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A NOTE ON SOME VARIANTS OF JENSEN'S INEQUALITY

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Abstract

In this paper, we present a refined Steffensen's inequality for convex functions and further prove some variants of Jensen's inequality using the new Steffensen's inequality.

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1. Introduction

The inequality

$$\int_{b-\lambda}^{b} g(x)dx \leq \int_{a}^{b} g(x)f(x)dx \leq \int_{a}^{a+\lambda} g(x)dx \tag{1}$$

was discovered in 1918 by Steffensen [10], where $\lambda = \int_a^b f(x)dx$, f and g are integrable functions defined on (a, b), g is decreasing and $0 \leq f(x) \leq 1$ for each $x \in (a, b)$. See also [6], [7], [8] and [9].

Let I be an interval in $\mathbb{R}$. If $\psi : I \to \mathbb{R}$ is convex, then for all x_1, $x_2 \in I$ and all positive numbers a_1 and a_2 satisfying $a_1 + a_2 = 1$, we have

$$\psi(a_1x_1 + a_2x_2) \leq a_1\psi(x_1) + a_2\psi(x_2).$$

Jensen [3] proved the inequality

$$\psi\left(\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i=1}^{n} a_i \psi(x_i), \tag{2}$$

where ψ is convex on an interval containing the real variables $x_1, x_2, ..., x_n$ and a_i $(1 \leq i \leq n)$ are positive weights such that $\sum_{i=1}^{n} a_i = 1$.

Mercer [5] proved the inequality

$$\psi\left(x_1 + x_n - \sum_{j=1}^{n} \lambda_j x_j\right) \leq \psi(x_1) + \psi(x_n) - \sum_{j=1}^{n} \lambda_j \psi(x_j), \tag{3}$$

where ψ is convex on an interval containing the real variables $x_1, x_2, ..., x_n$ with $0 < \lambda_j < 1$ $(j = 1, ..., n)$ such that $\sum_{j=1}^{n} \lambda_j = 1$.

The inequality (3) was first published by Mercer in 2003 as a variant of Jensen's inequality. One year later, Witkowski simply recovered the inequality in [11]. Thereafter, the inequality went through various refinements and generalizations. See for instance Matković [4] and the references therein.

The aim of this short note is to first provide a further proof of the following refined Steffensen's inequality (4) established in [2] and also recover the inequality (3) through the new inequality (4). Furthermore, another variant of the Jensen's inequality will be provided.

2. Preliminary Results

The following auxiliary results are presented.

Theorem 2.1 [2]. *Let* $0 \leq f(x) \leq 1$ *and* $\psi : [0, 1] \to \mathbb{R}$ *be a convex and differentiable function with* $\psi(0) = 0$. *If* $f : [0, 1] \to \mathbb{R}$ *is continuous*, *then*

$$\psi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\psi'(x)dx \tag{4}$$

for all $x \in [0, 1]$.

Proof. Let $\psi'(x)$ be the derivative of $\psi(x)$. Then $\psi'(x)$ is an increasing function on the interval $[0, 1]$ since $\psi(x)$ is an increasing function on $[0, 1]$. This implies that $-\psi'(x)$ is decreasing. Then by making the substitution $g(x) = -\psi'(x)$, $a = 0$ and $b = 1$ into (1) yields

$$\int_0^{\lambda}\psi'(x)dx \leq \int_0^1 f(x)\psi'(x)dx \leq \int_{1-\lambda}^1 \psi'(x)dx,$$

which simplifies to

$$\psi(\lambda) - \psi(0) \leq \int_0^1 f(x)\psi'(x)dx \leq \psi(1) - \psi(1-\lambda). \tag{5}$$

Since $\lambda = \int_0^1 f(x)dx$ and $\psi(0) = 0$, the first part of inequality (5) yields the required result

$$\psi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\psi'(x)dx.$$

Proposition 2.2 ([1], p. 62.). *Let* f_n *be a sequence of functions*. *If* $f_n \to f$ *in* L^1, *there is a subsequence* f_{n_j} *such that* $f_{n_j} \to f$ *almost everywhere* (*a.e.*).

3. Main Results

This section begins as follows:

Theorem 3.1. *Let* $f \in L^1([0, 1])$ *with* $0 \leq f(x) \leq 1$ *for all* $x \in [0, 1]$. *If* $\psi : [0, 1] \to \mathbb{R}$ *is a convex and differentiable function with* $\psi(0) = 0$, *then*

$$\psi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\psi'(x)dx. \tag{6}$$

Proof. Let $f_n \to f$ in L^1. Then by Proposition 2.2, there is subsequence $f_{n_j} \to f$, a.e.. Let there exists $h \in L^1$ such that $|f_{n_j}(x)| \leq h(x)$. Then by the dominated convergence theorem, $f \in L^1$ and

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\int_0^1 f_{n_j}(x)dx.$$

By the boundedness of $\psi'(x)$ on $[0, 1]$, we have

$$f_{n_j}(x)\psi'(x) \leq h(x)\psi'(x) \in L^1$$

which implies that

$$\int_0^1 f(x)\psi'(x)dx = \lim_{n\to\infty}\int_0^1 f_{n_j}(x)\psi'(x)dx. \tag{7}$$

Since f_n is continuous, then by (4), we have

$$\psi\left(\int_0^1 \lim_{n\to\infty} f_{n_j}(x)dx\right) \leq \lim_{n\to\infty}\int_0^1 f_{n_j}(x)\psi'(x)dx,$$

which yields the required result

$$\psi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\psi'(x)dx.$$

Example 3.2. Let $\psi(x) = e^x$. Then $\psi'(x) = e^x$. Thus

$$\exp\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\exp(x)dx.$$

Example 3.3. Let

$$f(x) = \begin{cases} b_1 & \text{if } 0 \leq x < x_1 \\ b_2 & \text{if } x_1 \leq x < x_2 \\ \vdots & \\ b_n & \text{if } x_{n-1} \leq x \leq 1, \end{cases}$$

where $0 < b_1 < b_2 < \cdots < b_n < 1$, $x_0 = 0$ and $x_n = 1$. Since

$$\int_{x_{i-1}}^{x_i} \psi'(x) dx = \psi(x_i) - \psi(x_{i-1}),$$

then by (6), we obtain

$$\psi\left(\sum_{i=1}^{n} b_i (x_i - x_{i-1})\right) \leq \sum_{i=1}^{n} b_i \{\psi(x_i) - \psi(x_{i-1})\}. \tag{8}$$

Theorem 3.4. *Let the function* ψ *be convex and differentiable on an interval containing an n-tuple* $x = (x_1, ..., x_n)$ *such that* $0 < x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$ *and* $a = (a_1, ..., a_n)$, *a positive n-tuple with* $\sum_{j=1}^{n} a_j = 1$. *If* $\psi(0) = 0$, *then*

$$\psi\left(x_1 + x_n - \sum_{j=1}^{n} a_j x_j\right) \leq \psi(x_1) + \psi(x_n) - \sum_{j=1}^{n} a_j \psi(x_j).$$

Proof. Consider $0 < a_j < 1$ $(1 \leq j \leq n)$ such that $\sum_{j=1}^{n} a_j = 1$. Let $b_n = 1 - a_n = \sum_{j=1}^{n-1} a_j$, such that $b_2 = a_1$ and $b_1 = 1$. Expansion of inequality (8) yields

$$\psi\{b_1(x_1 - x_0) + b_2(x_2 - x_1) + \cdots + b_n(x_n - x_{n-1})\}$$
$$\leq b_1[\psi(x_1) - \psi(x_0)] + b_2[\psi(x_2) - \psi(x_1)] + \cdots + b_n[\psi(x_n) - \psi(x_{n-1})].$$

Since $x_0 = 0$ and $\psi(0) = 0$,

$$\psi\{(b_1 - b_2)x_1 + (b_2 - b_3)x_2 + \cdots + (b_{n-1} - b_n)x_{n-1} + b_n x_n\}$$

$$\leq (b_1 - b_2)\psi(x_1) + (b_2 - b_3)\psi(x_2) + \cdots + (b_{n-1} - b_n)\psi(x_{n-1}) + b_n\psi(x_n).$$

Substitute $b_1 - b_2 = 1 - a_1$, $b_n = 1 - a_n$ and $-a_j = (b_j - b_{j+1})$ for $j = 2, ..., n-1$, we get

$$\psi\left((1 - a_1)x_1 + (1 - a_n)x_n - \sum_{j=2}^{n-1} a_j x_j\right)$$

$$\leq (1 - a_1)\psi(x_1) + (1 - a_n)\psi(x_n) - \sum_{j=2}^{n-1} a_j\psi(x_j).$$

Hence

$$\psi\left(x_1 + x_n - \sum_{j=1}^{n} a_j x_j\right) \leq \psi(x_1) + \psi(x_n) - \sum_{j=1}^{n} a_j\psi(x_j).$$

Remark 3.5. It is observed that

$$a_{n-1} = b_n - b_{n-1}$$

$$a_{n-2} = b_{n-1} - b_{n-2}$$

$$\vdots$$

$$a_2 = b_3 - b_2$$

$$a_1 - 1 = b_2 - b_1.$$

Thus

$$a_{n-1} + a_{n-2} + \cdots + a_1 - 1 = b_n - b_1$$

$$= 1 - a_n - 1.$$

Therefore,

$$\sum_{j=1}^{n} a_j = 1.$$

Next is a Lemma before another variant of the Jensen's inequality.

Lemma 3.6. *Let* $\psi(x)$ *be a convex and differentiable function on an interval I of real numbers. If* $\psi(0) = 0,$ *then*

$$\psi(x) \leq \psi'(x)x, \tag{9}$$

for all $x \in I.$

Proof. Let $x, y \in I.$ Since ψ is differentiable. Then by the definition of convexity, we have

$$\psi(y) - \psi(x) \geq \psi'(x)(y - x).$$

Putting $y = 0$ and $\psi(0) = 0,$ we obtain

$$\psi(x) \leq \psi'(x)x,$$

for all $x \in I.$

Theorem 3.7. *Let* ψ *be a convex and differentiable function on an interval containing an n-tuple* $x = (x_1, x_2, ..., x_n)$ *such that* $0 < x_1 \leq x_2 \leq \cdots \leq x_n \leq 1.$ *If* $\psi(0) = 0$ *and* $\sum_{j=1}^{n} a_j = 1$ *for* $0 < a_j < 1,$ *then*

$$\psi\left(\sum_{j=1}^{n} a_j x_j\right) \leq \sum_{j=1}^{n} a_j x_j \psi'(x_j). \tag{10}$$

Proof. Recall from (2) that

$$\psi\left(\sum_{j=1}^{n} a_j x_j\right) \leq \sum_{j=1}^{n} a_j \psi(x_j). \tag{11}$$

Then substitution of (9) into (11) yields the required result

$$\psi\left(\sum_{j=1}^{n} a_j x_j\right) \leq \sum_{j=1}^{n} a_j x_j \psi'(x_j).$$

Remark 3.8. The inequality (10) is reversed if ψ is concave.

Illustrative Example. Consider the convex function $\psi(x) = x^p$, $p > 1$, $x > 0$, then (10) becomes

$$\frac{1}{p}\left(\sum_{j=1}^{n} a_j x_j\right)^p \leq \sum_{j=1}^{n} a_j x_j^p. \tag{12}$$

For $p = 2$, we have

$$\frac{1}{2}\left(\sum_{j=1}^{n} a_j x_j\right)^2 \leq \sum_{j=1}^{n} a_j x_j^2.$$

4. Conclusion

This paper proved a refined Steffensen's inequality for convex functions. The Jensen-Mercer's inequality was also proved in this paper through exemplification of the refined Steffensen's inequality. A further variant of the Jensen's inequality was provided.

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References

[1] G. B. Folland, Real Analysis: Modern Techniques and their Applications, 2nd ed., John Willey and Sons, Inc., New York, 1999.

[2] M. M. Iddrisu, C. A. Okpoti and K. A. Gbolagade, A proof of Jensen's inequality through a new Steffensen's inequality, Adv. Inequal. Appl. 2014 (2014), Art. 29.

[3] J. L. W. V. Jensen, Sur les Fonctions Convexes et les inégalités entre les Valeurs Moyennes, Acta Mathematica 30 (1906), 175-193.

[4] A. Matković and J. Pečarić, On a variant of the Jensen-Mercer inequality for operators, J. Math. Ineq. 2 (2008), 299-307.

[5] A. McD. Mercer, A variant of Jensen's inequality, J. Ineq. Pure and Appl. Math. 4 (2003), Art. 73.

[6] D. S. Mitrinovic and J. E. Pečarić, On the Bellman generalization of Steffensen's inequality III, J. Math. Anal. and Appl. 135 (1988), 342-345.

[7] D. S. Mitrinovic, J. E. Pečarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, Boston, London, 1993.

[8] J. E. Pečarić, On the Bellman generalization of Steffensen's inequality, J. Math. Anal. and Appl. 88 (1982), 505-507.

[9] J. E. Pečarić, On the Bellman generalization of Steffensen's inequality II, J. Math. Anal. and Appl. 104 (1984), 432-434.

[10] J. F. Steffensen, On certain inequalities between mean values and their application to actuarial problems, Skandinavisk Aktuarietidskrift (1918), 82-97.

[11] A. Witkowski, A new proof of the monotonicity property of power means, J. Ineq. Pure and Appl. Math. 5 (2004), Art. 73.