Mathematica Aeterna, Vol. 4, 2014, no. 3, 263 - 268

Analogues of an Inequality for the m-th derivative of the Digamma Function

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Abstract

In this paper, we present the p, q and k analogues of a certain inequality established by Sulaiman in his paper. We also present a generalization of this inequality.

Mathematics Subject Classification: 33B15, 26A48.

Keywords: digamma function, p-analogue, q-analogue, k-analogue, Inequality.

1 Introduction and Preliminaries

We begin by recalling some basic definitions involving the Gamma function.

The digamma function $\psi(t)$ is defined as

$$\psi(t) = \frac{d}{dt}\ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0$$

where $\Gamma(t)$ is the well-known classical Euler's Gamma function defined by

$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}\,dx, \qquad t > 0.$$

The p-analogue of the digamma function, $\psi_p(t)$ is also defined as

$$\psi_p(t) = \frac{d}{dt}\ln(\Gamma_p(t)) = \frac{\Gamma_p'(t)}{\Gamma_p(t)}, \qquad t > 0.$$

where $\Gamma_p(t)$ is given by (see [2],[3])

$$\Gamma_p(t) = \frac{p!p^t}{t(t+1)\dots(t+p)} = \frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})}, \quad p \in N, \quad t > 0.$$

Similarly, the q-analogue of the digamma function, $\psi_q(t)$ is defined as,

$$\psi_q(t) = \frac{d}{dt}\ln(\Gamma_q(t)) = \frac{\Gamma_q'(t)}{\Gamma_q(t)}, \quad t > 0.$$

where $\Gamma_q(t)$ is given by (see [4])

$$\Gamma_q(t) = (1-q)^{1-t}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0.$$

Also, the k-analogue of the digamma function, $\psi_k(t)$ is defined as

$$\psi_k(t) = \frac{d}{dt}\ln(\Gamma_k(t)) = \frac{\Gamma_k'(t)}{\Gamma_k(t)}, \quad t > 0.$$

where $\Gamma_k(t)$ is given by (see [1],[5])

$$\Gamma_k(t) = \int_0^\infty e^{-\frac{x^k}{k}} x^{t-1}\,dx, \quad k > 0, \quad t > 0.$$

The functions $\psi(t)$, $\psi_p(t)$, $\psi_q(t)$ and $\psi_k(t)$ as defined above have the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty}\frac{1}{(1+n)(n+t)}, \quad t > 0$$

$$\psi_p(t) = \ln p - \sum_{n=0}^{p}\frac{1}{n+t}, \quad p \in N, \quad t > 0$$

$$\psi_q(t) = -\ln(1-q) + (\ln q)\sum_{n=1}^{\infty}\frac{q^{nt}}{1-q^n}, \quad q \in (0,1), \quad t > 0$$

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty}\frac{t}{nk(nk+t)}, \quad k > 0, \quad t > 0.$$

where γ is the Euler-Mascheroni's constant. For some properties of these functions, see [7], [3], [2] and [5] and the references therein.

By taking the m-th derivative of the above functions, it can be shown that the following statements are valid for $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+t)^{m+1}}, \quad t > 0$$

$$\psi_p^{(m)}(t) = (-1)^{m-1} m! \sum_{n=0}^{p} \frac{1}{(n+t)^{m+1}}, \quad p \in N, \quad t > 0$$

$$\psi_q^{(m)}(t) = (\ln q)^{m+1} \sum_{n=1}^{\infty} \frac{n^m q^{nt}}{1-q^n}, \quad q \in (0,1), \quad t > 0$$

$$\psi_k^{(m)}(t) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(nk+t)^{m+1}}, \quad k > 0, \quad t > 0.$$

In 2011, Sulaiman [6] presented the following results for $s, t > 0$ and for a positive odd integer m.

$$\psi^{(m)}(s)\psi^{(m)}(t) \geq \left[\psi^{(m)}(s+t)\right]^2 \tag{1}$$

The objective of this paper is to establish that the inequality (1) still holds true for the functions $\psi_p(t)$, $\psi_q(t)$ and $\psi_k(t)$. A generalization of this inequality is also presented.

2 Main Results

We now present the results of this paper.

Theorem 2.1. *Let $s, t > 0$ and $p \in N$. Suppose m is a positive odd integer, then the following inequality holds true.*

$$\psi_p^{(m)}(s)\psi_p^{(m)}(t) \geq \left[\psi_p^{(m)}(s+t)\right]^2 \tag{2}$$

Proof. We proceed as follows.

$$\begin{aligned}
\psi_p^{(m)}(s) - \psi_p^{(m)}(s+t) &= (-1)^{m-1} m! \sum_{n=0}^{p} \left[\frac{1}{(n+s)^{m+1}} - \frac{1}{(n+s+t)^{m+1}}\right] \\
&= m! \sum_{n=0}^{p} \left[\frac{1}{(n+s)^{m+1}} - \frac{1}{(n+s+t)^{m+1}}\right] \text{ (since } m \text{ is odd)} \\
&\geq 0.
\end{aligned}$$

Hence,
$$\psi_p^{(m)}(s) \geq \psi_p^{(m)}(s+t) \geq 0.$$
Similarly we have it that,
$$\psi_p^{(m)}(t) \geq \psi_p^{(m)}(s+t) \geq 0.$$
Multiplying the above inequalities yields,
$$\psi_p^{(m)}(s)\psi_p^{(m)}(t) \geq \left[\psi_p^{(m)}(s+t)\right]^2.$$

Theorem 2.2. *Let $s,t>0$ and $q \in (0,1)$. Suppose m is a positive odd integer, then the following inequality holds true.*
$$\psi_q^{(m)}(s)\psi_q^{(m)}(t) \geq \left[\psi_q^{(m)}(s+t)\right]^2 \tag{3}$$

Proof. We proceed as follows.
$$\begin{aligned}\psi_q^{(m)}(s) - \psi_q^{(m)}(s+t) &= (\ln q)^{m+1}\sum_{n=1}^{\infty}\left[\frac{n^m q^{ns}}{1-q^n} - \frac{n^m q^{n(s+t)}}{1-q^n}\right] \\ &= (\ln q)^{m+1}\sum_{n=1}^{\infty}\left[\frac{n^m q^{ns} - n^m q^{ns}.q^{nt}}{1-q^n}\right] \\ &= (\ln q)^{m+1}\sum_{n=1}^{\infty}\left[\frac{n^m q^{ns}(1-q^{nt})}{1-q^n}\right] \geq 0. \text{ (since } m \text{ is odd)}\end{aligned}$$

Hence,
$$\psi_q^{(m)}(s) \geq \psi_q^{(m)}(s+t) \geq 0.$$
Similarly we have it that,
$$\psi_q^{(m)}(t) \geq \psi_q^{(m)}(s+t) \geq 0.$$
Multiplying these inequalities yields,
$$\psi_q^{(m)}(s)\psi_q^{(m)}(t) \geq \left[\psi_q^{(m)}(s+t)\right]^2.$$

Theorem 2.3. *Let $s,t>0$ and $k>0$. Suppose m is a positive odd integer, then the following inequality holds true.*
$$\psi_k^{(m)}(s)\psi_k^{(m)}(t) \geq \left[\psi_k^{(m)}(s+t)\right]^2 \tag{4}$$

Proof. We proceed as follows.

$$\begin{aligned}\psi_k^{(m)}(s) - \psi_k^{(m)}(s+t) &= (-1)^{m+1} m! \sum_{n=0}^{\infty} \left[\frac{1}{(nk+s)^{m+1}} - \frac{1}{(nk+s+t)^{m+1}} \right] \\ &= m! \sum_{n=0}^{\infty} \left[\frac{1}{(nk+s)^{m+1}} - \frac{1}{(nk+s+t)^{m+1}} \right] \text{ (since } m \text{ is odd)} \\ &\geq 0.\end{aligned}$$

That is,

$$\psi_k^{(m)}(s) \geq \psi_k^{(m)}(s+t) \geq 0.$$

By a similar approach we have,

$$\psi_k^{(m)}(t) \geq \psi_k^{(m)}(s+t) \geq 0.$$

Multiplying these inequalities yields,

$$\psi_k^{(m)}(s)\psi_k^{(m)}(t) \geq \left[\psi_k^{(m)}(s+t) \right]^2.$$

Theorem 2.4. *Let $\alpha \in Z^+$ and $t_i > 0$ for each i. If m is a positive odd integer, then the following inequality holds true.*

$$\prod_{i=1}^{\alpha} \psi^{(m)}(t_i) \geq \left[\psi^{(m)} \left(\sum_{i=1}^{\alpha} t_i \right) \right]^{\alpha} \tag{5}$$

Proof. Proceed as follows.

$$\begin{aligned}\psi^{(m)}(t_1) - \psi^{(m)} \left(\sum_{i=1}^{\alpha} t_i \right) &= (-1)^{m+1} m! \sum_{n=0}^{\infty} \left[\frac{1}{(n+t_1)^{m+1}} - \frac{1}{(n+\sum_{i=1}^{\alpha} t_i)^{m+1}} \right] \\ &= m! \sum_{n=0}^{\infty} \left[\frac{1}{(n+t_1)^{m+1}} - \frac{1}{(n+\sum_{i=1}^{\alpha} t_i)^{m+1}} \right] \geq 0.\end{aligned}$$

Hence,

$$\psi^{(m)}(t_1) \geq \psi^{(m)} \left(\sum_{i=1}^{\alpha} t_i \right) \geq 0.$$

Continuing in a similar fashion yields,

$$\psi^{(m)}(t_2) \geq \psi^{(m)}\left(\sum_{i=1}^{\alpha} t_i\right) \geq 0,$$
$$\psi^{(m)}(t_3) \geq \psi^{(m)}\left(\sum_{i=1}^{\alpha} t_i\right) \geq 0,$$
$$\vdots \qquad \vdots$$
$$\psi^{(m)}(t_\alpha) \geq \psi^{(m)}\left(\sum_{i=1}^{\alpha} t_i\right) \geq 0.$$

Multiplying these inequalities yields,

$$\prod_{i=1}^{\alpha} \psi^{(m)}(t_i) \geq \left[\psi^{(m)}\left(\sum_{i=1}^{\alpha} t_i\right)\right]^{\alpha}.$$

Remark 2.5. If in (5) we set $t_1 = s$, $t_2 = t$ and $\alpha = 2$, then (1) is restored. Hence by this result, inequality (1) has been generalized.

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Received: March, 2014