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OPTIMAL CONTROL ANALYSIS OF AN HIV/AIDS MODEL WITH LINEAR INCIDENCE RATE

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Abstract. In this paper, a nonlinear Mathematical model is proposed to study the dynamics of HIV/AIDS in a variable size population involving two groups of infectives with different behavioral patterns and an infecting AIDS group. Basic Mathematical and epidemiological implications of the model, like the basic reproduction number and its sensitivity indexes with respect to its parameters, are derived. The basic model is modified into an optimal control problem by incorporating three controls, namely; Infection control, behavioral change efforts and administration of Highly Active Antiretroviral Therapy (HAART), aimed at controlling the spread of the disease. We examine the implementation of various combinations of the controls in order to determine the most cost effective strategy that can control the spread. Using the incremental cost-effective ratio for the various control strategies showed that the strategy that involves all the efforts is the most cost effective strategy. This reveals that the fight against the disease should be multidimensional, including treatment, education and others.

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1. Introduction

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HIV/AIDS is one of the most critical problems that have plagued the world, especially the under-developed and developing world, in the twentieth and twenty-first centuries. It has claimed and continue to claim several lives, despite numerous attempts to control its spread. An estimated 2.5 million people were newly infected with HIV in 2011 and 1.7 million people died of the disease [9]. Up to date, there has not been a cure for the disease and as such several studies, both theoretical and practical, have sought to find ways to curb its spread. Mathematical modeling has been extensively employed to study and provide strategies to control its spread. A wide range of researchers have done varied categories of studies in this area. Some have sought to provide models to study the dynamics of the disease. For example, May and Anderson [14] explored data on different categories of people at risk of getting the disease in the under-developed and developing world and proposed basic models to describe the dynamics of the disease. Anderson *et al.* [2] provided a preliminary study of the causative agent of the disease. Other researchers have sought to study the effects of various factors that can affect the transmission of the disease. In particular, Agraj *et al.* [1] developed a model to study the effect of screening of unaware infectives on the dynamics of the disease. Ratera *et al.* [17] studied the effects of screening and treatment on the dynamics of the disease, concluding that these two factors have the effect of reducing the spread of the disease. Since, HIV/AIDS is transmitted through interaction of infected and Susceptible individuals, the behavior of these people can greatly impact the spread. Thus, some researchers have sought to study the impact of various behavioral patterns on the transmission of the infection. For example, Daabo *et al.* [5] and Daabo and Seidu [6] studied the effect of irresponsible infectives on the spread of HIV/AIDS, concluding that increase in responsibility (positive attitude towards safe sex) can greatly reduce the spread of the infection. Research has shown that several factors affect the dynamics of HIV. This lead many researchers to consider the use of optimal control in the study of the dynamics of the disease. Some examples of such research include Karrachou *et al.* [11] who studied the impact of chemotherapy in controlling viral replication in HIV patients, Okosun *et al.* [15] who proposed a dynamical model to study the recruitment, preventive and treatment strategies for optimal workplace productivity in the presence of HIV/AIDS. In the present paper, we propose

a nonlinear dynamical system to study the optimal control of HIV/AIDS transmission in the presence of two categories of infective with different attitude towards sex, a sexually active AIDS group and one Susceptible group. We seek to study the most cost effective strategy, involving three control efforts, that can be used to fight the spread of HIV/AIDS in the presence of the three infective groups.

The remaining part of the paper is organized as follows. In section 2, we describe the formulation of the model, which consists of a system of ordinary differential equations representing the rate of change of the population sizes of the various sub-groups in the population. In section 3, some basic mathematical and epidemiological implications of the proposed model are presented. Section 4 presents the stability analysis of the model, while sections 5, 6 and 7 present a modification of the proposed model into an optimal control problem, Numerical simulation of the proposed models and Conclusions respectively.

2. Model formulation

In this paper, we propose a standard compartmental model that describes the dynamics of HIV/AIDS in a non-constant population. The model considers that the total population $N(t)$ is subdivided into four classes, namely; Susceptibles, $S(t)$, Careless infectives, $I_i(t)$, Careful Infectives, $I_r(t)$ and people with AIDS, $A(t)$. HIV/AIDS is assumed to be mainly transfered through contact with an infected person. Figure 1 is the schematic diagram of the model.

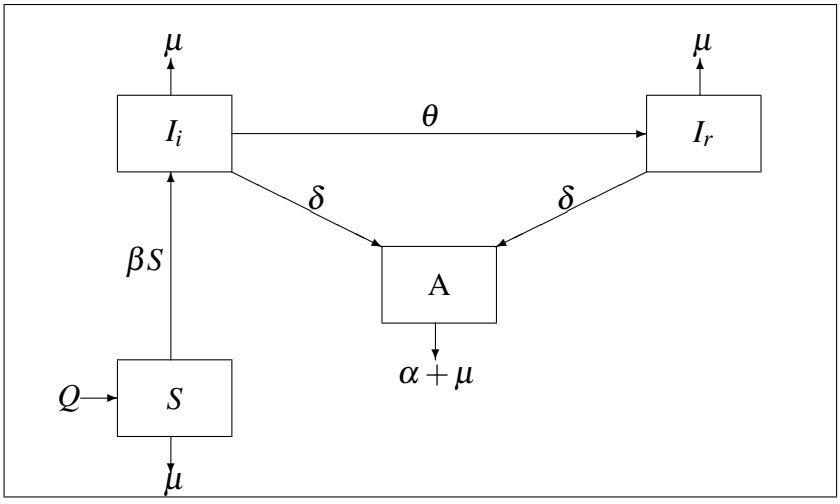

FIGURE 1. Schematic Diagram for our Model

Next, we describe, in this section, the model formulation process by describing how the populations of the various compartments evolve with time:

Susceptible Population

The population of this subclass grows through a constant inflow of Susceptibles at a rate Q into the population. The per-capita rate of contact between an infected person and a Susceptible one is β with η being modification parameter on infection due to responsible sexual lifestyle of the Careful Infectives and τ being the modification parameter on infection of AIDS individuals due to reduced sexual activity. We assume a bilinear incidence rate with an average number of sexual partners of an infectious individual being c. The natural death rate of Susceptibles and other subpopulations is μ. We assume that all infected persons are sexually active even though there is reduced contacts for responsible and AIDS patients due to right value judgment and weak sexuality respectively. Thus, we have he nonlinear differential equation $\frac{dS}{dt} = Q - c\beta\left(I_i + \eta I_r + \tau A\right)S - \mu S$

Careless Infectives Population

Efforts aimed at improving responsible sexual lifestyle of careless infectives succeed at a rate θ and Highly Active Antiretroviral Therapy treatment efforts lead to a reduced rate of progression into AIDS ,δ for both categories of infectives. Susceptibles are assumed to only become irresponsible infectives due to the fact that infected persons are often unaware of their HIV status and may still indulge in irresponsible sexual behavior.This leads to the equation $\frac{dI_i}{dt} = c\beta\left(I_i + \eta I_r + \tau A\right)S - (\delta + \theta + \mu)I_i$

Careful Infectives Population

The size of this subclass grows as a result of positive behavioral change of irresponsible infectives after they have become aware of their HIV status and thus we have $\frac{dI_r}{dt} = \theta I_i - (\delta + \mu)I_r$.

AIDS Patients' Population Equation

The size of the AIDS patients' subclass increases as a result of loss of immune system functionality of the infected persons. if δ is the rate of progression of the infected to AIDS, then the rate of growth of the AIDS persons is directly proportional to $\delta(I_i + I_r)$. If α is the rate of disease-induced death, then the population of AIDS persons decrease at a rate proportional to $\alpha + \mu$. this gives $\frac{dA}{dt} = \delta(I_i + I_r) - (\alpha + \mu)A$.

Thus, we have the following set of autonomous differential equations that model the spread of the disease in a typical homogeneously-mixed environment.

$$
\begin{cases}
\frac{dS}{dt} = Q - c\beta\left(I_i + \eta I_r + \tau A\right)S - \mu S \\
\frac{dI_i}{dt} = c\beta\left(I_i + \eta I_r + \tau A\right)S - (\delta + \theta + \mu)I_i \\
\frac{dI_r}{dt} = \theta I_i - (\delta + \mu)I_r \\
\frac{dA}{dt} = \delta(I_i + I_r) - (\alpha + \mu)A
\end{cases} \tag{1}
$$

3. Basic results about the model

In this section we present some basic properties of the model and the implications of such properties.

3.1 Biological feasibility

For all epidemiological models that deal with populations,it is necessary to ensure that the models are epidemiologically reasonable. There is the need to ensure that all variables are positive during the period of study. It is easy to show that all solutions of the model (1) always remain non-negative as long as the initial values are non-negative.

Theorem 3.1. *Let the initial state-values of the model (1) be non-negative (i.e.* $S(0) \geq 0, I_i(0) \geq 0, I_r(0) \geq 0, A(0) \geq 0$*). Then all solutions remain non-negative for* $t > 0$.

Proof

From $\frac{dA}{dt} = \delta(I_i + I_r) - (\alpha + \mu)A$ we have $\frac{dA}{dt} \geq -(\alpha + \mu)A$, which implies that $A(t) \geq A_0 e^{-(\alpha+\mu)t}$.
Similar arguments gives $I_r(t) \geq I_r(0)e^{-(\delta+\mu)t}$, $I_i(t) \geq I_i(0)e^{-(\delta+\theta+\mu)t}$ and $S(t) \geq S(0)e^{-f(I_i(t),I_r(t),A(t))}$
Thus, all solutions of the model (1) will remain positive provided the initial values are positive.

3.2 Boundedness of the model

For an epidemiological model to be considered realistic, it is expected that it does not describe a population whose size grows without bounds. This is due to the fact that populations are known to be limited in their growth due to such limiting factors as the carrying capacity of the environment, competition, death among others. Thus, we study, in this section, the boundedness

of our proposed model.

From the model (1) we have

$\frac{d}{dt}(S+I_i+I_r+A) = Q-\mu(S+I_i+I_r+A)-\alpha A \leq \mu\left[\frac{Q}{\mu}-(S+I_i+I_r+A)\right]$

Then $\lim_{t\to\infty} Sup(S+I_i+I_r+A) \leq \frac{Q}{\mu}$

Thus, we have

$\Omega = \left\{(S,I_i,I_r,A) : S+I_i+I_r+A \leq \frac{Q}{\mu}, S>0, I_i>0, I_r>0, A>0\right\}$ being the feasible region of the system (1).

From the result on positivity of the solutions of the model above, we can conclude that the region Ω is positively invariant with respect to system (1).

3.3 Basic reproduction number, $\mathscr{R}_0$

The basic reproduction number is a threshold parameter that is used in epidemiology to investigate the stability of critical points of models. It is defined as the number of secondary infections that arise as a result of the introduction of a single infectious agent in an initially susceptible population over the period of the infectiousness of the infectious agent. Thus, if this parameter is greater than one it is expected that the disease will continue to spread and will die off if the parameter is less than one.

Several researchers have studied this parameter [7, 8] and some have proposed techniques to evaluate it. We employ, here, the next generation matrix technique of [8], who showed that R_0 can be calculated as the spectral radius of the next generation matrix of the model at the disease-free equilibrium point.

Using the method in [8], the system (1) can be rewritten as

$\frac{dX}{dt} = F(X)-V(X)$ where

$$F(X) = \begin{bmatrix} 0 \\ c\beta(I_i+\eta I_r+\tau A)S \\ 0 \\ 0 \end{bmatrix} \text{ and } V(X) = \begin{bmatrix} -Q+c\beta(I_i+\eta I_r+\tau A)S+\mu S \\ (\delta+\theta+\mu)I_i \\ (\delta+\mu)I_r-\theta I_i \\ (\alpha+\mu)A-\delta(I_i+I_r) \end{bmatrix}$$

Evaluating the Jacobian matrices of $F(X)$ and $V(X)$ at the disease free equilibrium point gives respectively he following matrices.

$$\mathscr{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{c\beta\, Q}{\mu} & \frac{c\beta\,\eta\, Q}{\mu} & \frac{c\beta\,\tau\, Q}{\mu} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathscr{V} = \begin{bmatrix} \mu & \frac{c\beta\, Q}{\mu} & \frac{c\beta\,\eta\, Q}{\mu} & \frac{c\beta\,\tau\, Q}{\mu} \\ 0 & \delta+\theta+\mu & 0 & 0 \\ 0 & -\theta & \delta+\mu & 0 \\ 0 & -\delta & -\delta & \alpha+\mu \end{bmatrix}$$

The next generation matrix of the model is given by

$$\mathscr{D}\mathscr{V}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{c\beta\, Q}{\mu\,(\delta+\theta+\mu)} + \frac{c\beta\,\eta\, Q\theta}{\mu\,(\delta+\theta+\mu)(\delta+\mu)} + \frac{c\beta\,\tau\, Q\delta}{\mu\,(\delta+\mu)(\alpha+\mu)} & \frac{c\beta\,\eta\, Q}{\mu\,(\delta+\mu)} + \frac{c\beta\,\tau\, Q\delta}{\mu\,(\delta+\mu)(\alpha+\mu)} & \frac{c\beta\,\tau\, Q}{\mu\,(\alpha+\mu)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that basic reproduction number (defined as the spectral radius of $\mathscr{D}\mathscr{V}^{-1}$) will be given by

$$\mathscr{R}_0 = \frac{c\beta Q\,[(\alpha+\mu)\,(\delta+\mu+\eta\,\theta)+\tau\,\delta\,(\delta+\theta+\mu)]}{\mu(\delta+\theta+\mu)(\delta+\mu)(\alpha+\mu)}.$$

In the next subsection, we study the effect of changes of the model parameters on $\mathscr{R}_0$.

3.4 Sensitivity analysis of $\mathscr{R}_0$

Parameters that occur in mathematical models are often estimated from experiments and expected to be found within some ranges. Due to the fact that exact values of the parameters are not always available, it is often prudent to determine the effect of changes of the parameters on the predictions of the model. One procedure to do this is by the use of sensitivity analysis. The normalized forward sensitivity index is a typical measure used to study the sensitivity of a parameter relative to its dependent parameters.

Definition 3.1. Let $h = f(x_1, x_2, \ldots, x_n)$, be a differentiable function that depends on the parameters x_i, then the normalized forward sensitivity of h with respect to x_i is defines as $\gamma_{x_i}^{h} = \frac{x_i}{h}.\frac{dh}{dx_i}$. This index measures the relative change in h due to relative changes in x_i. The normalized forward sensitivity indexes of R_0 relative to its parameters are presented in Table 1. The implications of the values of the sensitivity indexes is that a unit increase in the rate of progresion to AIDs of infected individuals leads to a 0.71% decrease in R_0. The grater the absolute value of a sensitivity index of a parameter, the greater the effect of the parameter on its dependent

variable. Thus, μ is seen to have the greatest impact on $\mathscr{R}_0$, whilst α has the least impact.

TABLE 1. Sensitivity Indexes of parameters of $\mathscr{R}_0$

Parameter	Description	Sensitivity Index
Q	Rate of recruitment	+1.000
α	AIDs related death rate	-0.020
β	Contact rate between Susceptibles and Infectives	+1.000
c	Average number of contacts of a typical infective	+1.000
δ	Rate of progression of Infective into AIDS	-0.709
η	Modification parameter due to right-value judgment	+0.745
μ	Natural Death rate	-1.146
τ	Modification parameter due to weak sexual activity	+0.021
θ	Probability of positive behavioral change	-0.125

3.5 Existence of an endemic equilibrium point of the model

It is easily seen that the model(1) has a disease-free equilibrium point given by $E_0 = (\frac{Q}{\mu}, 0\,0, 0)$ and by simple algebraic calculations the model can be shown to also exhibit a unique endemic equillibrium point $E^*(S^*, I_i^*, I_r^*, A^*)$, given by

$S^* = \frac{Q}{\mu+\mathscr{R}_0-1}$, $I_i^* = \frac{Q}{\mu(\delta+\theta+\mu)}\left(1-\frac{1}{\mathscr{R}_0}\right)$, $I_r^* = \frac{\theta Q}{\mu(\delta+\theta+\mu)(\delta+\mu)}\left(1-\frac{1}{\mathscr{R}_0}\right)$ and $A^* = \frac{\delta Q}{\mu(\delta+\mu)(\alpha+\mu)}\left(1-\frac{1}{\mathscr{R}_0}\right)$.

Theorem 3.2 *The system (1) exhibits only two unique equilibrium points; namely the disease-free equilibrium and the endemic equilibrium, which are mutually exclusive. Hence the model does not exhibit any backward bifurcation.*

Proof. It is noted that $\mathscr{R}_0 \in (0,\infty)$ and the persistence of the disease depends on whether $\mathscr{R}_0$ is greater or less than unity. For $\mathscr{R}_0 < 1$ we have $(1-\frac{1}{\mathscr{R}_0}) < 0$ suggesting negative population sizes of infectives and AIDS subclasses, which does not make sense.For $\mathscr{R}_0 > 1$. This then suggests extinction of these subclasses and hence we have the disease-free equilibrium for $\mathscr{R}_0 <$ 1. Also $\mathscr{R}_0 = 1$ corresponds to the disease-free equilibrium. For $\mathscr{R}_0 > 1$, we have $(1-\frac{1}{\mathscr{R}_0}) > 0$,

making $I_i^* \neq 0$ which corresponds to the coexistence of Susceptibles and Infectives (i.e endemic Equilibrium). Thus, we realize that $\mathscr{R}_0 = \mathscr{R}_0^1 \cup \mathscr{R}_0^2$, where $\mathscr{R}_0^1 = (0,1]$, and $\mathscr{R}_0^2 = (1,\infty)$, which are disjoint sets and correspond to the existence of the disease-free and endemic equilibria respectively. this proves the mutual exclusivity of the equilibrium points.

4. Stability analysis of the model

To study the stability of the model we use the indirect Lyapunov method. The Jacobian matrix that linearizes the model is given by

$$J = \begin{bmatrix} -\mu - c\beta(I_i + \eta I_r + \tau A) & -c\beta S & -c\beta\eta S & -c\beta\tau S \\ c\beta(I_i + \eta I_r + \tau A) & c\beta S - (\delta+\theta+\mu) & c\beta\eta S & c\beta\tau S \\ 0 & \theta & -(\delta+\mu) & 0 \\ 0 & \delta & \delta & -(\alpha+\mu) \end{bmatrix} \tag{2}$$

To study the stability of the disease-free equilibrium, we evaluate the Jacobian at the point and determine the nature of the eigenvalues of the resulting matrix. Evaluating the Jacobian at the disease-free equilibrium gives:

$$J = \begin{bmatrix} -\mu & -\frac{c\beta Q}{\mu} & -\frac{c\beta\eta Q}{\mu} & -\frac{c\beta\tau Q}{\mu} \\ 0 & \frac{c\beta Q}{\mu} - (\delta+\theta+\mu) & \frac{c\beta\eta Q}{\mu} & \frac{c\beta\tau Q}{\mu} \\ 0 & \theta & -(\delta+\mu) & 0 \\ 0 & \delta & \delta & -(\alpha+\mu) \end{bmatrix} \tag{3}$$

While one of the eigenvalues of the Jacobian is known straight away to be negative (i.e $-\mu$), the remaining three eigenvalues are those of the matrix given by

$$J1 = \begin{bmatrix} \frac{c\beta Q}{\mu} - (\delta+\theta+\mu) & \frac{c\beta\eta Q}{\mu} & \frac{c\beta\tau Q}{\mu} \\ \theta & -(\delta+\mu) & 0 \\ \delta & \delta & -(\alpha+\mu) \end{bmatrix}. \tag{4}$$

However, this matrix has eigenvalues with negative real parts if $tr(J1)$, det(J) and det($J1^{[2]}$) are all negative. where $J1^{[2]}$ is the second additive matrix of $J1$ given by:

$$J1^{[2]} = \begin{bmatrix} \frac{c\beta Q}{\mu} - (\delta+\theta+\mu) - (\delta+\mu) & 0 & -\frac{c\beta\tau Q}{\mu} \\ \delta & \frac{c\beta Q}{\mu} - (\delta+\theta+\mu) - (\alpha+\mu) & \frac{c\beta\eta Q}{\mu} \\ -\delta & \theta & -(\alpha+\mu) - (\delta+\mu) \end{bmatrix}.$$

$$\begin{aligned} tr(J) &= \frac{c\beta Q}{\mu} - (\delta+\theta+\mu) - (\delta+\mu) - (\alpha+\mu) \\ &= (\delta+\theta+\mu)\left[\frac{\mathscr{R}_0(\alpha+\mu)(\delta+\mu)}{(\alpha+\mu)(\delta+\eta\theta+\mu)+\tau\delta(\delta+\theta+\mu)} - 1\right] - (\delta+\mu) - (\alpha+\mu) \\ &< (\delta+\theta+\mu)\left[\frac{\mathscr{R}_0(\alpha+\mu)(\delta+\mu)}{(\alpha+\mu)(\delta+\eta\theta+\mu)+\tau\delta(\delta+\theta+\mu)} - 1\right] \end{aligned}$$

It is easily seen that $tr(J) < 0$ if $\mathscr{R}_0 < 1$.

Also,

$$\begin{aligned} det(J1) &= (\alpha+\mu)(\delta+\mu)\left[\frac{c\beta Q}{\mu} - (\delta+\theta+\mu)\right] + \frac{c\beta\delta\tau\theta Q}{\mu} \\ &\quad + (\alpha+\mu)\frac{c\beta\eta\theta Q}{\mu} + (\delta+\mu)\frac{c\beta\delta\tau Q}{\mu} \\ &= (\alpha+\mu)(\delta+\mu)(\delta+\theta+\mu)(\mathscr{R}_0 - 1) \end{aligned}$$

We see that $det(J1) < 0$ if $\mathscr{R}_0 < 1$

The determinant of the second additive matrix $J1^{[2]}$ is also observed, after some algebraic manipulations, to be negative if $\mathscr{R}_0 < 1$. Thus we have the following result.

Theorem 4.1. *The disease-free equilibrium is locally asymptotically stable if the basic reproduction number $\mathscr{R}_0$ is less than unity.*

5. Model with optimal control

We present in this section, the model with three control strategies aimed at controlling the spread of the infection. The optimal control problem is developed in order to:

- Reduce infection with the control u_1
- Change behavior of infectives so as to reduce infection using the control measures, u_2
- Optimize the HAART treatment with the control measure u_3

Thus, the optimal control model is given as:

$$
\begin{cases}
\frac{dS}{dt} = Q - c\beta(1-u_1)\left(I_i + \eta I_r + \tau A\right)S - \mu S \\
\frac{dI_i}{dt} = c\beta(1-u_1)\left(I_i + \eta I_r + \tau A\right)S - ((1-u_2)\delta + u_3\theta + \mu)I_i \\
\frac{dI_r}{dt} = u_3\theta I_i - ((1-u_2)\delta + \mu)I_r \\
\frac{dA}{dt} = \delta(1-u_2)(I_i + I_r) - (\alpha + \mu)A
\end{cases} \tag{5}
$$

We investigate the optimal level of efforts needed to control the spread of the infection with minimal cost. This is done by minimizing the objective functional,

$$
J = \min_{u_1,u_2,u_3} \int_0^{t_f} (B_0 I_i + B_1 u_i^2 + B_2 u_2^2 + B_3 u_3^2)dt \tag{6}
$$

subject to our modified model. Where B_i are positive weights and $B_i u_i^2$ is the cost of applying control effort u_i. Our choice of the objective functional is similar to those used by other researchers in the field [3, 4, 13, 15]. Our aim is to seek an optimal control pair (u^*, X^*) where $u^* = (u_1^*, u_2^*, u_3^*)$ and $X^* = (S^*, I_i^*, I_r^*, A^*)$, that minimizes the Hamiltonian of the system given by

$$
\begin{cases}
H = & B_0 I_i + B_1 u_i^2 + B_2 u_2^2 + B_3 u_3^2 \\
& + \lambda_S \left[Q - c\beta(1-u_1)\left(I_i + \eta I_r + \tau A\right)S - \mu S\right] \\
& + \lambda_{I_i} \left[c\beta(1-u_1)\left(I_i + \eta I_r + \tau A\right)S - ((1-u_2)\delta + u_3\theta + \mu)I_i\right] \\
& + \lambda_{I_r} \left[u_3\theta I_i - ((1-u_2)\delta + \mu)I_r\right] \\
& + \lambda_A \left[\delta(1-u_2)(I_i + I_r) - (\alpha + \mu)A\right]
\end{cases} \tag{7}
$$

Using the Pontryagin's Minimum principle [16], and the existence result for optimal control from [10] the adjoint variable of the state variables satisfy the following set of differential equations.

$$
\begin{cases}
\frac{d\lambda_S}{dt} = -\frac{dH}{dS} = c\beta(1-u_1)(I_i + \eta I_r + \tau A)(\lambda_S - \lambda_{I_i}) + \mu\lambda_S \\
\frac{d\lambda_{I_i}}{dt} == -\frac{dH}{dI_i} - B_0 + c\beta S(1-u_1)(\lambda_S - \lambda_{I_i}) + \delta(1-u_2)(\lambda_{I_i} - \lambda_A) - (u_3\theta + \mu)\lambda_{I_i} \\
\frac{d\lambda_{I_r}}{dt} = -\frac{dH}{dI_r} = c\beta\eta S(1-u_2)(\lambda_S - \lambda_{I_i}) + \delta(1-u_2)(\lambda_{I_r} - \lambda_A) + \mu\lambda_{I_r} \\
\frac{d\lambda_A}{dt} = -\frac{dH}{dA} = c\beta\tau S(1-u_1)(\lambda_S - \lambda_{I_i}) + \lambda_A(\alpha + \mu)
\end{cases} \tag{8}
$$

with transversality conditions $\lambda_S(t_f) = \lambda_{I_i}(t_f) = \lambda_{I_r}(t_f) = \lambda_A(t_f) = 0$.

Equating $\frac{dH}{du_i}$ to zero [12] gives the following characterizations of the controls.

$$\tilde{u}_1 = \frac{c\beta S(I_i+\eta I_r+\tau A)(\lambda_{I_i}-\lambda_S)}{2B_1}, \quad \tilde{u}_2 = \frac{\delta\left[I_i(\lambda_A-\lambda_{I_i})+I_r(\lambda_A-\lambda_{I_r})\right]}{2B_2} \quad \text{and} \quad \tilde{u}_3 = \frac{\theta I_i(\lambda_{I_i}-\lambda_{I_r})}{2B_3} \tag{9}$$

by standard control arguments involving the bounds we have :

$$u_1^* = \begin{cases} 0 & \text{if} \quad \tilde{u}_1 \leq 0 \\ \tilde{u}_1 & \text{if} \quad 0 < \tilde{u}_1 \leq 1, \\ 1 & \text{if} \quad \tilde{u}_1 \geq 1 \end{cases} \quad u_2^* = \begin{cases} 0 & \text{if} \quad \tilde{u}_2 \leq 0 \\ \tilde{u}_2 & \text{if} \quad 0 < \tilde{u}_2 \leq 1 \\ 1 & \text{if} \quad \tilde{u}_2 \geq 1 \end{cases} \text{ and } \quad u_3^* = \begin{cases} 0 & \text{if} \quad \tilde{u}_3 \leq 0 \\ \tilde{u}_3 & \text{if} \quad 0 < \tilde{u}_3 \leq 1 \\ 1 & \text{if} \quad \tilde{u}_3 \geq 1 \end{cases}$$

6. Numerical simulation of optimal control the model

For numerical simulation purposes, we use the parameter values in table 2. The model 1 is solved by the use of the MATLAB code `ode45`, which is an implementation of the fourth-order Runge-Kutta method whiles the optimal control problem was solved using the algorithm below. Using this method, the optimal control problem is solved with the following algorithm.

Step 1:Initialize State, Costate and Control variables

$S(1) = S_0,\ I_i(1) = I_{i0}, I_r(1) = I_{r0}, A(1) = A_0$

$\lambda_S(tf) = 0, \lambda_{I_i}(tf) = 0, \lambda_{I_r}(tf) = 0, \lambda_A(tf) = 0$

$u_1(1) = 0, u_2(1) = 0$ and $u_3(1) = 0$

Step 2: Discretize the time domain.

$h = \frac{t_f}{n}$ where n is the number of discrete points

Step 3:Solve for State and Costate Equations

For *k=1 to n*

Solve for State Variables

$$S^{n+1} \leftarrow \frac{S^k+hQ}{1+h*\left[c\beta(1-u_1^k)(I_i^k+etaI_r^k+\tau A^k)+\mu\right]}$$

$$I_i^{k+1} \leftarrow \frac{I_i^k+h\left[c\beta(\eta I_r^k+\tau A^k)(1-u_1^k)S^{k+1}\right]}{1+h\left[c\beta(1-u_1^k)S^{k+1}-((1-u_2^k)\delta+u_3^k\theta+\mu)\right]}$$

$$I_r^{k+1} \leftarrow \frac{I_r^k+hu_3^k\theta I_i^{k+1}}{1+h\left[(1-u_2^k)\delta+\mu\right]}$$

$$A^{k+1} \leftarrow \frac{A^k+h\left[\delta(1-u_2^k)(I_i^{k+1}+I_r^{k+1})\right]}{1+h(\alpha+\mu)}$$

Solve the Costate Equations

$$\lambda_S^{n-k} \leftarrow \frac{\lambda_S^{n-k+1}+hc\beta(1-u_1^k)(I_i^{k+1}+\eta I_r^{k+1}+\tau A^{k+1})\lambda_{I_i}^{n-k+1}}{1+h\left[c\beta(1-u_1^k)(I_i^{k+1}+\eta I_r^{k+1}+\tau A^{k+1})+\mu\right]}$$

$$\lambda_{I_i}^{n-k} \leftarrow \frac{\lambda_{I_i}^{n-k+1}+h\left[B_0-c\beta(1-u_1^k)S^{k+1}\lambda_S^{n-k}+\delta(1-u_2^k)\lambda_A^{n-k+1}\right]}{1+h\left[\delta(1-u_2^k)-c\beta(1-u_1^k)S^{k+1}\right]}$$

$$\lambda_{I_r}^{n-k} \leftarrow \frac{\lambda_{I_r}^{n-k+1}-h\left[c\beta\eta(1-u_1^k)(\lambda_A^{n-k}-\lambda_{I_i}^{n-k})S^{k+1}-\delta(1-u_2^k)\lambda_A^{n-k+1}\right]}{1+h\left[\delta(1-u_2^k)+\mu\right]}$$

$$\lambda_A^{n-k} \leftarrow \frac{\lambda_A^{n-k+1}+h\left[c\beta\tau(1-u_2^k)(\lambda_S^{n-k}-\lambda_{I_i}^{n-k})S^{k+1}\right]}{1+h(\alpha+\mu)}$$

Update Controls

$$R_1^k = \frac{1}{2B_1}\left[c\beta S^{k+1}(I_i^{k+1}+\eta I_r^{k+1}+\tau A^{k+1})(\lambda_{I_i}-\lambda_S)\right]$$

$$R_2^k = \frac{\delta}{2B_2}\left[I_i^{k+1}(\lambda_A^{n-k}-\lambda_{I_i}^{n-k})+I_r^{k+1}(\lambda_A^{n-k}-\lambda_{I_i}^{n-k})\right]$$

$$R_3^k = \frac{\theta I_i^{k+1}(\lambda_{I_i}^{n-k}-\lambda_{I_r}^{n-k})}{2B_3}$$

$$u_1^{k+1} = \begin{cases} 0 & \text{if} \quad R_1^k \le 0 \\ R_1^k & \text{if} \quad 0 < R_1^k \le 1 \\ 1 & \text{if} \quad R_1^k \ge 1 \end{cases}$$

$$u_2^{k+1} = \begin{cases} 0 & \text{if} \quad R_2^k \le 0 \\ R_2^k & \text{if} \quad 0 < R_2^k \le 1 \\ 1 & \text{if} \quad R_2^k \ge 1 \end{cases}$$

$$u_3^{k+1} = \begin{cases} 0 & \text{if} \quad R_3^k \le 0 \\ R_3^k & \text{if} \quad 0 < R_3^k \le 1 \\ 1 & \text{if} \quad R_3^k \ge 1 \end{cases}$$

end

Algorithm for Solving Optimality System

We solve the optimal control problem, considering different combinations of strategies in our model. Thus, we implement the following strategy combinations:

Strategy A: Using only infection control(i.e. $u_1 \neq 0, u_2 = u_3 = 0$)

Strategy B: Using only behavioral change efforts.(i.e. $u_1 = 0, u_2 \neq 0, u_3 = 0$)

Strategy C: Using only HAART treatment.(i.e. $u_1 = u_2 = 0\, u_3 \neq 0$)

Strategy D: Using Infection control and behavioral change only (i.e. $u_1 \neq 0, u_2 \neq 0, u_3 = 0$)

Strategy E: Using Infection control and HAART treatment only $u_1 \neq 0, u_3 \neq 0, u_2 = 0$)

Strategy F: Using Behavioral change and HAART treatment only (i.e. $u_1 = 0, u_2 \neq 0, u_3 \neq 0$)

Strategy G: Using all Control efforts (i.e. $u_1 \neq 0, u_2 \neq 0, u_3 \neq 0$)

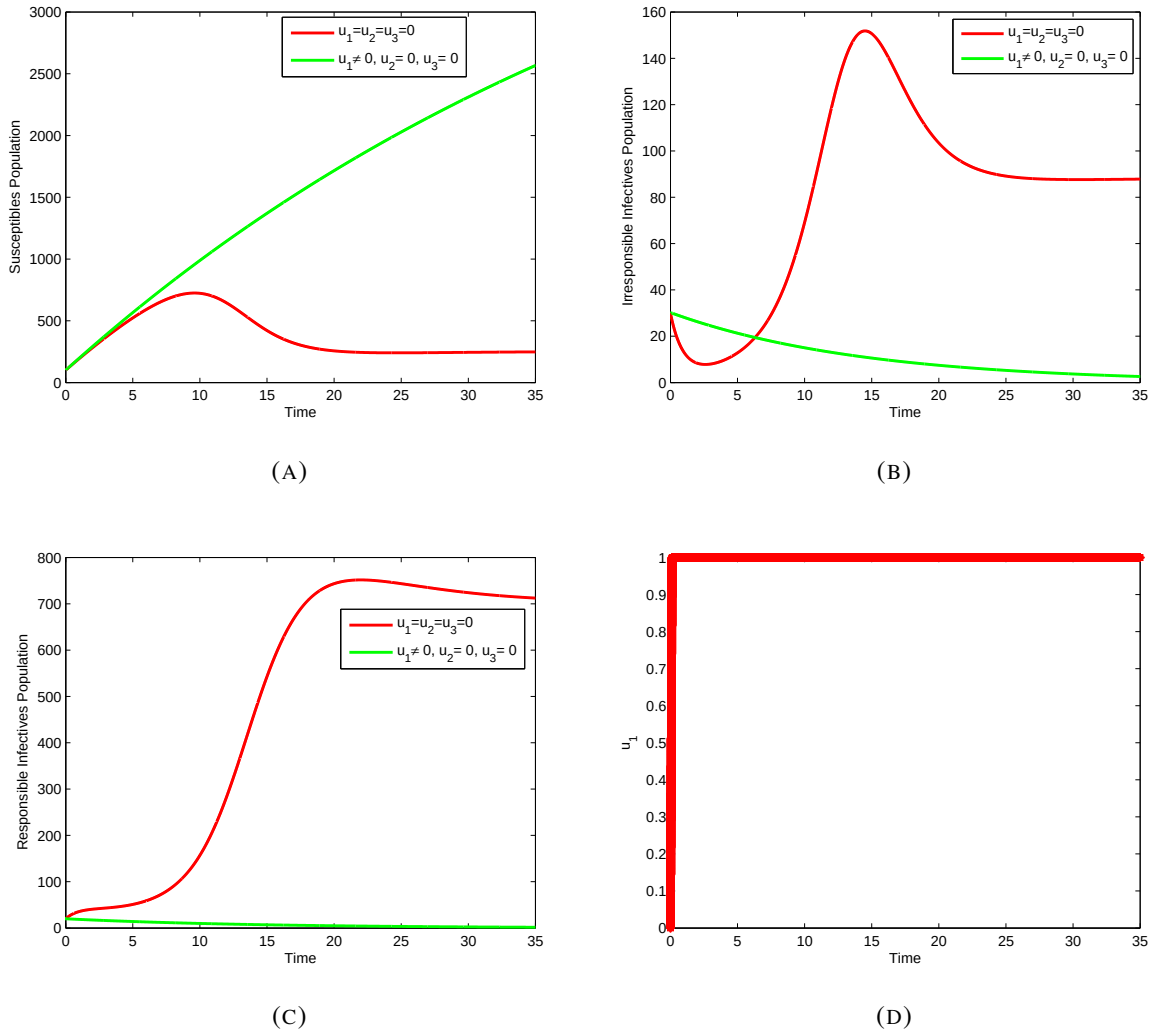

FIGURE 2. Simulations of the optimal control problem implementing Strategy A. The control profile is seen to rise sharply to the upper bound and remain so for the duration of the control strategy.

Some results of the numerical simulations of the optimal control problem are presented in Figures 2 and 3

To compare our intervention strategies, we make use of the incremental cost effectiveness ratio among the possible interventions. We calculate the incremental cost effectiveness ratio of each strategy over all other strategies. The incremental cost effectiveness ratio (ICER) of strategy 1 over 2 can be defined as the additional cost incurred per additional outcome in implementing strategy 1 instead of 2 given by

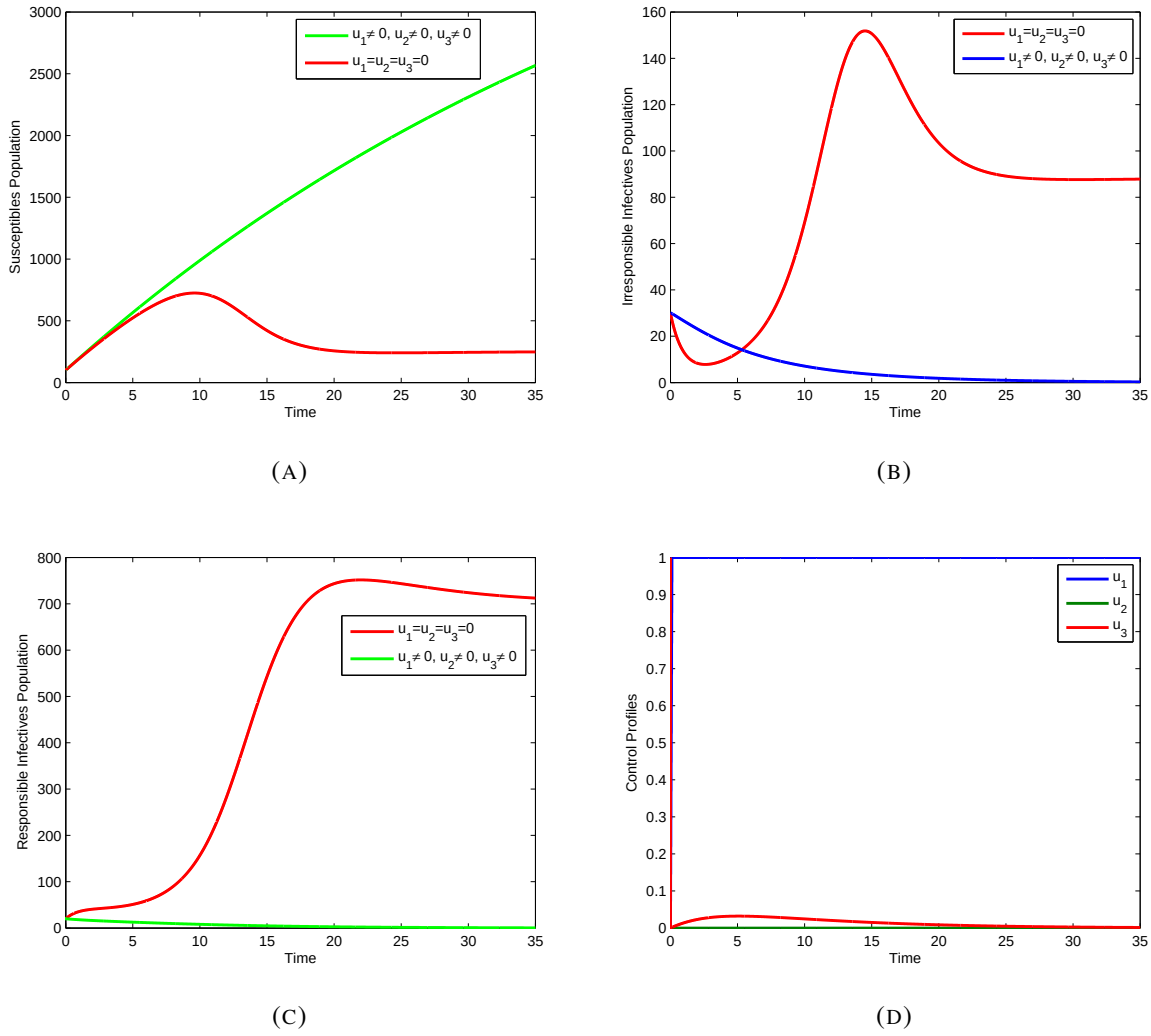

FIGURE 3. Simulations of the optimal control problem implementing Strategy G.

$$ICER = \frac{\text{Cost of Strategy 1-Cost of Strategy 2}}{\text{Total Number of infections averted using 1-Total Number of infections averted using 2}}.$$

Thus, the smaller the ICER of strategy 1 over 2 implies that it is better to implement strategy 1 and vice versa. We rank our strategies by comparing two at a time. Thus, we compare strategy A and B, and then the best one is compared with C. This process continues until we compare the best with the final strategy, G. Thus, from most to least effective we have Strategy G, F , D, B, C and A. Hence it is better to implement all our intervention strategies if we want to reduce the infection and minimize cost.

TABLE 2. Parameter descriptions and values used in the model.

Parameter	**Parameter Description**	**Values**	**Ref.**
Q	Rate of recruitment	120.000	-
α	AIDs related death rate	1.000	[15]
β	Contact rate between Susceptibles and Infectives	0.344	[1]
c	Average number of contacts of a typical infective	3.000	-
δ	Rate of progression of Infective into AIDS	0.100	[1]
η	Modification parameter due to right-value judgment	0.400	[15]
μ	Natural Death rate	0.020	[1]
τ	Modification parameter due to weak sexual activity	0.100	-
θ	Probability of positive behavioral change	0.955	[5]
B_1	Cost per unit condom(For infection reduction)	\$0.24	-
B_2	Cost per individual educated(for attitudinal change)	\$33.13	
B_3	Cost of HAART per individual	\$869.12	-

Strategy	Infections Averted	Total Cost	ICER
Strategy A	796	239,450	-38606.15
Strategy B	-277	3,018,800	-2,590.66

It is seen that Strategy A is more cost effective than Strategy B. So we eliminate B and compare A with C.

Strategy	Infections Averted	Total Cost	ICER
Strategy A	796	239,450	
Strategy C	-2	652,260,000.00	-817,239.08

Thus we eliminate A and compare C with D. Continuing this process we realize that Strategy G is the most cost effective in combating the spread of the disease.

7. Conclusions

In this paper, we proposed and studied a nonlinear deterministic model for the dynamics of HIV/AIDS in a variable sized population. Basic dynamics of the proposed model such as the basic reproduction ratio were discussed and it was shown that the disease can be eradicated if the basic reproduction is less or equal to unity. We also sought to examine possible intervention strategies aimed at reducing infection with minimum cost. For this, we incorporated three control efforts (namely, Infection control, Behavioral change control and HAART treatment) into our model to obtain an optimal control problem. Conditions for optimal control of the disease were derived and analysed using the Pontryagin's Minimum principle and the resulting State, Co-state and bounds on the controls we numerically solved. To find the most cost effective way of controlling the disease, the incremental cost effectiveness ratio was calculated for all possible implementations of our control efforts and ICER of each strategy compared with the rest strategies. It is realized that implementing all the control efforts will better than all other possible combinations of controls. Our findings are in suggests that the ABC campaign against HIV/AIDS is good but should not be overemphasized, but augmented by treatment for effective combat against the disease.

Conflict of Interests

The authors declare that there is no conflict of interests.

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