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THE (p, q)-ANALOGUES OF SOME INEQUALITIES FOR THE DIGAMMA FUNCTION

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Abstract. In this paper, we present the (p, q)-analogues of some inequalities concerning the digamma function. Our results generalize some earlier results.

1. Introduction and Preliminaries

The classical Euler's Gamma function, $\Gamma(t)$ and the digamma function, $\psi(t)$ are commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\,dx \quad \text{and} \quad \psi(t) = \frac{d}{dt}\ln\Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$

The p-analogues of the Gamma and digamma functions are respectively defined as follows.

$$\Gamma_p(t) = \frac{p! p^t}{t(t+1)\dots(t+p)} \quad \text{and} \quad \psi_p(t) = \frac{d}{dt}\ln\Gamma_p(t) = \frac{\Gamma_p'(t)}{\Gamma_p(t)}, \quad t > 0.$$

where $\lim_{p\to\infty}\Gamma_p(t) = \Gamma(t)$ and $\lim_{p\to\infty}\psi_p(t) = \psi(t)$. For some more insights and properties of these functions, see [1], [3] and the references therein.

Similarly, the q-analogues of the Gamma and digamma functions are respectively defined for $q \in (0, 1)$ as (see also [1] and [3])

$$\Gamma_q(t) = (1-q)^{1-t}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{t+n}} \quad \text{and} \quad \psi_q(t) = \frac{d}{dt}\ln\Gamma_q(t) = \frac{\Gamma_q'(t)}{\Gamma_q(t)}, \quad t > 0.$$

where $\lim_{q\to 1^-}\Gamma_q(t) = \Gamma(t)$ and $\lim_{q\to 1^-}\psi_q(t) = \psi(t)$.

In 2012, Krasniqi [2] defined the (p, q)-analogue of the Gamma function, $\Gamma_{p,q}(t)$ as

$$\Gamma_{p,q}(t) = \frac{[p]_q^t [p]_q!}{[t]_q[t+1]_q\dots[t+p]_q}, \quad t > 0, \quad p \in N, \quad q \in (0, 1).$$

where $[p]_q = \frac{1-q^p}{1-q}$. For several properties and characteristics of this function, we refer to [4]

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Similarly, the (p, q)-analogue of the digamma function $\psi_{p,q}(t)$ is defined as

$$\psi_{p,q}(t) = \frac{d}{dt} \ln \Gamma_{p,q}(t) = \frac{\Gamma'_{p,q}(t)}{\Gamma_{p,q}(t)}, \quad t > 0, \quad p \in N, \quad q \in (0, 1).$$

The functions $\psi(t)$ and $\psi_{p,q}(t)$ as defined above have the following series representations.

$$\psi(t) = -\gamma + (t-1) \sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)}, \quad t > 0$$

$$\psi_{p,q}(t) = \ln[p]_q + (\ln q) \sum_{n=1}^{p} \frac{q^{nt}}{1-q^n}, \qquad t > 0.$$

where γ is the Euler-Mascheroni's constant.

By taking the m-th derivative of these functions, it can easily be shown that the following statements are valid for $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+t)^{m+1}}, \quad t > 0$$

$$\psi_{p,q}^{(m)}(t) = (\ln q)^{m+1} \sum_{n=1}^{p} \frac{n^m q^{nt}}{1-q^n}, \quad t > 0.$$

In 2011, Sulaiman [10] presented the following results.

$$\psi(s+t) \geq \psi(s) + \psi(t) \tag{1.1}$$

for $t > 0$ and $0 < s < 1$.

$$\psi^{(m)}(s+t) \leq \psi^{(m)}(s) + \psi^{(m)}(t) \tag{1.2}$$

for $s, t > 0$ and for a positive odd integer m.

$$\psi^{(m)}(s+t) \geq \psi^{(m)}(s) + \psi^{(m)}(t) \tag{1.3}$$

for $s, t > 0$ and for a positive even integer m.

$$\psi^{(m)}(s)\psi^{(m)}(t) \geq \left[\psi^{(m)}(s+t)\right]^2 \tag{1.4}$$

for $s, t > 0$ and for a positive odd integer m.

Prior to Sulaiman's results, Mansour and Shabani by using different techniques established similar inequalities for the function $\psi_q(t)$. These can be found in [5].

Our objective in this paper is to establish that the inequalities (1.1), (1.2), (1.3) and (1.4) still hold true for the (p, q)-analogue of the digamma function.

2. Main Results

We now present the results of this paper.

Theorem 2.1. *Let $t > 0$, $0 < s \leq 1$, $q \in (0, 1)$ and $p \in N$. Then the following inequality is valid.*

$$\psi_{p,q}(s+t) \geq \psi_{p,q}(s) + \psi_{p,q}(t). \tag{2.1}$$

Proof. Let $\mu(t) = \psi_{p,q}(s+t) - \psi_{p,q}(s) - \psi_{p,q}(t)$. Then fixing s we have,

$$\mu'(t) = \psi'_{p,q}(s+t) - \psi'_{p,q}(t) = (\ln q)^2 \sum_{n=1}^{p} \left[\frac{nq^{n(s+t)}}{1-q^n} - \frac{nq^{nt}}{1-q^n} \right]$$
$$= (\ln q)^2 \sum_{n=1}^{p} \frac{nq^{nt}(q^{ns}-1)}{1-q^n} \leq 0.$$

That implies μ is non-increasing. Furthermore,

$$\lim_{t\to\infty} \mu(t) = \lim_{t\to\infty} [\psi_{p,q}(s+t) - \psi_{p,q}(s) - \psi_{p,q}(t)]$$
$$= -\ln[p]_q + (\ln q) \lim_{t\to\infty} \sum_{n=1}^{p} \left[\frac{q^{n(s+t)}}{1-q^n} - \frac{q^{ns}}{1-q^n} - \frac{q^{nt}}{1-q^n} \right]$$
$$= -\ln[p]_q + (\ln q) \lim_{t\to\infty} \sum_{n=1}^{p} \left[\frac{q^{ns}.q^{nt} - q^{ns} - q^{nt}}{1-q^n} \right]$$
$$= -\ln[p]_q - (\ln q) \sum_{n=1}^{p} \frac{q^{ns}}{1-q^n} \geq 0.$$

Therefore $\mu(t) \geq 0$ concluding the proof. $\square$

Theorem 2.2. *Let $s,t > 0$, $q \in (0,1)$ and $p \in N$. Suppose that m is a positive odd integer, then the following inequality is valid.*

$$\psi^{(m)}_{p,q}(s+t) \leq \psi^{(m)}_{p,q}(s) + \psi^{(m)}_{p,q}(t). \tag{2.2}$$

Proof. Let $\eta(t) = \psi^{(m)}_{p,q}(s+t) - \psi^{(m)}_{p,q}(s) - \psi^{(m)}_{p,q}(t)$. Then fixing s we have,

$$\eta'(t) = \psi^{(m+1)}_{p,q}(s+t) - \psi^{(m+1)}_{p,q}(t)$$
$$= (\ln q)^{m+2} \sum_{n=1}^{p} \left[\frac{n^{m+1}q^{n(s+t)}}{1-q^n} - \frac{n^{m+1}q^{nt}}{1-q^n} \right]$$
$$= (\ln q)^{m+2} \sum_{n=1}^{p} \left[\frac{n^{m+1}q^{nt}(q^{ns}-1)}{1-q^n} \right] \geq 0. \text{ (since } m \text{ is odd)}$$

That implies η is non-decreasing. Furthermore,

$$\lim_{t\to\infty} \eta(t) = (\ln q)^{m+1} \lim_{t\to\infty} \sum_{n=1}^{p} \left[\frac{n^m q^{n(s+t)}}{1-q^n} - \frac{n^m q^{ns}}{1-q^n} - \frac{n^m q^{nt}}{1-q^n} \right]$$
$$= (\ln q)^{m+1} \lim_{t\to\infty} \sum_{n=1}^{p} \left[\frac{n^m q^{ns}.q^{nt}}{1-q^n} - \frac{n^m q^{ns}}{1-q^n} - \frac{n^m q^{nt}}{1-q^n} \right]$$
$$= -(\ln q)^{m+1} \sum_{n=1}^{p} \frac{n^m q^{ns}}{1-q^n} \leq 0. \text{ (since } m \text{ is odd)}$$

Therefore $\eta(t) \leq 0$ concluding the proof. $\square$

Theorem 2.3. *Let $s,t > 0$, $q \in (0,1)$ and $p \in N$. Suppose that m is a positive even integer, then the following inequality is valid.*

$$\psi^{(m)}_{p,q}(s+t) \geq \psi^{(m)}_{p,q}(s) + \psi^{(m)}_{p,q}(t). \tag{2.3}$$

Proof. Let $\lambda(t) = \psi^{(m)}_{p,q}(s+t) - \psi^{(m)}_{p,q}(s) - \psi^{(m)}_{p,q}(t)$. Then fixing s we have,

$$\begin{aligned}\lambda'(t) &= \psi^{(m+1)}_{p,q}(s+t) - \psi^{(m+1)}_{p,q}(t)\\ &= (\ln q)^{m+2} \sum_{n=1}^{p} \left[\frac{n^{m+1} q^{n(s+t)}}{1-q^n} - \frac{n^{m+1} q^{nt}}{1-q^n} \right]\\ &= (\ln q)^{m+2} \sum_{n=1}^{p} \left[\frac{n^{m+1} q^{nt}(q^{ns}-1)}{1-q^n} \right] \leq 0. \text{ (since } m \text{ is even)}\end{aligned}$$

That implies λ is non-decreasing. Furthermore,

$$\begin{aligned}\lim_{t\to\infty} \lambda(t) &= (\ln q)^{m+1} \lim_{t\to\infty} \sum_{n=1}^{p} \left[\frac{n^m q^{n(s+t)}}{1-q^n} - \frac{n^m q^{ns}}{1-q^n} - \frac{n^m q^{nt}}{1-q^n} \right]\\ &= -(\ln q)^{m+1} \sum_{n=1}^{p} \frac{n^m q^{ns}}{1-q^n} \geq 0. \text{ (since } m \text{ is even)}\end{aligned}$$

Therefore $\lambda(t) \geq 0$ concluding the proof. □

Theorem 2.4. *Let $s, t > 0$, $q \in (0,1)$ and $p \in N$. Suppose m is a positive odd integer, then the following inequality holds true.*

$$\psi^{(m)}_{p,q}(s)\psi^{(m)}_{p,q}(t) \geq \left[\psi^{(m)}_{p,q}(s+t)\right]^2 \tag{2.4}$$

Proof. We proceed as follows.

$$\begin{aligned}\psi^{(m)}_{p,q}(s) - \psi^{(m)}_{p,q}(s+t) &= (\ln q)^{m+1} \sum_{n=1}^{p} \left[\frac{n^m q^{ns}}{1-q^n} - \frac{n^m q^{n(s+t)}}{1-q^n} \right]\\ &= (\ln q)^{m+1} \sum_{n=1}^{p} \left[\frac{n^m q^{ns}(1-q^{nt})}{1-q^n} \right] \geq 0. \text{ (since } m \text{ is odd)}\end{aligned}$$

That implies,

$$\psi^{(m)}_{p,q}(s) \geq \psi^{(m)}_{p,q}(s+t) \geq 0.$$

Similarly we have,

$$\psi^{(m)}_{p,q}(t) \geq \psi^{(m)}_{p,q}(s+t) \geq 0.$$

Multiplying these inequalities yields the desired results. Thus,

$$\psi^{(m)}_{p,q}(s)\psi^{(m)}_{p,q}(t) \geq \left[\psi^{(m)}_{p,q}(s+t)\right]^2.$$

□

3. Concluding Remarks

Remark. *If in inequalities (2.1), (2.2), (2.3) and (2.4) we allow $p \to \infty$ as $q \to 1^-$, then we repectively recover the inequalities (1.1), (1.2), (1.3) and (1.4). We have thus generalized the earlier results as in* [5] *and* [10]*. The k, p and q analogues of (1.1), (1.2) and (1.3) can be found in the papers* [7], [8] *and* [9]*. Also, the (q, k)-analogues of (2.1), (2.2), (2.3) and (2.4) can be found in* [6].

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References

[1] V. Krasniqi, T. Mansour and A. Sh. Shabani, *Some Monotonicity Properties and Inequalities for Γ and ζ Functions*, Mathematical Communications **15**(2)(2010), 365-376.

[2] V. Krasniqi and F. Merovci, *Some Completely Monotonic Properties for the (p,q)-Gamma Function*, Mathematica Balkanica, New Series **26**(2012), 1-2.

[3] V. Krasniqi, A. Sh. Shabani, *Convexity Properties and Inequalities for a Generalized Gamma Function*, Applied Mathematics E-Notes **10**(2010), 27-35.

[4] Valmir B. Krasniqi, Hari M. Srivastava and Sever S. Dragomir, *Some complete monotonicity properties for the (p,q)-gamma function*, Appl. Math. Comput. **219**(2013), 10538-10547.

[5] T. Mansour and A. Sh. Shabani, *Some Inequalities for the q-Digamma Function*, J. Ineq. Pure Appl. Math. **10**(1)(2009), Art. 12.

[6] K. Nantomah, *The (q,k)-analogues of some inequalities for the Digamma function*, South Asian J. Math. **4**(2)(2014), 64-68.

[7] K. Nantomah and E. Prempeh, *The k-Analogues of Some Inequalities for the Digamma Function*, Math. Aeterna, **4**(3)(2014), 269-273.

[8] K. Nantomah and E. Prempeh, *The p-Analogues of Some Inequalities for the Digamma Function*, Math. Aeterna, **4**(3)(2014), 275-279.

[9] K. Nantomah and E. Prempeh, *The q-Analogues of Some Inequalities for the Digamma Function*, Math. Aeterna, **4**(3)(2014), 281-285.

[10] W. T. Sulaiman, *Turan inequalites for the digamma and polygamma functions*, South Asian J. Math. **1**(2)(2011), 49-55.

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