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GEOMETRICAL PROOF OF NEW STEFFENSEN'S INEQUALITY AND APPLICATIONS

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Abstract. In this paper, we give a geometrical proof of a new Steffensen's inequality for convex functions. In addition, we present applications of the Steffensen's inequality leading to the determination of Fourier coefficients.

Keywords: Steffensen's inequality, geometrical proof, Fourier coefficient.

2010 AMS Subject Classification: 26D15.

1. Introduction

The the following inequality was discovered in 1918 by Steffensen [9]

$$\int_{b-\lambda}^{b} g(s)ds \leq \int_{a}^{b} g(s)f(s)ds \leq \int_{a}^{a+\lambda} g(s)ds, \tag{1}$$

where $\lambda = \int_a^b f(s)ds$, f and g are integrable functions defined on (a,b), g is monotone decreasing and for each $s \in (a,b)$, $0 \leq f(s) \leq 1$; see also [5], [8], [7] and [6] and the references therein. Godunova and Levin in [3] noted that the generalisation of (1) by Bellman in [2] was incorrect.

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Pecaric [8] corrected the Bellman generalisation with a narrow subclass. The corrected result is

$$\left(\int_0^1 f(s)g(s)ds\right)^p \leq \int_0^\lambda g(s)^p ds, \tag{2}$$

where $\lambda = \left(\int_0^1 f(s)ds\right)^p$, $g : [0,1] \longrightarrow \Re$ is a nonnegative and nonincreasing function, $f : [0,1] \longrightarrow \Re$ is an integrable function with $0 \leq f(s) \leq 1$ ($\forall s \in [0,1]$) and $p \geq 1$, for the proof; see [8] and the references therein.

The purpose of this paper is to present a refinement of inequality (2) with proofs consisting of both analytical and geometrical.

2. Preliminaries

We begin with convex functions.

Definition 2.1. (Convex functions) Let I be an interval in $\Re$. Then $\psi : I \longrightarrow \Re$ is said to be convex if for all $t_1, t_2 \in I$ and for all positive λ and μ satisfying $\lambda + \mu = 1$, we have

$$\psi(\lambda t_1 + \mu t_2) \leq \lambda \psi(t_1) + \mu \psi(t_2). \tag{3}$$

A convex function necessarily is continuous for $t_1, t_2 \in I$.

A function ψ is said to be strictly convex if for all $t_1 \neq t_2$, ψ is said to be strictly convex.

Remark 2.1. The convexity of a function $\psi : I \longrightarrow \Re$ means geometrically that, the function ψ falls below (or lies on and not above) the chord joining the endpoints $(t_1, \psi(t_1))$ and $(t_2, \psi(t_2))$, for every $t_1, t_2 \in I$.

Intuitively, a convex function has a tangent line at each point and lies above of its tangent lines. That is, for each $t \in I$ there exists a slope C_t such that

$$\psi(s) \geq \psi(t) + C_t(s - t), \quad \forall x \in I.$$

We remark here that if ψ is differentiable at t then $C_t = \psi'(t)$.

Definition 2.2. A function ψ is said to be concave if $-\psi$ is convex (i.e. if the inequality (3) is reversed). If it is strict for all $t_1 \neq t_2$, ψ is said to be strictly concave.

Remark 2.2. If $\psi''(t)$ exists at each point of the interval I, then a necessary and sufficient condition that $\psi(t)$ is convex is that $\psi''(t) \geq 0$ for all $t \in I$.

For the above discussion, we refer authors to [5] and [1]. Some examples of convex functions are: $|t|$, t^k for $k > 1$ and $-t^k$ for $0 < k < 1$, e^t, $t\log t$, $-\log t$ and concave functions are: t^k for $0 < k < 1$, $\log t$, $\sqrt{t}$ for $t \geq 0$ and so on.

3. Main results

We first present a refinement of inequality (2) here.

Theorem 3.1. *Let the function $f : [0,1] \longrightarrow \Re$ be continuous such that $0 \leq f(s) \leq 1$. If $\psi : [0,1] \longrightarrow \Re$ is a convex, differentiable function with $\psi(0) = 0$, then*

$$\psi\left(\int_0^1 f(s)ds\right) \leq \int_0^1 f(s)\psi'(s)ds \tag{4}$$

for all $s \in [0,1]$.

Proof. Let $p = 1$. Since the differential of $\psi(s)$ denoted $\psi'(s)$ is increasing and $-\psi'(s)$ is nonincreasing for all $s \in [0,1]$, substitution of $g(s) = -\psi'(s)$ in (2) gives

$$-\int_0^1 f(s)\psi'(s)ds \leq \int_0^\lambda -\psi'(s)ds.$$

This simplifies to

$$\int_0^\lambda \psi'(s)ds \leq \int_0^1 f(s)\psi'(s)ds,$$

$$\psi(\lambda) - \psi(0) \leq \int_0^1 f(s)\psi'(s)ds. \tag{5}$$

Since $\lambda = \int_0^1 f(s)ds$ and $\psi(0) = 0$, thus (5) becomes

$$\psi\left(\int_0^1 f(s)ds\right) \leq \int_0^1 f(s)\psi'(s)ds$$

This completes the proof.

Let us consider a case of a simple function f on an interval $[s_0, s_2]$ such that $0 \leq s_0 < s_2 \leq 1$. We give some definitions

Definition 3.1. Let a_1 and a_2 be real numbers. Define a function $f : [s_0, s_2] \to \Re$ by

$$f(s) = \begin{cases} a_1 & \text{if} \quad s_0 \leq s < s_1, \\ \\ a_2 & \text{if} \quad s_1 \leq s \leq s_2. \end{cases}$$

Then f is called a simple function since for every $s \in [s_0, s_2]$, we have $f(s) = a_j$ for $j = 1, 2$.

Let us obtain a continuous function f_ε from f. Let $\varepsilon > 0$, we have the partition $\{[s_0, s_1 - \varepsilon), [s_1 - \varepsilon, s_1 + \varepsilon), [s_1 + \varepsilon, s_2]\}$ of $[s_0, s_2]$.

Definition 3.2. Let a_1 and a_2 be real numbers. Define a function $f_\varepsilon : [s_0, s_2] \to \Re$ by

$$f_\varepsilon(s) = \begin{cases} a_1 & \text{if} \quad s_0 \leq s < s_1 - \varepsilon, \\ \\ \frac{a_2 - a_1}{2\varepsilon}(s - s_1 + \varepsilon) + a_1 & \text{if} \quad s_1 - \varepsilon \leq s < s_1 + \varepsilon, \\ \\ a_2 & \text{if} \quad s_1 + \varepsilon \leq s \leq s_2. \end{cases}$$

Remark 3.1. Let us remark that f_ε is continuous in $[s_0, s_2]$ since $\lim_{s \to s^*} f_\varepsilon(s) = f_\varepsilon(s^*)$ for every $s^* \in [s_0, s_2]$.

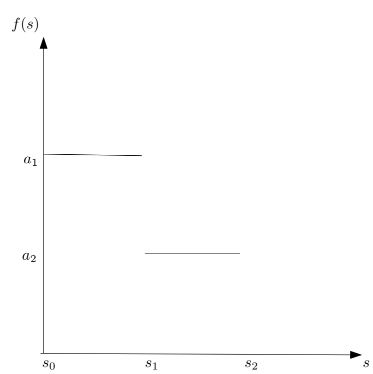

Figure1.

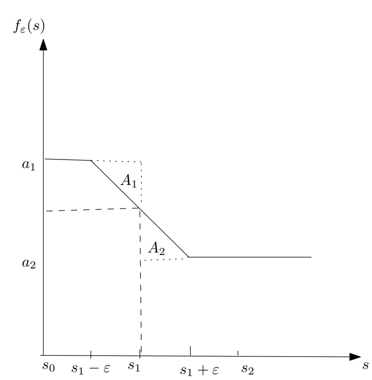

Figure2.

Lemma 3.1. *Let $f(s)$ and $f_\varepsilon(s)$ be functions as in Definitions 3.1 and 3.2 respectively. Then*

$$\int_{s_0}^{s_2} f(s)ds = \int_{s_0}^{s_2} f_\varepsilon(s)ds. \tag{6}$$

Proof. The midpoint of the line

$$f_\varepsilon(s) = \frac{a_2 - a_1}{2\varepsilon}(s - s_1 + \varepsilon) + a_1 \;\; for \;\; s_1 - \varepsilon \le s < s_1 + \varepsilon$$

is $P = \left(s_1, \frac{a_1+a_2}{2}\right)$. (See Figure 2). Therefore, the areas

$$A_1 = \frac{\varepsilon}{2}\left[a_1 - \left(\frac{a_1+a_2}{2}\right)\right] = \frac{\varepsilon}{4}(a_1 - a_2),$$

$$A_2 = \frac{\varepsilon}{2}\left[\left(\frac{a_1+a_2}{2}\right) - a_2\right] = \frac{\varepsilon}{4}(a_1 - a_2).$$

Therefore, we have

$$A_1 = A_2.$$

Lemma 3.2. *Let $f(s)$ and $f_\varepsilon(s)$ be functions as in Definitions 3.1 and 3.2 respectively. If $\psi(s)$ is a convex, differentiable function with $\psi(0) = 0$, then*

$$\int_{s_0}^{s_2} [f_\varepsilon(s) - f(s)]\psi'(s)ds = \frac{a_1 - a_2}{2\varepsilon}\int_{s_1-\varepsilon}^{s_1+\varepsilon} [\psi(s) - \psi(s_1)]ds.$$

Proof. Write

$$\int_{s_0}^{s_2} [f_\varepsilon(s) - f(s)]\psi'(s)ds = \int_{s_0}^{s_2} f_\varepsilon(s)\psi'(s)ds - \int_{s_0}^{s_2} f(s)\psi'(s)ds. \tag{7}$$

The second term on the right side of (7) gives

$$\int_{s_0}^{s_2} f(s)\psi'(s)ds = \int_{s_0}^{s_1} a_1\psi'(s)ds + \int_{s_1}^{s_2} a_2\psi'(s)ds,$$

$$\int_{s_0}^{s_2} f(s)\psi'(s)ds = (a_1 - a_2)\psi(s_1) + a_2\psi(s_2) - a_1\psi(s_0). \tag{8}$$

Also, the first term on the right side of (7) is expressed as

$$\begin{aligned}\int_{s_0}^{s_2} f_\varepsilon(s)\psi'(s)ds = &\int_{s_0}^{s_1-\varepsilon} a_1\psi'(s)ds \\ &+ \int_{s_1-\varepsilon}^{s_1+\varepsilon}\left[\frac{a_2-a_1}{2\varepsilon}(s - s_1 + \varepsilon) + a_1\right]\psi'(x)ds \\ &+ \int_{s_1+\varepsilon}^{s_2} a_2\psi'(s)ds.\end{aligned}$$

Applying integration by parts, we obtain

$$\int_{s_0}^{s_2} f_\varepsilon(s)\psi'(s)ds = a_1[\psi(s_1-\varepsilon)-\psi(s_0)] + \frac{a_2-a_1}{2\varepsilon}\left[2\varepsilon\psi(s_1+\varepsilon) - \int_{s_1-\varepsilon}^{s_1+\varepsilon}\psi(s)ds\right]$$
$$+a_1\psi(s_1+\varepsilon) - a_1\psi(s_1-\varepsilon) + a_2\psi(s_2) - a_2\psi(s_1+\varepsilon),$$

which simplifies to

$$\int_{s_0}^{s_2} f_\varepsilon(s)\psi'(s)ds = \frac{a_1-a_2}{2\varepsilon}\int_{s_1-\varepsilon}^{s_1+\varepsilon}\psi(s)ds + a_2\psi(s_2) - a_1\psi(s_0). \tag{9}$$

Thus, the difference between inequalities (8) and (9) gives

$$\int_{s_0}^{s_2}[f_\varepsilon(s)-f(s)]\psi'(s)ds = \frac{(a_1-a_2)}{2\varepsilon}\int_{s_1-\varepsilon}^{s_1+\varepsilon}[\psi(s)-\psi(s_1)]ds$$

as required.

Lemma 3.3. *Let $g(s)$ be a continuous function on the interval $[s_0, s_2]$. Then*

$$\lim_{\eta\to 0}\frac{1}{2\eta}\int_{s_1-\eta}^{s_1+\eta} g(s)ds = g(s_1).$$

Proof. Let $\eta > 0$ and set

$$I(\eta) = \frac{1}{2\eta}\int_{s_1-\eta}^{s_1+\eta} g(s)ds.$$

Continuity of g at s_1. Let $\varepsilon > 0$, there exists $\delta > 0$ such that $|g(s)-g(s_1)| < \varepsilon$ whenever $|s-s_1| < \delta$. Since

$$|I(\eta)-g(s_1)| \le \frac{1}{2\eta}\int_{s_1-\eta}^{s_1+\eta}|g(s)-g(s_1)|ds$$

for $\eta < \delta$, we have

$$s_1-\eta \in (s_1-\delta, s_1+\delta)$$

and

$$s_1+\eta \in (s_1-\delta, s_1+\delta).$$

Thus, $|s-s_1| < \eta$ and hence $|I(\eta)-g(s_1)| < \varepsilon$. Therefore $I(\eta)\to g(s_1)$ as $\eta\to 0$.

Lemma 3.4. *Let f be a simple function defined as in Definition 3.1 such that $0 \le f \le 1$. If ψ is a convex, differentiable function with $\psi(0)=0$, then*

$$\psi\left(\int_{s_0}^{s_2} f(s)ds\right) \le \int_{s_0}^{s_2} f(s)\psi'(s)ds.$$

Proof. Let $f_\varepsilon(s)$ be continuous as in Definition 3.2. Then by Lemma 3.1, Theorem 3.1 and Lemma 3.2 respectively, we have

$$\begin{aligned}\psi\left(\int_{s_0}^{s_2} f(s)ds\right) &= \psi\left(\int_{s_0}^{s_2} f_\varepsilon(s)ds\right)\\ &\leq \int_{s_0}^{s_2} f_\varepsilon(s)\psi'(s)ds\\ &\leq \int_{s_0}^{s_2} f(s)\psi'(s)ds + \int_{s_0}^{s_2}[f_\varepsilon(s)-f(s)]\psi'(s)ds\\ &\leq \int_{s_0}^{s_2} f(s)\psi'(s)ds + \frac{(a_1-a_2)}{2\varepsilon}\int_{s_1-\varepsilon}^{s_1+\varepsilon}[\psi(s)-\psi(s_1)]ds.\end{aligned}$$

Thus, by Lemma 3.3, when $\varepsilon \to 0$, we obtain

$$\psi\left(\int_{s_0}^{s_2} f(s)ds\right) \leq \int_{s_0}^{s_2} f(s)\psi'(s)ds$$

as required.

Theorem 3.1. *Let f be a simple function on $[0,1]$ such that $0 \leq f(s) \leq 1$ for all $s \in [0,1]$. If ψ is a convex, differentiable function with $\psi(0) = 0$, then*

$$\psi\left(\int_0^1 f(s)ds\right) \leq \int_0^1 f(s)\psi'(s)ds.$$

Proof. Let f be a simple function. There exists $\{0 = s_0, s_1, \cdots, s_n = 1\}$ and $\{a_1, a_2, \cdots, a_n\}$ such that $f(s) = a_j$ on $[s_j, s_{j+1})$ for $0 \leq j \leq n-1$. Let $0 < \varepsilon < \min|s_{j+1} - s_j|$ and define

$$f_\varepsilon(s) = f(s)$$

if

$$s \in [0, s_1-\varepsilon) \cup [s_1+\varepsilon, s_2-\varepsilon) \cup \cdots \cup [s_j+\varepsilon, s_{j+1}-\varepsilon) \cup \cdots \cup [s_{n-1}+\varepsilon, 1).$$

And

$$f_\varepsilon(s) = \frac{a_{(j+1)} - a_j}{2\varepsilon}(s - s_j + \varepsilon) + a_j$$

if

$$s \in [s_j - \varepsilon, s_j + \varepsilon)$$

where $j = 1, \cdots, n-1$. (See Figure 3 and Figure 4). Then, following Lemma 3.4, we have

$$\int_0^1 f(s)ds = \int_0^1 f_\varepsilon(s)ds$$

and

$$\begin{aligned}\psi\left(\int_0^1 f(s)ds\right) &= \psi\left(\int_0^1 f_\varepsilon(s)ds\right)\\ &\leq \int_0^1 f(s)\psi'(s)ds + \sum_{j=1}^{n-1} \frac{a_j - a_{j+1}}{2\varepsilon} \int_{s_j-\varepsilon}^{s_j+\varepsilon} [\psi(s) - \psi(s_j)]ds.\end{aligned}$$

Therefore

$$\psi\left(\int_0^1 f(s)ds\right) \leq \int_0^1 f(s)\psi'(s)ds$$

as required.

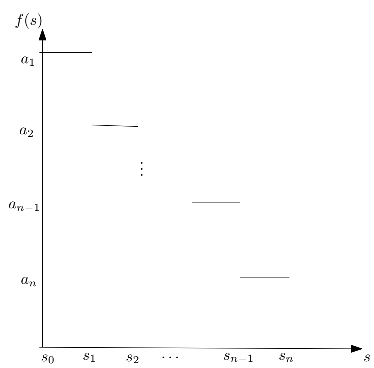

Figure3

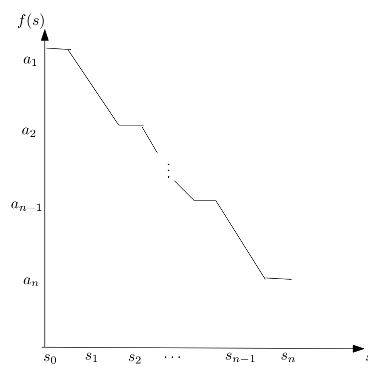

Figure4

4. Applications

In Theorem 3.1, replace 1 by $a > 0$. Thus

$$\psi\left(\int_0^{2\pi} f(s)ds\right) \leq \int_0^{2\pi} f(s)\psi'(s)ds.$$

We estimate the Fourier coefficients of ψ :

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \psi(s)\cos(ns)ds$$

and

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \psi(s)\sin(ns)ds$$

for $n \geq 1$. For the estimate of b_n, let $f(s) = \frac{1}{2}(1+\varepsilon\cos ns)$ for $\varepsilon = 1$ or -1. Thus

$$\frac{1}{2}\int_0^{2\pi}(1+\varepsilon\cos ns)ds = \pi$$

and

$$\frac{1}{2}\int_0^{2\pi}(1+\varepsilon\cos ns)\psi'(s)ds = \frac{\psi(2\pi)}{2}(1+\varepsilon) + \frac{n\varepsilon}{2}\int_0^{2\pi}\psi(s)\sin(ns)ds$$

Hence

$$\psi(\pi) \leq \frac{\psi(2\pi)}{2}(1+\varepsilon) + \frac{n\varepsilon}{2}\int_0^{2\pi}\psi(s)\sin(ns)ds$$

Take $\varepsilon = 1$ or -1 and we obtain

$$\frac{\psi(\pi)-\psi(2\pi)}{n\pi} \leq b_n \leq \frac{-\psi(\pi)}{n\pi}.$$

Also, for the estimate of a_n, let $f(s) = \frac{1}{2}(1+\varepsilon\sin ns)$ for $\varepsilon = 1$ or -1. Thus

$$\frac{1}{2}\int_0^{2\pi}(1+\varepsilon\sin ns)ds = \pi$$

and

$$\frac{1}{2}\int_0^{2\pi}(1+\varepsilon\sin ns)\psi'(s)ds = \frac{\psi(2\pi)}{2} - \frac{n\varepsilon}{2}\int_0^{2\pi}\psi(s)\cos(ns)ds.$$

Hence

$$\psi(\pi) \leq a_n \leq \frac{\psi(2\pi)}{2} - \frac{n\varepsilon}{2}\int_0^{2\pi}\psi(s)\cos(ns)ds.$$

Take $\varepsilon = 1$ or -1 and we obtain

$$\frac{\psi(\pi)-\psi(2\pi)/2}{n\pi} \leq a_n \leq \frac{\psi(2\pi)/2-\psi(\pi)}{n\pi}.$$

Example 3.1. If

- $\psi(s) = s$, then $a_n = 0$ and $b_n = -\frac{1}{n}$.
- $\psi(s) = s^2$, then $\frac{-\pi}{n} \leq a_n \leq \frac{\pi}{n}$ and $\frac{-3\pi}{n} \leq b_n \leq \frac{-\pi}{n}$.

4. Conclusion

The new Steffensen's inequality (4) is thus proved for continuous functions as well as simple (discontinuous) functions and also valid for all functions $f \in L^1([0,1])$. An application of the inequality has also been established for the determination of Fourier coefficients.

Conflict of Interests

The authors declare that there is no conflict of interests.

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