac{1}{e^t+e^{-t}}\quad\text{and}\quad h(t)=e^{-t}.$$

Then, $\frac{f(t)}{h(t)}=\frac{1}{1+e^{-2t}}$ is increasing and by Lemma 3, $\frac{F(x)}{H(x)}$ is increasing as well. Thus, for $0<x\le y$, we have

$$\frac{F(x)}{H(x)}\le\frac{F(y)}{H(y)}\quad\Longleftrightarrow\quad F(x)H(y)\le F(y)H(x)$$

which implies

$$\int_0^\infty \frac{t^{x-1}}{e^t+e^{-t}}\,dt\cdot\int_0^\infty t^{y-1}e^{-t}\,dt \le \int_0^\infty \frac{t^{y-1}}{e^t+e^{-t}}\,dt\cdot\int_0^\infty t^{x-1}e^{-t}\,dt$$

which further implies

$$\beta(x)\Gamma(x)\Gamma(y)\le\beta(y)\Gamma(y)\Gamma(x).$$

Thus

$$\beta(x)\le\beta(y)$$

as required.

Corollary 2. *Let $x_i>0$ for $i=1,2,3,\ldots,n$. Then the inequality*

$$\prod_{i=1}^{n}\beta(x_i)\le\left[\beta\left(\sum_{i=1}^{n}x_i\right)\right]^n \tag{11}$$

is valid.

Proof. Let $x_i>0$ for $i=1,2,3,\ldots,n$. Then since $\beta(x)$ is increasing, we have

$$\begin{aligned}
0<\beta(x_1)&\le\beta\left(\sum_{i=1}^{n}x_i\right),\\
0<\beta(x_2)&\le\beta\left(\sum_{i=1}^{n}x_i\right),\\
&\vdots\\
0<\beta(x_n)&\le\beta\left(\sum_{i=1}^{n}x_i\right).
\end{aligned}$$

Taking products yields

$$\prod_{i=1}^{n}\beta(x_i)\le\left[\beta\left(\sum_{i=1}^{n}x_i\right)\right]^n$$

as required.

Remark. In particular, if $n=2$, $x_1=x$ and $x_2=y$ in (11), then we obtain

$$\beta(x)\beta(y)\le[\beta(x+y)]^2.$$

Theorem 4. *The inequality*

$$(x+1)\frac{\beta(x+2)}{\beta(x+1)}\ge x\frac{\beta(x+1)}{\beta(x)} \tag{12}$$

holds for $x>0$.

Proof. Let $x>0$, $g(t)=\frac{1}{e^t+e^{-t}}$, $f(t)=t$, $m=x-1$, $n=x+1$, $a=0$ and $b=\infty$. Then by Lemma 4, we have

$$\int_0^\infty \frac{t^{x-1}}{e^t+e^{-t}}\,dt\cdot\int_0^\infty \frac{t^{x+1}}{e^t+e^{-t}}\,dt\ge\left[\int_0^\infty \frac{t^{x}}{e^t+e^{-t}}\,dt\right]^2$$

which implies

$$\beta(x)\Gamma(x)\cdot\beta(x+2)\Gamma(x+2)\ge[\beta(x+1)\Gamma(x+1)]^2. \tag{13}$$

By using the functional equation (2), the relation (13) becomes

$$(x+1)\beta(x)\beta(x+2)\ge x(\beta(x+1))^2$$

which gives the required result.

*Remark.*We deduce from inequality (12) that the function

$$\phi(x) = x\frac{\beta(x+1)}{\beta(x)}$$

is increasing on $(0,\infty)$. This implies

$$\frac{\beta(x+1)}{\beta(x)} + x\left[\frac{\beta(x+1)}{\beta(x)}\right]' \geq 0$$

or equivalently,

$$\beta(x+1)\left[1 - x\frac{\beta'(x)}{\beta(x)}\right] + x\beta'(x+1) \geq 0$$

for $x > 0$.

Corollary 3.*The inequality*

$$\frac{4G}{\pi} < x\frac{\beta(x+1)}{\beta(x)} < \frac{\pi^3}{16G} \qquad (14)$$

holds for $x \in (1,2)$, where G is the Catalan's constant.

*Proof.*Since $\phi(x) = x\frac{\beta(x+1)}{\beta(x)}$ is increasing, then for $x \in (1,2)$, we have $\phi(1) < \phi(x) < \phi(2)$ which results to (14).

Theorem 5.*Let $\alpha \geq 1$ and $x \in (0,1)$. Then,*

$$\frac{G^{\alpha}}{\beta(1+\alpha)\Gamma(1+\alpha)} \leq \frac{[\beta(1+x)\Gamma(1+x)]^{\alpha}}{\beta(1+\alpha x)\Gamma(1+\alpha x)} \leq \left(\frac{\pi}{4}\right)^{\alpha-1} \qquad (15)$$

where G is the Catalan's constant. The inequality is reversed if $0 < \alpha \leq 1$.

*Proof.*Let $f(x) = \beta(1+x)\Gamma(1+x)$. Then $f(x)$ is differentiable and by Remark 3, it is logarithmically convex. Then by Lemma 5, the function $g(x) = \frac{[\beta(1+x)\Gamma(1+x)]^{\alpha}}{\beta(1+\alpha x)\Gamma(1+\alpha x)}$ is decreasing for $\alpha \geq 1$. Hence for $x \in (0,1)$, we have $g(1) \leq g(x) \leq g(0)$ yielding the result (15). If $0 < \alpha \leq 1$, then $g(x)$ is increasing and for $x \in (0,1)$, we have $g(0) \leq g(x) \leq g(1)$ which gives the reverse inequality of (15).

4 Conclusion

In this study, we have established some inequalities involving the Dirichlet beta and Euler's Gamma functions. We have also discussed the monotonicity of the Dirichlet beta function. The generalized forms of the Hölder's and Minkowski's inequalities among other analytical techniques were employed.

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