UNIVERSITY FOR DEVELOPMENT STUDIES

MODIFICATIONS OF THE WEIGHTED WEIBULL DISTRIBUTION

DONNE KOMLA MUDDEY

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MODIFICATIONS OF THE WEIGHTED WEIBULL DISTRIBUTION

BY

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JULY, 2021

DECLARATION

Student

I hereby declare that this thesis is the result of my original work and that no part of it has been presented for another degree in this University or elsewhere..

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We hereby declare that the preparation and presentation of this thesis was supervised in accordance with the guidelines on supervision of thesis laid down by the University for Development Studies.

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DEDICATION

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ABSTRACT

In modeling real-life events with respect to probability theory, two particular characteristics are considered, either the probability distribution is Flexible or the distribution is Tractable. Statistically, in order to retain the originality of the data, appropriate probability distribution needs to be employed rather than to transform the existing dataset. Classical distributions lack the ability to model and describe some important real-life events. Hence, the derived compound distributions are most appropriately employed to increase flexibility and capability to model real-life datasets. This study modified the weighted Weibull distribution with respect to Exponentiated Weighted Weibull and Geometric Weighted Weibull distributions which were obtained and derived having an interest in statistical theory. The shapes of the probability density function and hazard rate functions are investigated, as well as some structural statistical properties of the distribution. The study reports the use of the maximum likelihood estimation to determine unknown parameters by means of the Machov Chain Monte Carlo simulation and application using four illustrative datasets. The study shows the two derived modifications are obtained, which are the Geometric Weighted Weibull and the Exopnentiated Weighted Weibull. It is further reported that, the two derived distributions have superior performance compared with other modifications of the distributions. The The processes were performed using the *R*-Software. It is recommended that further study can be extended based on the derived distributions to construct autoregressive processes.

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CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Flexibility and tractability are essential ingredients at the heart of modeling probability distributions (Oguntunde et al., 2016; Ofosu et al., 2016). Oguntunde et al. (2016) contrast tractability and flexibility. They explained tractable as how a function of the distribution is transform with less effort and normally applies in the simulation of random samples. On the other hand, flexibility means the addition of parameters to the distribution to improve on the model.

Statistically, it is necessary to retain the originality of the data set by employing appropriate probability distribution for a good fit rather than to transform the existing data. Consequently, previous literature had shown that there is a need to ensure existing classical distributions are extended and modified (Merovci , 2015), this could enhance the capability and increase flexibility to model real-life data sets. To broaden a current standard distribution, different methodologies could be employed. For example, the flexibility of a statistical distribution can be expanded utilizing generalisation which includes utilizing generalised family of distribution. The flexibility can likewise be expanded by altering the existing distribution.

Weibull distribution has become more useful compared with other distributions, more especially in the practice of lifetime data modeling. In the study by Abbas et al. (2019), they stated that in the field like; financial, biomedical, engineering, insurance, actuarial and environmental sciences it is not suitable to adopt classical distribution. In these fields, classical models become only relevant when there is a need to select models for applications. In that case, many researchers have come up with several extensions of models. Notwithstanding that, there is still a call to improve upon existing models for flexibility in data modeling.

Generalisation of Weibull distribution has attracted several researchers to seek an extension of the distribution which increases flexibility in modeling real-life data. The Weibull distribution is not a suitable model to explain non-monotone hazard rate function (HRF), such as unimodal, U-shape, or bathtub form (Abbas et al., 2019; Aleem el al., 2013; Sarhan and Apaloo , 2013). To extend the Weibull distribution, Mudholkar and Srivastava (1993) developed exponentiated Weibull distribution that exhibits unimodal and bathtub shape. Pal et al. (2006) have extensively dealt with this distribution by estimating unknown parameters and mathematical properties. According to Al-Saleh and Agarwal (2006), hazard function can exhibit unimodal and bathtub shapes in their extended version of the Weibull distribution. Further extended Weibull distribution was also proposed by Zhang and Xie (2007) by executing the Marshall and Olkin (1997) approach of including another parameter. They developed extended Weibull by finding the product of cumulative hazard function and $e^{\lambda x}$. New weighted Weibull (WW) distributions were proposed by implementing the Nasiru family of distribution (Nasiru , 2015; Abbas et al., 2019).

It is very important to develop and improve Weibull distribution because; firstly, it enhances the theoretical interpretation of the data. Secondly, improving judgment on model fit, and presenting a model whose empirical fit better suits a specific dataset. This implies that it provides a solution for data interpretation and model specification. The theory of weighted distribution provides a technique for fitting models to the unknown weight functions (Saghira et al., 2017).

When modeling monotone hazard rates, the Weibull distribution has been a powerful probability distribution which has received appreciable usage in reliability analysis, while weighted distributions are used to adjust the probabilities of the events as observed and recorded. The Weibull distribution provides a good alternative to exponential, gamma, and lognormal distribution in biological studies and life testing (Nasiru et al., 2018; Almalki and Yuan , 2013; Zhang and Xie , 2007).

However, certain lifetime data; business life cycles, human mortality records, and graduate unemployment require non-monotonic shapes like the modified unimodal, bathtub shape, and unimodal. Such bathtub hazard curve have almost level center segments and the corresponding densities have a positive anti mode. A case of a bathtub shape failure rate is the human mortality involvement in a high newborn child death rate which decreases quickly to arrive at a low. It at that point stays at that level for many years before getting once more. Unimodal failure rates can be seen in course of a sickness whose mortality arrives at a top after some limited period and afterward gradually declines. Modified weighted Weibull distributions often fit the first and middle phases of the hazard functions. However, it lacks the credibility to fit the last phase of the bathtub and the modified unimodal shapes.

Recently, various weighted versions of Weibull and exponential distributions have been proposed in the literature. For example, Azzalini (1985) proposed a method of obtaining weighted model by adding shape parameter. Azzalinis method has been used extensively for several symmetric and non-symmetric distributions (Mudholkar and Srivastava , 1993; Abbas et al., 2019). Nasiru (2015) used a similar approach to introduce a weighted version of the Weibull distribution and a three-parameter weighted Weibull distribution studied and its statistical properties. The capability of the new distribution was exhibited by applying it to a lifetime dataset. Badmus and Bamiduro (2014) had introduced an exponentiated weighted Weibull model which is introduced to result a model that is superior to both weighted Weibull and Weibull distributions as far as the estimate of their characteristics. An expansion of exponential distribution has been given by Nadarajah and Kotz (2006), utilizing the logit of Beta distribution and the link function of the Beta. From that point forward broad work has been done utilizing the logit of beta distribution in literature. For example, Gupta et al. (1998) proposed a generalised exponential distribution that gives an option in contrast to exponential and Weibull distributions. Mudholkar and Srivastava (1993) studied an exponentiated Weibull.

The new modified weighted Weibull distributions were derived from the concept of the weighted Weibull distribution, to provide a better flexibility extension and tractability of the mathematical concepts that can be used in much wider situations. This research modified the weighted Weibull distribution proposed by the weighted version of (Nasiru , 2015) concerning the following; exponentiated generalised weighted Weibull distribution and geometric weighted Weibull distribution, were obtained and its statistical properties were studied which interest to statistical theory.

1.2 Statement of the Problem

The standard distributions such as Weibull, exponentiated and gamma distributions have monotonic hazard rate functions in solving reliability problems, in any case, they do not give parametric fit to practical applications to model the complex lifetime of a system. Statistically, it is necessary to retain the originality of the data set by employing appropriate probability distribution for goodness-of-fit rather than to transform the existing data. Literature has shown that there is a need to ensure that classical distributions are modified and extended (Merovci , 2015).

However, certain lifetime data; business life cycles, human mortality records, and graduate unemployment require non-monotonic shapes including; modified unimodal, bathtub shape, and unimodal. For example, if a man begins a business, it is normal that the risk would not be consistent, the risk may be high at the underlying stage and decrease over the long haul. Clearly, classical distributions cannot describe and model some significant real-life events. Hence, compound distributions could be most appropriate, this research is all about implementing the extended Nasiru (2015) weighted Weibull distribution to model compound lifetime of a system. Model specification and interpretation are the hallmark of weighted distributions for efficient and effective modeling of statistical data (Saghira et al., 2017). Modified weighted Weibull distributions often fit the first and middle phases of the hazard functions. However, it lacks the credibility to fit the last phase of the bathtub and the modified unimodal shapes.

The aim of this generalisation is to introduce new modified weighted Weibull distribution that was derived from the concept of the weighted distribution to provide a better flexibility extension and tractability of the mathematical concepts of the weighted Weibull (WW) distribution.

1.3 The Objectives of the Study

1.3.1 General Objective

The general objective is to study and develop some modifications of the weighted Weibull distribution.

1.3.2 Specific Objectives

1. To develop geometric weighted Weibull distribution.

2. To develop exponentiated weighted Weibull distribution.

3. To develop estimators for estimating the parameters.

4. Demonstrate the application of various modifications using real data.

1.4 Significance of the Study

Bourguignon et al. (2014) posit that in the recent past, many standard distributions for modeling real-life data set have been employed but not rarely in applied areas such as insurance modeling and lifetime analysis. To address the challenges encountered on some real-life datasets in terms of flexibility, it is relevant to add shape parameters to the baseline distribution. This makes it more flexible in examining the tail characteristic. Thus, to fit the third phase of the modified unimodal well.

The knowledge of the appropriate distribution of real datasets greatly improves the sensitivity, power, and efficiency of the statistical test associated with the datasets. Hence, developing a new extended distribution with additional shape parameters to improve their goodness-of-fit is paramount. The new compound distribution can be valuable in the areas of distribution theory in statistics, insurance, and modeling real-life dataset relating to demography to construct a life table. Also, the compound distribution can be helpful in describing real-life phenomena whose failure rate is not constant. Thus, non-monotonic hazard rate functions are well fitted to describe the datasets with the following patterns; modified unimodal, unimodal, and bathtub hazard rate.

1.5 Scope of the Study

The baseline distribution the researcher implemented in this study was Nasirus family of distribution because is an improvement over the exponentiated generalised Weibull distribution. The study was restricted to the modification of Nasiru (2015) weighted Weibull distribution to achieve the stated objectives.

1.6 Definition of Terms

Asymmetry: It implies the distribution needs balance or it is lopsided.

Baseline Distribution: Parent distribution that is being extended.

Bathtub Shape: The shapes describe a curve that has three parts, namely; decreasing, constant, and increasing.

Degree of Freedom: Is the number of observation (pieces of information) in the data that are free to vary when estimating statistical parameters.

Failure Rate: It can also be called hazard function, hazard rate, the force of mortality, and risk. It is the frequency at which a component or system fails.

Inverted Bathtub Shape: This is the direct inverse of a bathtub shape. Such a bend likewise has three sections however in this methodical way; increasing, constant, and decreasing.

Monotonic Function: Refers to whether the function is increasing or decreasing. Observing the graph and check its derivative can be used to determine if a function is monotonic.

Weighted distribution: Is used to adjust the probability of events as observed and recorded.

1.7 Limitations of the Study

The developed models are well fitted for skewed datasets which lack the capability for modeling symmetrical types of datasets.

1.8 Organization of the Study

Chapter one introduces the basic concepts that resulted in the development of the compound distributions which comprises; the statement of the problem, general and specific objectives, the significance of the study, and the scope of the study. Chapter two introduces the families of weighted Weibull distribution which are extensively discussed. The literature search on the study variable or the mathematical and statistical properties were reviewed appropriately with the various modifications distributions. Chapter three highlights the various statistical methods employed to achieve the stated objectives. The major statistical and mathematical concepts implemented are; parameter estimation method of maximum likelihood estimation (MLE) which is used to estimate unknown parameters. We consider criteria like the Kolmogorov-Smirnov test statistics (K-S), Cramr-Von Mises (W*), and Log-likelihood Ratio Test (lnL) as its goodness-of-fit. Information criteria such as Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) are also considered. Chapter four focuses on the development of the new geometric weighted Weibull (GWW) distribution that satisfied objective 1, objective 3, and objective 4. Chapter five introduces a new modification of the exponentiated weighted Weibull (EWW) distribution and its cumulative distribution function (CDF) and probability density function (PDF), which satisfied objective 2. Chapter six, comprises the summary, conclusions of the thesis, and presents possible future work.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Several studies that have been completed in the past which are identified with the stated objectives of the study are discussed in this chapter. The researcher delved into some competitive models to the introduced distributions and their characteristics functions. Exponentiated generalised weighted Weibull distributions and geometric weighted Weibull distributions were thoroughly reviewed.

2.2 Historical Development of Weibull Distribution

The pioneer of Weibull distribution was the Swedish physicist Waloddi Weibull (Weibull , 1951). According to Hallinan Jr (1993), Weibull was the first to introduce a location parameter and scale parameter to make the distribution meaningful. With the improvement of the distribution, Waloddi was considered as the pioneer of Weibull distribution. Weibull used the distribution in 1939 in examining breaking strength of material (Lai et al. , 2003). Several researchers have used his work in different areas; climate and weather, forestry, engineering, quality control, maintenance and replacement, geography, and other fields. Failures rate of a phenomena being observed and recorded can be modelled by using Weibull distribution (Rinne , 2008; Murthy et al., 2004). Fisher (1934) used the method of weighted to weight the count of component to form distribution of recorded observations.

Recently, authors utilized the idea of weighted methodology for various purposes (Nasiru , 2015; Zhang and Xie , 2007; Marshall and Olkin , 1997; Lai et al. , 2003; Aryal et al. , 2016; Almalki and Yuan , 2013; Abbas et al., 2019), were extended generalised proposed models, since Weibull distribution is not a suitable model to explain the non-monotone hazard rate function such as unimodel, U-shaped or bathtub form. The researcher observed that there is a gap in the literature in view of the fact that extensive works have not been done toward the path of weighted Weibull distributions for statistical theory and practices.

2.3 Some Important Modifications of the Weibull Distribution

Modified forms of the Weibull distributions were proposed by many authors purposely for non-monotonic shapes. For instance, two-parameter truncated and flexible Weibull extension developed have a bathtub shaped hazard function (Bebbington et al., 2007; Zhang and Xie , 2007). Exponentiated Weibull distribution introduced for three-parameter by Mudholkar and Srivastava (1993). Analysts have created different modified types of the Weibull distribution to accomplish non-monotonic shapes. The two-parameter flexible Weibull extension introduced by Badmus and Bamiduro (2014) has a force of mortality exhibits bathtub shape. Exponentiated modified Weibull extension distribution by Sarhan and Apaloo (2013) which exhibits a bathtub-shape pattern. Generalisations of the Weibull distribution have been impetus for researchers to propose extended distribution of modeling lifetime data set to increase flexibility. Almalki and Yuan (2013) have also applied their distribution in a serial system by putting together the modified Weibull and Weibull distribution. Al-Saleh and Agarwal (2006) showed that the risk function can display unimodal and bathtub shapes by extended version of the Weibull distribution.

2.4 Conceptualization of the Weighted Weibull Distributions

Weibull distribution commonly applied lifetime distributions in different areas or fields with many applications. The PDF and CDF are traceable and closed forms which result into simple expressions of survival and risk rates. Suppose the random variable r follows the Weibull distribution. Then, probability density function and the cumulative distribution of the Weibull distribution (Weibull , 1951) respectively;

$$q(r) = \alpha\theta r^{\theta-1} \exp\left(\alpha r^{\theta}\right), \ r > 0 \tag{2.1}$$

$$Q(r) = 1 - \exp\left(\alpha r^{\theta}\right), \ r > 0. \tag{2.2}$$

where $\alpha > 0$ and $\theta > 0$ are the scale and shape parameter respectively. The corresponding survival function:

$$\bar{Q}(r) = \exp\left(\alpha r^{\theta}\right), \ r > 0. \tag{2.3}$$

Its hazard rate h(r):

$$h(r) = \frac{q(r)}{S(r)}$$

$$h(r) = \alpha\theta r^{\theta-1}, \ r > 0 \tag{2.4}$$

Hazard rate can exhibit; increasing, decreasing or constant depending on $\theta > 0, \ \theta < 0$ or $\theta = 1$ this does not show any kind of non-monotonic shapes. Aleem el al. (2013) proposed and derived modified weighted distribution with its PDF as:

$$f_w(r) = \frac{[1 - w(tr)]^c f(r)}{E[1 - w(tr)]^c}, \ r \in R, \tag{2.5}$$

$$F_r(\theta r) = w(tr)$$

The weighted Weibull derived by Nasiru (2015) family as:

$$q(r;\lambda,\alpha,\theta) = K g_0(r) G_0(\lambda r), \ r > 0, \tag{2.6}$$

Its PDF and CDF of another weighted Weibull distribution (WWD), respectively

$$q(r) = (1+\lambda^{\theta})\alpha\theta r^{\theta-1} e^{-(\alpha r^{\theta}+\alpha(\lambda r)^{\theta}}, \ r > 0 \tag{2.7}$$

and

$$Q(r) = 1 - e^{\alpha r^{\gamma(1+\lambda^{\gamma})}}, \ r,\alpha,\lambda > 0, \tag{2.8}$$

Scale parameter is denoted as α, whilst γ, λ are shape parameters. Corresponding survival function of the model:

$$\bar{Q}(r;\gamma,\alpha,\lambda) = 1 - F(r;\gamma,\alpha,\lambda) = e^{-\alpha r^{\gamma(1+\lambda^{\gamma})}} \tag{2.9}$$

Hazard rates:

$$h(r) = \frac{q(r)}{\bar{Q}(r)} = (1+\lambda^{\gamma})\alpha\gamma r^{\gamma-1} \tag{2.10}$$

2.5 Some Important Weighted Weibull Probability Distributions

This section review some important parametric and non-parametric families of weighted Weibull distributions and their characteristics, including shapes of their hazard functions, probability density function, and cumulative distribution functions of the various study.

2.5.1 Exponentiated Weibull Distribution

Mudholkar and Srivastava (1993) have used their exponential Weibull (EW) distribution with applications to flood and bus-motor failure data which were two-parameter traceable modification of the distribution. They introduced parameter as a shape to the distribution.

With its PDF and CDF of the EW distribution respectively:

$$q(r:\alpha,\theta,\lambda) = \lambda\alpha\theta r^{\theta-1} e^{-\alpha r^{\theta}}\left(1-e^{-\alpha r^{\theta}}\right)^{\lambda-1},\ r>0 \tag{2.11}$$

and

$$Q(r:\alpha,\ \theta,\lambda) = \left(1-e^{-\alpha r^{\theta}}\right)^{\lambda},\ r>0 \tag{2.12}$$

where $\alpha,\theta,\gamma > 0$, θ and λ represents shape parameters and scale parameter. Setting =1, it transforms the distribution to Weibull distribution. The EW distribution transforms to generalised Rayleigh distribution (GR) when $\theta = 2$ Surles and Padgett (2002). Setting $\theta = 1,2$ and $\lambda = 1$ particular cases of the exponential and Rayleigh distributions are derived.

2.5.2 Modified Weibull Extension

Xie et al. (2002) developed modified Weibull extension having a J-shape force of mortality rate. They derived PDF and CDF respectively as follows:

$$g(r:\alpha,\theta,\lambda)=\lambda\alpha^{\frac{\theta-1}{\theta}}r^{\theta-1}e^{\alpha r^{\theta}}exp\left[\lambda\alpha^{\frac{-1}{\theta}}\left(1-e^{-\alpha r^{\theta}}\right)\right],\ r>0 \tag{2.13}$$

and

$$G(r:\alpha,\theta,\lambda)=1-exp\left[\lambda\alpha^{\frac{-1}{\theta}}\left(1-e^{-\alpha r^{\theta}}\right)\right],\ r>0 \tag{2.14}$$

where $\alpha,\theta,\gamma>0$. In order to transform modified Weibull to Weibull distribution, must be set to a smaller value in a way that $1-e^{-\alpha r^{\theta}}$ is approximately equal to $-\alpha r^{\theta}$ (Xie et al., 2002).

The failure function is:

$$h(r;\alpha,\theta,\lambda)=\lambda\alpha^{\frac{\theta-1}{\theta}}r^{\theta-1}e^{\alpha r^{\theta}} \tag{2.15}$$

θ is the shape parameter. The PDF of the modified Weibull shows decreasing, unimodal or decreasing followed by unimodal.

2.5.3 Odd Weibull Distribution

This distribution was proposed by Cooray (2006). He obtained this distribution by combining inverse Weibull distributions and odds of the Weibull to form three-parameter unknown model.

$$Q(r;\alpha,\theta,\lambda)=1-\left[1+\left(e^{\alpha r^{\theta}}-1\right)^{\lambda}\right]^{-1},\ r>0 \tag{2.16}$$

where $\alpha>0,\ \theta,\gamma>0$, shape parameter are and . PDF and hazard function as follows, respectively:

$$Q(r;\alpha,\theta,\lambda)=\alpha\theta\lambda r^{\theta-1}e^{\alpha r^{\theta}}\left(e^{\alpha r^{\theta}}-1\right)^{\lambda-1}\left[1+\left(e^{\alpha r^{\theta}}-1\right)^{\lambda}\right]^{-2},\ r>0 \tag{2.17}$$

and

$$h(r;\alpha,\theta,\lambda) = \alpha\theta\lambda r^{\theta-1} e^{\alpha r^{\theta}} \left(e^{\alpha r^{\theta}} - 1\right)^{\lambda-1} \left[1 + \left(e^{\alpha r^{\theta}} - 1\right)^{\lambda}\right]^{-1}, \; r > 0 \tag{2.18}$$

The failure function accommodates the non-monotonic shapes:

2.5.4 Additive Weibull distribution

The additive Weibull distribution was proposed by Xie and lai (1996) which exhibits bathtub shaped of the failure function. The derived PDF, CDF and the failure function of the model are, respectively:

$$q(r) = (\alpha\theta r^{\theta-1} + \beta\gamma r^{\gamma-1}) e^{-\alpha r^{\theta} - \beta r^{\gamma}}, \; r > 0, \tag{2.19}$$

$$Q(r) = 1 - e^{-\alpha r^{\theta} - \beta r^{\gamma}}, \; r > 0, \tag{2.20}$$

$$h(r) = \alpha\theta r^{\theta-1} + \beta\gamma r^{\gamma-1} \tag{2.21}$$

where $\gamma < 1$ and α, θ, $\beta > 0$. It follows Weibull distribution when $= 0$ or $= 0$. The failure rate is monotonically increases when and are greater than one. For special case, additive Weibull will transform to Sarhan and Apaloo (2013) when $\gamma > 0$ and $\theta = 1$.

2.5.5 Modified Weibull distribution

Lai et al. (2003) derived three-parameter distribution known as modified Weibull distribution. They studied some of its statistical properties and adopted maximum likelihood estimation method to estimate the unknown parameters.

$$Q(r) = 1 - e^{-\beta r^{\gamma} e^{\lambda r}}, \; r > 0, \tag{2.22}$$

The corresponding PDF and failure rate are, respectively

$$q(r) = \beta(\gamma + \lambda r) r^{\gamma-1} e^{\lambda r} e^{-\beta r^{\gamma} e^{\lambda r}}, \; r > 0. \tag{2.23}$$

and

$$h(r) = \beta(\gamma + \lambda x) r^{\gamma-1} e^{\lambda r}, \; r > 0. \tag{2.24}$$

2.5.6 Kumaraswamy Modified Weibull distribution

Cordeiro et al. (2012) developed Kumaraswamy Modified Weibull distribution which was an improvement on the modification of the Weibull distribution. The CDF, PDF and the hazard function are respectively:

$$Q(r) = 1 - \{1 - [1 - \exp(-\alpha r^{\theta} e^{\lambda r})]^{a}\}^{b}, \ r > 0 \tag{2.25}$$

$$q(r) = ab\alpha r^{\theta-1}(\theta + \lambda r)(\lambda r - \alpha r^{\theta} e^{\lambda r}[1 - \exp(\alpha r^{\theta} e^{\lambda r})]^{a-1} \times K, \tag{2.26}$$

where $K = \{1 - [1 - \exp(-\alpha r^{\theta} e^{\lambda r})]^{a}\}^{b-1}$

and

$$h(r) = \frac{ab\alpha r^{\theta-1}(\theta + \lambda r)(\lambda r - \alpha r^{\theta} e^{\lambda r}[1 - \exp(\alpha r^{\theta} e^{\lambda r})]^{a-1}}{\{1 - [1 - \exp(-\alpha r^{\theta} e^{\lambda r})]^{a}\}}, \ r > 0 \tag{2.27}$$

where $a, \ b, \ \alpha, \ \theta, \ \lambda > 0$. Special cases include the generalised modified Weibull distribution, modified Weibull, Log Weibull and the Kumaraswamy Weibull distribution.

2.5.7 Weighted Weibull distribution

Nasiru (2015) proposed another generalisation of the Weibull distribution with the baseline distribution of modified weighted version of Azzalini (1985). The probability density function and cumulative density of the new class of the weighted Weibull distribution are respectively:

$$q(r) = \left(1 + \lambda^{\gamma}\right) \alpha \gamma r^{\gamma-1} e^{-\alpha r^{\gamma}(1+\lambda^{\gamma})}, \ r, \ \alpha, \ \lambda > 0 \tag{2.28}$$

and

$$Q(r) = 1 - e^{(-\alpha r^{\gamma} + \alpha(\lambda r)^{\gamma}} \tag{2.29}$$

where, α is a scale parameter and γ, λ are shape parameter. The corresponding survival function of the distribution is given by:

$$\bar{Q}(r;\gamma,\alpha,\lambda) = 1 - Q(r;\gamma,\alpha,\lambda) = e^{-\alpha r^{\gamma}(1+\lambda^{\gamma})} \tag{2.30}$$

and the hazard function is:

$$h(r) = \frac{q(r)}{\bar{Q}(r)} = \left(1+\lambda^{\gamma}\right)\alpha\gamma r^{\gamma-1} \tag{2.31}$$

2.5.8 The Exponentiated generalised family of distribution

Gupta et al. (1998) proposed an exponentiated generalised family of distribution technique to enhance flexibility of the existing model for modelling life-time dataset. Suppose a random variable X has an arbitrary baseline distribution G(x), the CDF and PDF of the distributions are repectively

$$Q(r) = [G(r)]^{\beta},\ r > 0. \tag{2.32}$$

and

$$q(r) = \beta[G(r)]^{\beta-1}g(r) \tag{2.33}$$

where $r > 0$ "and" $\beta > 0$ represent shape parameter, G(r) and g(r) are the cdf and pdf respectively of the parent distribution. In Oguntunde et al. (2016), proposed exponentiated generalised Weibull and its statistical properties including the PDF, CDF, survival, failure rate, quantile and estimation of parameters were derived. The Exponentiated Generalised Frechet distribution was explored by (Abd-Elfattah et al., 2016).

2.5.9 Techniques for Generating new Discrete Distribution

Najarzandegan and Alamatsaz (2017) state techniques for generating new discrete distributions. Firstly, by using existing baseline distribution, its probability mass function (PMF) is well defined and using any probability density function (pdf) or survival function of a parent continuous distribution. Over the last years, new classes of the distributions have received considerable attentions. For instance, the discrete normal distribution was introduced by Lisman et al. (1972). Similarly, generalised inverse Weibull-Poisson Mahmoudi and Torki (2011), alpha-skew-Laplace Harandi and Torki (2015), discrete beta-exponential Nekoukhou et al. (2015) exponential geometric and generalisation of weighted geometric distribution Nekoukhou et al. (2015) were also introduced and studied. Secondly, is to define weighted version of a baseline distribution, as introduced by Patil Rao (1978). For example, Bhati and Joshi (2018) proposed the weighted geometric distribution using this method.

In this study, the researcher implemented the second approach on geometric distribution with the appropriate weighted function defined by Nekoukhou et al. (2012). Supposed sample N say $X_1, X_2, X_3 \dots X_N$ are independent and identically distributed (iid) random variables from WW distribution. Consider the N is distributed according to the geometric distribution with PDF

$$P(N=n)=\theta^{(n-1)}(1-\theta),\ n=1,2,...,\ 0\leq\theta<1. \tag{2.34}$$

Nekoukhou et al. (2012) proposed discrete geometric exponential distribution with the CDF of X for $x>0$ can be obtained as follows:

$$Q(r)=P(X\leq r)=\sum_{n=1}^{\infty}P(X\leq r, N=n)$$

$$Q(r)=\sum_{n=1}^{\infty}P(X\leq r|N=n)P(N=n) \tag{2.35}$$

The marginal CDF of X is given by

$$Q(r)=P(X=r)=\frac{(1-\theta)\,Q(r)}{1-\theta[Q(r)]} \tag{2.36}$$

Let $W=max\{r_1, r_2, r_3 \dots r_N\}$ then the conditional CDF of W—N=n is given by

$$Q_{W|N=n}(w)=\left(1-e^{-\alpha w^{\beta}\left(1+\lambda^{\beta}\right)}\right)^{n}. \tag{2.37}$$

2.6 Information Criteria

The researcher at a point would increase the number of parameters, which would improve the fitting of the data set that will directly result in increase of the likelihood. Maximum likelihood preferred when there are many parameters in the model for better comparison. Then, an information criteria can be used to make a comparison between different other statistical models which may have different numbers of parameters. The measures; Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (AICc), and Consistent Akaike Information Criterion (CAIC) are widely used information criteria for selecting the appropriate model among different others models.

CHAPTER 3

METHODOLOGY

3.1 Introduction

This chapter highlights on various methods, the researcher employed to achieve the stated objectives. The major statistical and mathematical concepts the researcher adopted were; parameters estimation, goodness of fit test, hazard function, quantile, survival function, entropy, and order statistics. Appropriate statistical software including; Mathematica and R software were used in executing the statistical analyses in this study.

3.2 Parameter Estimation

According to Ramachandran and Tsokos (1978) there are several methods to estimate unknown parameters; Maximum Likelihood Estimation, Least Square Estimation, Method of Moment (MOM), and Bayesian Estimation. In this study, the maximum likelihood estimation approach was utilized to estimate unknown parameters.

3.3 The Method of Maximum Likelihood Estimation (MLE)

Maximum likelihood is mostly used to estimate parameters that are unknown in statistical modeling. This method of parameter estimates was pioneered by Fisher (1934). MLE is the most widely used classical approach for the estimation of parameters of a probability distribution. It is based on a likelihood function. The likelihood function attains its maximum at a specific value of the parameters. It enjoys many desirable properties including; invariance property, asymptotic normality, consistency, and asymptotic efficiency (Nassar et al. , 2018). MLE focuses on the joint PDF for all observed counts.

Let $r_1, r_2, \ldots, r_n$ be independent identically distributed random variables of size n with pdf or pmf $q(r;\phi), \phi$ unknown parameter (Ramachandran and Tsokos , 1978). The following are steps in the estimation of parameters:

The likelihood function is derived.

$$q(r_1 \ldots, r_n|\Phi) = \prod_{i=1}^{n} q(r_i; \Phi) \tag{3.1}$$

Taking logarithm of the Likelihood function

$$L(\phi|r_1 \ldots, r_n) = \prod_{i=1}^{n} q(r_i; \Phi)$$

$$\mathcal{L}(\phi|r_1 \ldots, r_n) = \sum_{i=1}^{n} \log q(r_i; \Phi) \tag{3.2}$$

Differentiating eqn (3.2) and equating to zero.

$$\frac{\partial \mathcal{L}(\phi|r_1 \ldots, r_n)}{\partial \phi} = 0, \; i = 1, 2 \ldots k \tag{3.3}$$

Solving system of non-linear likelihood equations simultaneously for ϕ_i.

3.4 Goodness-of-Fit

This is the statistical technique employed by the researcher to examine whether the random sample came from a particular distribution. In this study, the researcher adopted; the log-likelihood ratio test, Kolmogorov-Smirnov test, Information criteria, and Total Time on Test.

3.4.1 The Log-likelihood Ratio Test (LRT)

The Log-likelihood Ratio Test can be used to compare two models that are nested.

Let $r_r, r_2, \ldots, r_n$ be independent identically distributed random variables of size n with pdf $q(r; \phi)$ and Φ_0, Φ_1 are nested.

Formulation of the hypothesis required:

$$H_0 : \Phi \in \Phi_0$$

$$H_0 : \Phi \in \Phi_1$$

The LRT depends on the likelihood ratio as:

$$w = 2\ln\left(\frac{L_0(\Phi)}{L_1(\Phi)}\right) \tag{3.4}$$

The decision is drawn, based on the distribution with the largest likelihood value as the best model fit.

3.4.2 Kolmogorov-Smirnov Test (K-S)

This test computes the distance between the empirical distribution function $Q_n(r)$ of the given data and the estimated cumulative distribution function $Q^*(r)$ of the candidate distribution.

The calculated K-S statistics are then computed with the tabulated K-S significance levels, α. If there are more than one distribution to be compared, the distribution with the smallest K-S is more appropriate.

The value of K-S test statistics is defined by

$$K - S\,\text{test} = Max_{1\leq i\leq n}\left|Q_n(r_i) - \frac{i-1}{n}, \frac{1}{n} - Q^*(r)\right| \tag{3.5}$$

3.4.3 Total Time on Test

The total time on test can be represented as TTT- transform, and is a visual technique to test whether the sample data set follows the distribution of the bathtub shape of the hazard rate function. TTT-transform provides a visualization of the shape of the hazard rate. Aarset (1987) used the TTT-transform to test if a random sample belongs to a distribution with a bathtub shaped hazard rate.

The TTT-transform of distribution with CDF C is defined as

$$H^{-1}(p) = \int_0^{C^{-1}(p)} S(u)\,du, p \in 0,1 \tag{3.6}$$

The scaled TTT-transform of the distribution is

$$\varphi C(p) = \frac{H^{-1}(p)}{H^{-1}(1)} \tag{3.7}$$

The curve $\varphi C(P)$ versus $0 \leq P \leq 1$ is called the scaled TTT-transform

3.4.4 Conceptualization of the Weighted Weibull (Nasiru, 2015)

$$f(x,\lambda,\alpha,\theta) = Kg_0(x)G_0(\lambda x), x > 0, \tag{3.8}$$

$$f(x) = (1+\lambda^{\theta})\alpha\theta x^{\theta-1}e^{-(\alpha x^{\theta}+\alpha(\lambda x))^{\theta}}, x > 0 \tag{3.9}$$

$$F(x) = 1 - e^{(\alpha x^{\theta}+\alpha(\lambda x))^{\theta}}, x, \alpha, \lambda \tag{3.10}$$

Corresponding survival function and hazard function, respectively.

$$\bar{F}(x;\gamma,\alpha,\lambda) = 1 - (x;\gamma,\alpha,\lambda) = e^{-\alpha x^{\gamma(1+\lambda^{\gamma})}} \tag{3.11}$$

$$h(x) = \frac{f(x)}{\bar{F}} = (1+\lambda^{\gamma})\alpha\gamma x^{\gamma-1} \tag{3.12}$$

3.4.5 Exponentiated Generalised Family of Distribution

Gupta et al. (1998) proposed an exponentiated generalised family of distribution technique to enhance flexibility of the existing model for modelling life-time dataset. Suppose a random variable X has an arbitrary baseline distribution G(x), the CDF and PDF of the distributions are repectively

$$Q(r) = [G(r)]^{\beta},\ r > 0. \tag{3.13}$$

and

$$q(r) = \beta[G(r)]^{\beta-1} g(r) \tag{3.14}$$

where $r > 0$ "and" $\beta > 0$ represent shape parameter, G(r) and g(r) are the cdf and pdf respectively of the parent distribution. In Oguntunde et al. (2016), proposed exponentiated generalised Weibull and its statistical properties including the PDF, CDF, survival, failure rate, quantile and estimation of parameters were derived. The Exponentiated Generalised Frechet distribution was explored by (Abd-Elfattah et al., 2016).

3.4.6 Nekoukhou Generator, (2012)

In this study, the researcher implemented the second approach on geometric distribution with the appropriate weighted function defined by Nekoukhou et al. (2012). Supposed sample N say $X_1, X_2, X_3 \ldots X_N$ are independent and identically distributed (iid) random variables from WW distribution. Consider the N is distributed according to the geometric distribution with PDF

$$P(N = n) = \theta^{(n-1)}(1-\theta),\ n = 1, 2, ...,\ 0 \leq \theta < 1. \tag{3.15}$$

Nekoukhou et al. (2012) proposed discrete geometric exponential distribution with the CDF of X for $x > 0$ can be obtained as follows:

$$Q(r) = P(X \leq r) = \sum_{n=1}^{\infty} P(X \leq r, N = n)$$

$$Q(r) = \sum_{n=1}^{\infty} P(X \leq r | N = n) P(N = n) \tag{3.16}$$

The marginal CDF of X is given by

$$Q(r) = P(X = r) = \frac{(1-\theta)\, Q(r)}{1 - \theta[Q(r)]} \tag{3.17}$$

3.4.7 Information Criteria

The outcome of adding parameters to the probability distribution of the model number of parameters which would improve the fitting of the data set that will directly increase of the likelihood. Maximum likelihood is preferred when there are many parameters in the model for better comparison. The Akaike Information Criterion (AIC), (Akaike , 1974) was used instead of some other discrimination criteria because it is asymptotically efficient. The Akaike Information Criterion (AIC), computed as:

$$AIC = -2L(\Phi; y_i) + 2k \tag{3.18}$$

Φ represents MLE of ϕ and k is the number of parameters. The model with the minimum AIC would be the best model to fit the data set.

The Bayesian Information Criteria is appropriate in a situation when the sample size is not large or the model has too many parameters. This method was due to Schwarz (1978) and is defined by

$$BIC = k\ln(n) - 2L(\Phi, y_i) \tag{3.19}$$

3.5 Build up of the structure of the model

The two main models of the thesis were arrived at by compounding the Weighted Weibull distribution with Gupta et al. (1998) to develop the EWW distribution. Also, the Weighted Weibull distribution was equally compounded with the Nekoukhou et al. (2012) to develop the GWW, these procedures are depicted in the Figure 3.1.

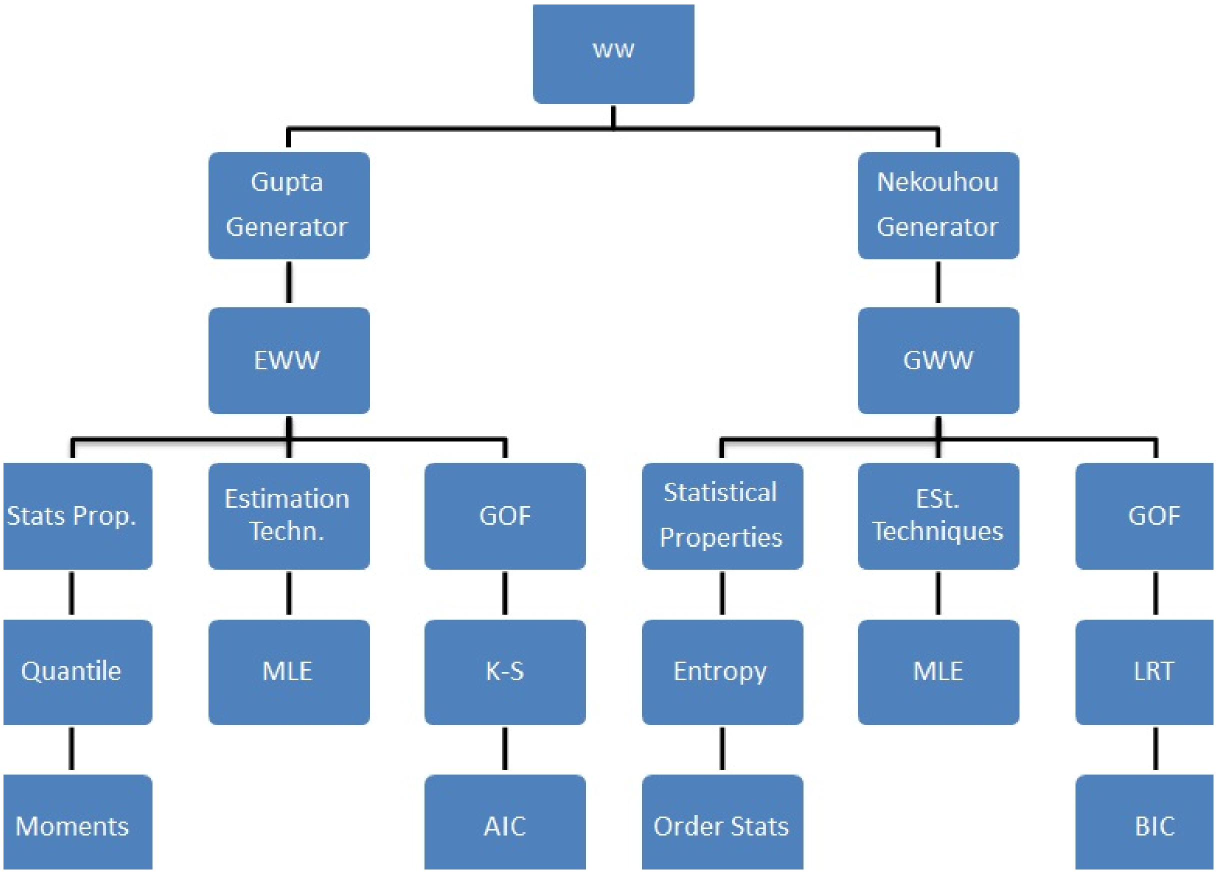

Figure 3.1: The structure of the models

3.6 Sources of data

This study is a quantitative research and the datasets were obtained from secondary sources, extracted from textbooks and Joournal for further analysis. The applications of the models were demonstrated using four real datasets to prove the flexibility of the GWW and the EWW distributions.

3.6.1 Application 1: Real data for strength of 1.5cm Glass Fibers

The data comprises 63 observations on the strength of 1.5cm glass fibers measured at the National Physical Laboratory, England.

Table 3.1: Real data for strength of 1.5cm Glass fiber

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.64	1.68	1.73
1.81	2.0	0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61
1.66	1.68	1.76	1.82	2.01	0.77	1.11	1.28	1.42	1.50
1.54	1.60	1.62	1.66	1.69	1.76	1.84	2.24	0.81	1.13
1.29	1.48	1.5	1.55	1.61	1.62	1.66	1.70	1.77	1.84
0.84	1.24	1.30	1.48	1.51	1.55	1.61	1.63	1.67	1.70
1.78	1.89	1.61							

3.6.2 Application 2: Failure and running times dataset

The dataset represents failure and running times for sample devices from an Eld Tracking Study. The dataset has been previously studied by Merovci (2015). The dataset has thirty observations and can be found in Table 3.2.

Table 3.2: Failure and running times of devices

2.75	0.13	1.47	0.23	1.81	0.30
0.65	0.10	3.00	1.73	1.06	3.00
3.00	2.12	3.00	3.00	3.00	0.02
2.61	2.93	0.88	2.47	0.28	1.43
3.00	0.23	3.00	0.80	2.45	2.66

3.6.3 Application 3: Electronic Components Failure Rate Data

Lifetime data of 20 electronic components studied by Nasiru (2015). The data is shown in Table 3.3

Table 3.3: Lifetimes of 20 electronics components

0.03	0.22	0.73	1.25	1.52	1.8	2.38	2.87	3.14	4.72
0.12	0.35	0.79	1.41	1.79	1.94	2.4	2.99	3.17	5.09

3.6.4 Application 4: Weight of the Diamond Stones in Carat

Data obtained from Singfat (1996) which represents the weight of the diamond stones in carat. To show the applicability and the assessment of the merit of the proposed model, we use the data set indicated in Table 3.4.

Table 3.4: Weight of the diamond stones in carat

1.4575	0.3092	0.3642	0.0119	0.0664	2.6125	0.6027	0.1693	0.5894	0.1558
0.7701	0.0626	0.5350	0.1352	0.4024	0.2872	1.2177	2.6257	0.3954	0.4107

CHAPTER 4

GEOMETRIC WEIGHTED WEIBULL DISTRIBUTION

4.1 Introduction

In this chapter, Geometric weighted Weibull (GWW) distribution is introduced. The GWW distribution was obtained by compounding the geometric and weighted Weibull distribution which satisfied objective one. The probability density function, cumulative distribution function, survival function, and failure rate of the GWW were defined. The model has four unknown parameters, and the hazard function can take different shapes. Thus, decreasing, increasing, bathtub-shape, and unimodal. Some structural statistical properties of the model are discussed such as quantile, moments, moment generating function, incomplete moment, Renyi entropy, mean residual life, and order statistics. The method of MLE was adopted to estimate unknown parameters of the GWW four-parameter distribution. Monte Carlo simulation method is discussed using the quantile function.

4.2 Developing the Geometric Weighted Weibull Distribution

Supposed sample N say $X_1, X_2, X_3 \ldots X_N$ are independent and identically distributed (iid) random variables from WW distribution. Consider the N is distributed according to the geometric distribution with PDF: Nekoukhou et al. (2012) proposed discrete geometric exponential distribution with the CDF of X for $x > 0$ can be obtained as:

$$Q(r) = \sum_{n=1}^{\infty} P(X \leq r|N = n)P(N = n). \tag{4.1}$$

The marginal CDF of X is given by

$$Q(r) = P(X = r) = \frac{(1-\theta)\,Q(r)}{1-\theta[Q(r)]}. \tag{4.2}$$

Let $W = max\{r_1, r_2, r_3 \dots r_N\}$ then the conditional CDF of W—N=n is given by

$$Q_{W|N=n}(w) = \left(1 - e^{-\alpha w^{\beta}\left(1+\lambda^{\beta}\right)}\right)^{n}. \tag{4.3}$$

That is the weighted Weibull cumulative distribution with parameters n, θ, λ , and α

The Geometric Weighted Weibull distribution (GWW) is defined by the marginal CDF of w,

$$Q(w) = \frac{(1-\theta)\left(1 - e^{-\alpha w^{\beta}(1+\lambda^{\beta})}\right)}{1 - \theta\left(1 - e^{-\alpha w^{\beta}(1+\lambda^{\beta})}\right)}, \ \lambda, \ \alpha, \ \beta > 0, \ w > 0. \tag{4.4}$$

The associated PDF is given by:

$$q(w) = \frac{(1-\theta)\alpha\beta(1+\lambda^{\beta})w^{\beta-1}e^{-\alpha w^{\beta}(1+\lambda^{\beta})}}{\left[1 - \theta\left(1 - e^{-\alpha w^{\beta}(1+\lambda^{\beta})}\right)\right]^{2}}, \ w > 0. \tag{4.5}$$

Figure 4.1 shows that the PDF of GWW distribution is quite flexible for modeling survival data. This implies that the distribution exhibits symmetrical, right-skewed, left-skewed, unimodal shapes with small and large values of skewness and kurtosis measure.

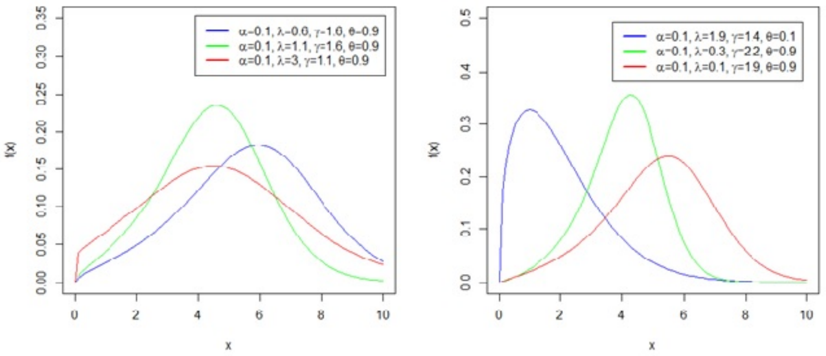

Figure 4.1: Plots of the density curves for different parameters values

The corresponding survival and hazard rate functions are respectively given:

$$S(w) = \frac{e^{-\alpha w^{\beta}(1+\lambda^{\beta})}}{1 - \theta\left(1 - e^{-\alpha w^{\beta}(1+\lambda^{\beta})}\right)}, \ w > 0, \tag{4.6}$$

and

$$h(w) = \frac{(1-\theta)\alpha\beta(1+\lambda^{\beta})w^{\beta-1}}{\left[1-\theta\left(1-e^{-\alpha w^{\beta}(1+\lambda^{\beta})}\right)\right]},\ w > 0. \tag{4.7}$$

The GWW model shows flexibility in accommodating all forms of the hazard rate function as shown in Figure 4.2. Such as monotonically decreasing, monotonically increasing, unimodal and inverted bathtub shapes for different combination of the values of the parameters.

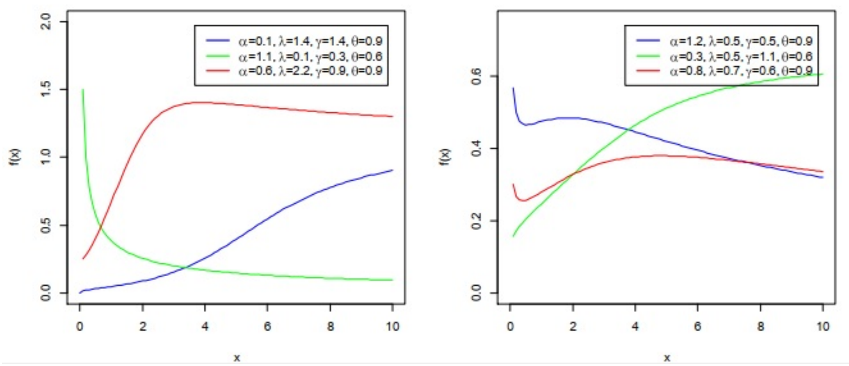

Figure 4.2: Plots of the failure rate for some parameter values

4.3 Mixture Representation

Proposition 4.3.1. *The density of GWW distribution can be written as*

$$q_w(w) = (1-\theta)(1+\lambda^{\beta})\alpha\beta w^{\beta-1}\sum_{i=0}^{\infty}(i+1)\theta^{i}e^{-\alpha(i+1)w^{\beta}(1+\lambda^{\beta})}. \tag{4.8}$$

using expansion of power series

Proof. By Equation (4.6)

$$(1-x)^{-n} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}x^{i},\ \ |x| < 1 \text{and } n > 0,$$

expanding the denominator of Equation (4.6) yields

$$\left[1-\theta(1-e^{-\alpha w^{\beta}(1+\lambda^{\beta})})\right]^{-2}=\sum_{i=0}^{\infty}\begin{pmatrix}2+i-1\\ i\end{pmatrix}\theta^{i}e^{-\alpha w^{\beta}(1+\lambda^{\beta})i},$$

$$=\sum_{i=0}^{\infty}\theta^{i}\begin{pmatrix}i+1\\ i\end{pmatrix}e^{-\alpha i w^{\beta}(1+\lambda^{\beta})}.$$

By applying the relation $\begin{pmatrix}n\\ r\end{pmatrix}=\frac{n!}{(n-r)!r!}$ we have

$$\left[1-\theta(1-e^{-\alpha w^{\beta}(1+\lambda^{\beta})})\right]^{-2}=\sum_{i=0}^{\infty}\frac{(i+1)!\theta^{i}}{(i+1-i)!i!}e^{-\alpha i w^{\beta}(1+\lambda^{\beta})}$$

$$\left[1-\theta(1-e^{-\alpha w^{\beta}(1+\lambda^{\beta})})\right]^{-2}=\sum_{i=0}^{\infty}(i+1)\theta^{i}e^{-\alpha i w^{\beta}(1+\lambda^{\beta})}.$$

The density function in equation (4.6) can be written as

$$q_{w}(w)=(1-\theta)(1+\lambda^{\beta})\alpha\beta w^{\beta-1}e^{-\alpha w^{\beta}(1+\lambda^{\beta})}*\sum_{i=0}^{\infty}(i+1)\theta^{i}e^{-\alpha i w^{\beta}(1+\lambda^{\beta})}.$$

Hence

$$q_{w}(w)=(1-\theta)(1+\lambda^{\beta})\alpha\beta w^{\beta-1}\sum_{i=0}^{\infty}(i+1)\theta^{i}e^{-\alpha(i+1)w^{\beta}(1+\lambda^{\beta})}.$$

□

4.4 Statistical Properties

Some important statistical characteristics of the derived model such as quantile, moments, incomplete moment, moment generating function, Rényi entropy, mean residual life, and order statistics were discussed.

4.4.1 Quantile

Quantile is the inverse cumulative distribution function used to generate random samples of the distribution. It can also serve as an alternative to the probability density function.

Proposition 4.4.1. *The quantile function z of the Geometric weighted Weibull distribution can be expressed as*

$$Q_w(z) = \left[-\frac{1}{\alpha(1+\lambda^{\beta})}\log\left(\frac{1+\theta(z-1)-z}{1+\theta(z-1)}\right)\right]^{\frac{1}{\beta}}, z \in [0,1]. \tag{4.9}$$

Proof. Using the CDF of the Geometric weighted Weibull distribution as defined in equation (4.5), it represented as $Q_w(z) = Q^{-1}(z)$ □

The quantile function is obtained by solving the equation

$$\frac{(1-\theta)\left(1-e^{-\alpha w^{\beta}(1+\lambda^{\beta})}\right)}{1-\theta\left(1-e^{-\alpha w^{\beta}(1+\lambda^{\beta})}\right)} = z.$$

By cross multiplying, we have

$$e^{-\alpha w^{\beta}(1+\lambda^{\beta})} = \frac{1+\theta z-\theta-z}{1+\theta(z-1)}.$$

Taking the logarithm of both sides gives

$$-\alpha w^{\beta}(1+\lambda^{\beta}) = \log\left(\frac{1-z+\theta(z-1)}{1+\theta(z-1)}\right).$$

Further,

$$w^{\beta} = -\frac{1}{\alpha(1+\lambda^{\beta})}\log\left(\frac{1-z+\theta(z-1)}{1+\theta(z-1)}\right).$$

and the quantile is given as

$$Q_w(z) = \left[-\frac{1}{\alpha(1+\lambda^{\beta})}\log\left(\frac{1+\theta(z-1)-z}{1+\theta(z-1)}\right)\right]^{\frac{1}{\beta}}, z \in [0,1].$$

4.4.2 Moments

Moments are used to compute various measures of central tendency, variation, kurtosis, and skewness among others.

Proposition 4.4.2. *The sth moments of W say μ_s, is given by*

$$\mu'_s = (1-\theta)\sum_{i=0}^{\infty}\frac{(i+1)\theta^i\Gamma\left(\frac{s}{\beta}+1\right)}{\left[\alpha(1+\lambda^{\beta})\right]^{\frac{s}{\beta}}(i+1)^{\frac{s}{\beta}+1}}.\ s = 1,2,\ldots \tag{4.10}$$

Proof. By definition, the sth moments of GWW model

$$\mu'_s = \int_0^{\infty} w^s q(w)dw.$$

Using the mixture representation GWW density in equation (4.9)

$$\mu'_s = \int_0^{\infty} w^s(1-\theta)(1+\lambda^{\beta})\alpha\beta w^{\beta-1}\sum_{i=0}^{\infty}(i+1)\theta^i e^{-\alpha(i+1)w^{\beta}(1+\lambda^{\beta})}dw.$$

Furthermore,

$$\mu'_s = (1-\theta)(1+\lambda^{\beta})\alpha\beta\sum_{i=0}^{\infty}(i+1)\theta^i\int_0^{\infty} w^s w^{\beta-1}e^{-\alpha(i+1)w^{\beta}(1+\lambda^{\beta})}dw. \tag{4.11}$$

Let z $=\alpha(1+\lambda^{\gamma})(1+i)w^{\beta}$, as w $\to 0, z \to 0$, as w $\to \infty, z \to \infty$.

Also,

$$w = \left[\frac{z}{\alpha(1+\lambda^{\gamma})(1+i)}\right]^{\frac{1}{\beta}}$$

$$\frac{dz}{dw} = \alpha(1+\lambda^{\gamma})\beta(1+i)w^{\beta-1}$$

$$\frac{dz}{(i+1)} = \alpha(1+\lambda^{\beta})\beta(1+i)w^{\beta-1}dw.$$

Hence, equation (4.13) becomes

$$\mu'_s = (1-\theta)\sum_{i=0}^{\infty}(i+1)\theta^i\int_0^{\infty}\left[\left(\frac{z}{\alpha(i+1)(1+\lambda^{\beta})}\right)^{\frac{1}{\beta}}\right]^r e^{-z}\frac{dz}{(i+1)}.$$

Further,

$$\mu'_s = (1-\theta)\sum_{i=0}^{\infty}(i+1)\theta^i\int_0^{\infty}\frac{z^{\frac{s}{\beta}+1-1}}{\left[\alpha(i+1)(1+\lambda^{\beta})\right]^{\frac{s}{\beta}}}e^{-z}\frac{dz}{i+1}.$$

Applying some mathematical algebra, we obtain

$$\mu'_s = (1-\theta)\sum_{i=0}^{\infty}\frac{(i+1)\theta^i\Gamma\left(\frac{s}{\beta}+1\right)}{\left[\alpha(1+\lambda^{\beta})\right]^{\frac{s}{\beta}}(i+1)^{\frac{s}{\beta}+1}},\ s=1,2,\dots\ .$$

□

4.4.3 Moment Generating Function (MGF)

This involves putting together all the moments for a random variable in a single expression. With the MGF it is easier to obtain the moments of higher powers of the probability distribution.

Proposition 4.4.3. *The MGF of a random variable follows GWW can be expressed as*

$$\mathrm{M}_w(t) = (1-\theta)\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\frac{(i+1)\theta^i\Gamma\left(\frac{s}{\beta}+1\right)}{\left[\alpha(1+\lambda^{\beta})\right]^{\frac{s}{\beta}}(i+1)^{\frac{s}{\beta}+1}}.\ s=1,2,... \tag{4.12}$$

Proof. By definition, the MGF is defined as

$$\mathrm{M}_w(t) = \mathrm{E}(e^{tW}) = \int_0^{\infty} e^{tw}q(w)dw.$$

Using Taylor series,

$$\mathrm{M}_w(t) = \mathrm{E}\left[\sum_{s=0}^{\infty}\frac{t^s w^s}{s!}\right],$$

$$\mathrm{M}_w(t) = \sum_{s=0}^{\infty}\frac{t^s}{s!}\mathrm{E}\left[w^s\right],$$

$$\mathrm{M}_w(t) = \sum_{s=0}^{\infty}\frac{t^s}{s!}\mu'_s\,.$$

Substituting μ'_s into it and after some algebra, we have moments generating the function of GWW. Since $\mathrm{M}_w(t) = \sum_{s=0}^{\infty}\frac{t^s}{s!}\mathrm{E}\left[w^s\right]$, we have

$$\mathrm{M}_w(t) = (1-\theta)\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\frac{(i+1)\theta^i\Gamma\left(\frac{s}{\beta}+1\right)}{\left[\alpha(1+\lambda^{\beta})\right]^{\frac{s}{\beta}}(i+1)^{\frac{s}{\beta}+1}}.$$

□

4.4.4 Incomplete Moment

The incomplete moments are also known as the conditional moments that describes the shape of many distributions. It plays an important role in measuring inequality, for example; income quantiles, Lorenz, and Bonferroni curves.

Proposition 4.4.4. *The sth incomplete moments of W can be expressed as*

$$\varphi'_s(w) = (1-\theta)\sum_{i=0}^{\infty} \frac{(i+1)\theta^i \gamma(\frac{s}{\beta}+1, \alpha(i+1)(1+\lambda^{\beta})w^{\beta})}{\left[\alpha(i+1)(1+\lambda^{\beta})\right]^{\frac{s}{\beta}}(i+1)^{\frac{s}{\beta}+1}}, \; s = 1,2,\dots . \tag{4.13}$$

Proof. By definition

$$\varphi'_s(w) = \int_0^w t^s q(t)dt$$

Substitute in Equation (4.9),

$$\varphi'_s(w) = \int_0^y t^s(1-\theta)(1+\lambda^{\beta})\alpha\beta t^{\beta-1}\sum_{i=0}^{\infty}(i+1)\theta^i e^{-\alpha(i+1)t^{\beta}(1+\lambda^{\beta})}dt,$$

$$\varphi'_s(w) = (1-\theta)(1+\lambda^{\beta})\alpha\beta\sum_{i=0}^{\infty}(i+1)\theta^i\int_0^w t^s t^{\beta-1}e^{-\alpha(i+1)(1+\lambda^{\beta})t^{\beta}}dt.$$

Similarly, as w $\to 0, z \to 0$, as t $= w, z = \alpha(i+1)(1+\lambda^{\beta})w^{\beta}$

$$dy = \frac{dz}{\alpha(i+1)(1+\lambda^{\gamma})w^{\beta-1}},$$

$$w = \left[\frac{z}{\alpha(i+1)(1+\lambda^{\beta})}\right]^{\frac{1}{\beta}},$$

The incomplete moments, after some simplification gives

$$\varphi'_s(w) = (1-\theta)\sum_{i=0}^{\infty}(i+1)\theta^i\int_0^{\alpha(i+1)(1+\lambda^{\beta})w^{\beta}} \frac{z^{\frac{s}{\beta}+1-1}}{\left[\alpha(i+1)(1+\lambda^{\beta})\right]^{\frac{s}{\beta}}}e^{-z}\frac{dz}{i+1}.$$

Using the definition of gamma function,

$$\varphi'_s(w) = (1-\theta)\sum_{i=0}^{\infty}\frac{(i+1)\theta^i\gamma(\frac{s}{\beta}+1,\alpha(i+1)(1+\lambda^\beta)w^\beta)}{\left[\alpha(i+1)(1+\lambda^\beta)\right]^{\frac{s}{\beta}}(i+1)^{\frac{s}{\beta}+1}},\ s=1,2,\dots .$$

□

4.4.5 Rényi Entropy

The entropy is used to measure the randomness of the systems. To examine the uncertainty of the random variable Rényi entropy can be employed.

Proposition 4.4.5. *Rényi entropy of GWW is given as*

$$I_R = \frac{1}{1-\delta}\log\left[(1-\theta)^\delta(1+\lambda^\beta)^\delta(\alpha\beta)^\delta\sum_{i=0}^{\infty}\binom{2\delta+i-1}{i}\theta^i\frac{\Gamma\left(\frac{\delta(\beta-1)-\beta+1}{\beta}\right)}{\left[\alpha(i+1)(1+\lambda^\beta)\right]^{\left(\frac{\delta(\beta-1)+1}{\beta}\right)}}\right]. \tag{4.14}$$

Proof. By definition

$$I_R = \frac{1}{1-\delta}\log\left[\int_0^\infty q_{q^\delta}(w)dw\right],\ \delta>0,\delta\neq 1.$$

Based on the GWW density function in equation (4.9)

$$[q(r)]^\delta = \frac{(1-\theta)^\delta(1+\lambda^\gamma)^\delta(\alpha\beta)^\delta w^{\delta(\beta-1)}e^{-\alpha\delta w^\beta(1+\lambda^\beta)}}{\left[1-\theta(1-e^{-\alpha w^\beta(1+\lambda^\beta)})\right]^{2\delta}},$$

Using the mixture representation concept, we have

$$\left[1-\theta(1-e^{-\alpha w^\beta(1+\lambda^\beta)})\right]^{-2\delta} = \sum_{i=0}^{\infty}\binom{2\delta+i-1}{i}\theta^i e^{-\alpha i(1+\lambda^\beta)w^\beta}.$$

Thus,

$$[q(w)]^\delta = (1-\theta)^\delta(1+\lambda^\beta)^\delta(\alpha\beta)^\delta\sum_{i=0}^{\infty}\binom{2\delta+i-1}{i}\theta^i w^{\delta(\beta-1)}e^{-\alpha(i+1)(1+\lambda^\beta)w^\beta}.$$

This simplified to

$$I_R = \frac{1}{1-\delta}\log\left[A*\sum_{i=0}^{\infty}\binom{2\delta+i-1}{i}\theta^i\int_0^\infty w^{\delta(\beta-1)}e^{-\alpha(i+1)(1+\lambda^\beta)w^\beta}dw\right],$$

Where $A = (1-\theta)^{\delta}(1+\lambda^{\beta})^{\delta}(\alpha\beta)^{\delta}$.

Let z $= \alpha(1+\lambda^{\gamma})(1+i)w^{\beta}$, as w $\to 0, z \to 0$, as w $\to \infty$, $z \to \infty$.

$$w = \left[\frac{z}{\alpha\beta(1+\lambda^{\beta})(1+i)}\right]^{\frac{1}{\beta}}$$

$$\frac{dz}{dw} = \alpha\beta(1+\lambda^{\beta})(1+i)w^{\beta-1}dw$$

$$dw = \frac{dz}{\alpha\beta(1+\lambda^{\beta})(i+1)w^{\beta-1}}.$$

Substituting w and the differential of z with respect to w, we obtain

$$I_R = \frac{1}{1-\delta}\log\left[A\sum_{i=0}^{\infty}\begin{pmatrix}2\delta+i-1\\ i\end{pmatrix}\theta^{i}\int_0^{\infty}\left[\left(\frac{z}{\alpha(i+1)(1+\lambda^{\beta})}\right)^{\frac{1}{\beta}}\right]^{\delta(\beta-1)-\beta+1} e^{-z}\frac{dz}{\alpha(i+1)(1+\lambda^{\beta})}\right],$$

After some simplification, we have

$$I_R = \frac{1}{1-\delta}\log\left[A\sum_{i=0}^{\infty}\begin{pmatrix}2\delta+i-1\\ i\end{pmatrix}\theta^{i}\int_0^{\infty}\left(\frac{z^{\delta(\beta-1)-\beta+1}}{\left[\alpha(i+1)(1+\lambda^{\beta})\right]^{\frac{\delta(\beta-1)-\beta+1}{\beta}+1}}\right)e^{-z}dz\right].$$

We obtain

$$I_R = \frac{1}{1-\delta}\log\left[A\sum_{i=0}^{\infty}\begin{pmatrix}2\delta+i-1\\ i\end{pmatrix}\theta^{i}\frac{\Gamma\frac{\delta(\beta-1)-\beta+1}{\beta}}{\left[\alpha(i+1)(1+\lambda^{\beta})\right]^{\frac{\delta(\beta-1)+1}{\beta}}}\right], \delta > 0, \delta \neq 1.$$

□

4.4.6 Mean Residual Life (MRL)

In reliability studies, the life expectancy is an important characteristic of the model (Gupta et al. , 1998). MRL uniquely determines the distribution function and describes the aging process. MRL is appropriately used for non-parametric and parametric modeling. Actuaries applied

MRL for the determination of premiums and insurance claims for life insurance products.

Proposition 4.4.6. *The MRL of the GWW distribution is given by*

$$m_W(w) = \frac{\left[\mu'_1 - (1-\theta)\sum_{i=0}^{\infty}\frac{(i+1)\theta^i\gamma(\frac{1}{\beta}+1,\alpha(i+1)(1+\lambda^\beta)w^\beta)}{[\alpha(i+1)(1+\lambda^\beta)]^{\frac{1}{\beta}}(i+1)^{\frac{1}{\beta}+1}}\right]}{S(t)} - t. \tag{4.15}$$

Proof. By definition

$$m_W(t) = E[W-t|W>t] = \frac{1}{S(t)}\int_t^\infty (w-t)^r q(w)dw,\ t>0,$$

where

$$S(t) = 1 - Q(t),$$

is the survival function.

$$m_W(t) = \frac{\mu'_1 - \varphi_1(w)}{s(t)} - t$$

Substituting the first incomplete moments yield

$$m_W(w) = \frac{\left[\mu'_1 - (1-\theta)\sum_{i=0}^{\infty}\frac{i\theta^i\gamma(\frac{1}{\beta}+1,\alpha(i+1)(1+\lambda^\beta)w^\beta)}{[\alpha(i+1)(1+\lambda^\beta)]^{\frac{1}{\beta}}(i+1)^{\frac{1}{\beta}+1}}\right]}{S(t)} - t.$$

□

4.4.7 Order Statistics

Order statistics assume a significant function in quality control testing and reliability to predict the failure of future items based on the times of a few early failures. This technique has in recent times extensively used in statistical inferences partly because some of their properties do not depend on the distribution from which the random sample is obtained.

Proposition 4.4.7. *The density of the Order Statistics of a random sample from a GWW can be* *expressed as*

$$q_{k:n}(w) = \frac{n!}{(k-1)!(n-k)!}\sum_{i=0}^{n-k}(1)^{i}\begin{pmatrix} n-k \\ i \end{pmatrix}[Q(w)]^{k+i-1}q(w),\ k = 1,2,3,\ldots,n. \tag{4.16}$$

Proof. By definition

$$g_{k:n}(w) = \frac{n!}{(k-1)!(n-k)!}q(w)[Q(w)]^{k-1}[1-Q(w)]^{n-k}, for\ k = 1,2,3,\ldots,n$$

Using the binomial expansion

$$g_{k:n}(w) = \frac{n!}{(k-1)!(n-k)!}\sum_{i=0}^{n-k}(1)^{i}\begin{pmatrix} n-k \\ i \end{pmatrix}[Q(w)]^{k+i-1}q(x), k = 1,2,3,\ldots,n.$$

□

4.5 Parameter Estimation

In this section, MLE method was employed to estimate unknown parameters of the GWW four-parameter distribution.

4.5.1 Maximum Likelihood Estimation

The maximum likelihood estimation method is the most widely used classical approach for estimating the parameters of a probability distribution model and is based on a likelihood function. The likelihood function attains its maximum at a specific value of the parameters. It enjoys many desirable properties including asymptotic normality, asymptotic efficiency, invariance property, and consistency (Nasiru et al., 2018).

Ley $y_1 ..., y_n$ be a random sample of size from the GWW $(\theta, \alpha, \beta, \lambda)$ distribution. The log-likelihood function for the vector of parameters $\xi = (\theta, \alpha, \beta, \lambda)^T$ becomes

$$L = n\log(1-\theta) + n\log(\alpha) + n\log(\beta) + n\log\left(1+\lambda^{\beta}\right) + (\beta-1)\sum_{i=1}^{n}\log y_i \times B, \tag{4.17}$$

where

$$B = -\sum_{i=1}^{n}\left(\alpha y_i^{\beta}\left(1+\lambda^{\beta}\right)\right) - 2\sum_{i=0}^{n}\log(1-\theta\left(1-e^{-\alpha\times y_i^{\beta}\times\left(1+\lambda^{\beta}\right)}\right).$$

Components of the score vector $U(\xi)$ are obtained by partially differentiating the log-likelihood function.

The associated components of the score function $U(\xi) = \left[\frac{\partial L}{\partial\theta}, \frac{\partial L}{\partial\alpha}, \frac{\partial L}{\partial\beta}, \frac{\partial L}{\partial\lambda}\right]^T$ are

$$\frac{\partial L}{\partial\theta} = -\frac{n}{1-\theta} - 2\sum_{i=0}^{n}\frac{-1+e^{-\alpha\left(1+\lambda^{\beta}\right)y_i^{\beta}}}{1-\left(1-e^{-\alpha\left(1+\lambda^{\beta}\right)y_i^{\beta}}\right)\theta}. \tag{4.18}$$

$$\frac{\partial L}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(1+\lambda^{\beta}\right)y_i^{\beta} - 2\sum_{i=0}^{n} - \frac{e^{-\alpha\left(1+\lambda^{\beta}\right)y_i^{\beta}}\theta\left(1+\lambda^{\beta}\right)\log[e]y_i^{\beta}}{1-\left(1-e^{-\alpha\left(1+\lambda^{\beta}\right)y_i^{\beta}}\right)\theta}. \tag{4.19}$$

$$\frac{\partial L}{\partial\beta} = \frac{n}{\beta} + \frac{n\lambda^{\beta}\log[\lambda]}{1+\lambda^{\beta}} + \sum_{i=1}^{n} log y_i - V - W, \tag{4.20}$$

where

$$V = 2\sum_{i=0}^{n} + \frac{e^{-\alpha\left(1+\lambda^{\beta}\right)y_i^{\beta}}\theta\log[e]\left(-\alpha\lambda^{\beta}\log[\lambda]y_i^{\beta} - \alpha\left(1+\lambda^{\beta}\right)\log[y_i]y_i^{\beta}\right)}{1-\left(1-e^{-\alpha\left(1+\lambda^{\beta}\right)y_i^{\beta}}\right)\theta}$$

and

$$W = \sum_{i=1}^{n}\left(\alpha\lambda^{\beta}\log[\lambda]y_i^{\beta} + \alpha\left(1+\lambda^{\beta}\right)\log[y_i]y_i^{\beta}\right).$$

and

$$\frac{\partial L}{\partial\lambda} = \frac{n\beta\lambda^{-1+\beta}}{1+\lambda^{\beta}} - \sum_{i=1}^{n}\alpha\beta\lambda^{-1+\beta}y_i^{\beta} - 2\sum_{i=0}^{n} - \frac{e^{-\alpha\left(1+\lambda^{\beta}\right)y_i^{\beta}}\alpha\beta\theta\lambda^{-1+\beta}\log[e]y_i^{\beta}}{1-\left(1-e^{-\alpha\left(1+\lambda^{\beta}\right)y_i^{\beta}}\right)\theta}. \tag{4.21}$$

The MLE can be derived by solving the nonlinear equations numerically for the unknown parameters. A good set of initial values is essential. The numerical result is derived directly with the application of lifetime data set using statistical software R. Details code have been provided in appendix A2.

4.6 Monte Carlo Simulation Geometric Weighted Weibull

In this section, the Monte Carlo simulation experiment was executed to examine the properties of the maximum likelihood estimator for the parameters of the GWW distribution. All the computations were done by R-software (see appendix A1 for R-code). Random samples for the simulation were generated using the quantile function of the GWW distribution. The simulation was performed with four different combinations of the parameter values of α,θ,β and λ.

The properties of the estimators were investigated by calculating average estimates of the parameters and the corresponding RMSE. In particular, we have considered samples sizes =25, 50, 75 and 100. the experiments, on each of the samples were replicated 1000 times. In such case, four parameter were used, these are (α,θ,β and λ) = (0.1, 0.3, 0.2, 0.1), (α,θ,β and λ) = (0.1, 0.3, 0.2, 0.2), (α,θ,β and λ) = (0.1, 0.3, 0.2, 0.4) and (α,θ,β and λ) = (0.1, 0.3, 0.4, 0.5). Table 4.1 and 4.2 respectively displays the average bias (AB) and root mean square error (RMSE) for the maximum likelihood estimators.The AB for some estimators of the parameters decreases as the sample size increases while it fluctuates for others. The RMSE for estimators of all the parameters showed a decreasing values with increasing sample size. We observe that all the estimators satisfy desirable properties of the MLE.

Table 4.1: Monte Carlo Simulation Results, AB and RMSE (in parentheses) for GWW Distribution

Parameter Val				***n***	**ABIAS/RMSE**			
A	**θ**	**β**	**λ**		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
0.1	**0.3**	**0.2**	**0.1**	***25***	0.0261	0.0055	0.0204	-0.0154
					(0.0967)	(0.0447)	(0.0683)	(0.3265)
				50	0.0227	0.0021	0.0247	-0.0206
					(0.0870)	(0.0362)	(0.0808)	(0.3148)
				75	0.0273	-0.0018	0.02559	-0.0042
					(0.0919)	(0.0320)	(0.0792)	(0.3102)
				100	0.0273	-0.0023	0.0246	0.0058
					(0.0890)	(0.0301)	(0.0810)	(0.3025)
Parameter Val				***n***	**ABIAS/RMSE**			
0.1	**0.3**	**0.2**	**0.2**	***25***	0.0242	0.0083	0.0272	-0.0204
					(0.1012)	(0.0462)	(0.1244)	(0.3300)
				50	0.0246	0.0006	0.0255	-0.0069
					(0.0862)	(0.0342)	(0.1081)	(0.3125)
				75	0.0241	-0.0003	0.0338	-0.0026
					(0.0855)	(0.0317)	(0.1141)	(0.3123)
				100	0.0314	-0.0041	0.0403	0.0226
					(0.0916)	(0.0316)	(0.1207)	(0.3114)

Table 4.2: Monte Carlo Simulation Results, AB and RMSE (in parentheses) for GWW Distribution

Parameter Val				***n***	**ABIAS/RMSE**			
A	**θ**	**β**	**Λ**		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
0.1	**0.3**	**0.2**	**0.4**	***25***	0.0226	0.0182	0.0227	-0.0211
					(0.0942)	(0.0941)	(0.1435)	(0.327)
				50	0.0280	0.0015	0.0376	-0.0010
					(0.0936)	(0.0752)	(0.1503)	(0.328)
				75	0.0247	-0.0023	0.0319	-0.0065
					(0.0848)	(0.0644)	(0.1332)	(0.3121)
				100	0.0227	-0.0042	-0.0327	-0.0072
					(0.0792)	(0.0601)	(0.1331)	(0.3020)
0.1	**0.3**	**0.4**	**0.5**	***25***	0.0179	0.0205	0.0451	-0.0333
					(0.0926)	(0.0934)	(0.2412)	(0.3246)
				50	0.0214	0.0050	0.0484	-0.0219
					(0.0850)	(0.0714)	(0.2271)	(0.3144)
				75	0.0213	-0.0006	0.0507	-0.0010
					(0.0789)	(0.0628)	(0.2220)	(0.3093)
				100	0.0235	-0.0031	0.0459	-0.0049
					(0.0807)	(0.059)	(0.2144)	(0.3016)

4.7 APPLICATION

The applications of the GWW distribution was demonstrated in this section using two real datasets to illustrate the potentiality of the GWW model, the fit of the GWW distribution was compared with that of other existing competitive models. The smaller these statistics are, the better the fit. The result is obtained using algorithm provided in *appendix A*3.

4.7.1 Application 1: Real Data for Strength of 1.5cm Glass Fibers

An application to a real data set to prove the flexibility of the GWW distribution was carried out. The geometric weighted Weibull (GWW), weighted Weibull (WW), Exponentiated Weibull (EW), exponentiated generalised weighted Weibull (EGWW), Weibull (W), new Weibull exponential (NWE), and Additive Weibull (AddW) were fit to a real data set taken from Smith and Naylor (1987).

Table 4.3 shows the summary statistics of the strength of 1.5cm glass fibers with a measured average of 1.51. The skewness for this dataset is 0.92, positive skewness assumes the size of the right-handed tail is longer than the tail of left-hand. Since the skewness is between 0.1 and 1, the data are moderately skewed.

Table 4.3: Descriptive Statistics of Strength of 1.5cm Glass Fiber

Minimum	Mean	Maximum	Variance	Skewness	Kurtosis
0.55	1.51	2.24	0.11	0.92	1.10

The data shows an increasing failure rate since the plot is concave above 45 degree line as displayed in Figure 4.3

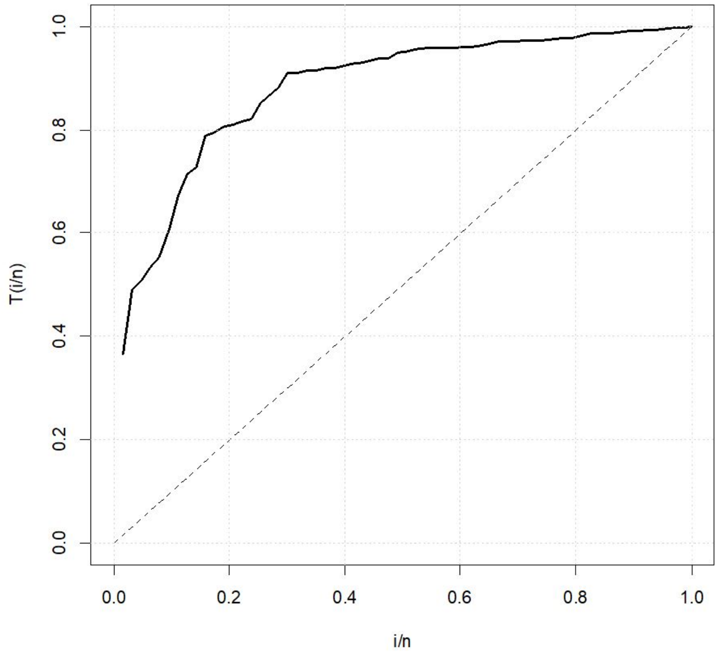

Figure 4.3: TTT-Transform for the strength of 1.5cm Glass Fiber

The estimates for the parameters of the fitted models with their corresponding standard errors, z-value, and p-value are provided in Table 4.4. The significance of the parameters was verified by using the standard error test at a 5% significance level. For the WW model, MLE for the three parameters was also significant and for EW, the parameter was not significant. Only $\hat{b}$ was significant for EGWW and for AddW model only $\hat{\theta}$ was significant.

Table 4.4: Maximum Likelihood Estimates and the Standard Errors for Strength of 1.5cm Glass Fiber

Distribution	Estimate	Std. error	z-value	*p-value*
GWW	$\widehat{\alpha}$ =0.34375	1.1418e-03	-2.1969	0.02802*
	$\widehat{\lambda}$ =1.00583	1.3702e-09	1.4761e+07	< 2e-16*
	$\widehat{\gamma}$ =3.20315	8.5446e-01	8.5353e+00	< 2e-16*
	$\widehat{\theta}$ =0.93980	1.9822e-02	9.8956e+02	< 2e-16*
WW	$\widehat{\alpha}$ =0.05978	0.02052	2.9136	0.003573*
	$\widehat{\lambda}$ =0.06885	7.3061e-08	9.4239e+05	<2.2e-16*
	$\widehat{\theta}$ = 5.7797	5.7619e-01	1.0031e+01	<2.2e-16*
EW	$\widehat{\lambda}$ = 0.6837	0.25757	2.6547	0.00793*
	$\widehat{\gamma}$ = 0.02069	0.026005	0.7959	0.42608
	$\widehat{\beta}$ = 7.19678	1.71433	4.1980	2.693e-05*
EGWW	$\widehat{\alpha}$ =0.043908	3.51283	2.1147e+06	0.57256
	$\widehat{\lambda}$ = 0.779181	16.605228	0.0469	0.962574
	$\widehat{\theta}$ = 7.28690	1.708452	4.2652	1.997e-05
	$\widehat{a}$ = 0.37950	9.23147	6.2546	1.71662
	$\widehat{b}$ =0. 671015	0.248950	2.6954	0.007031*
W	$\widehat{\gamma}$ = 5.78012	0.020505	2.9146	0.00356*
	$\widehat{\beta}$ = 0.05977	0.57601	10.0346	<2.2e-16*
NWE	$\widehat{\alpha}$ = 0.24964	11.00443	0.0227	0.9819
	$\widehat{\lambda}$ = 1.65840	117.18543	0.0142	0.9887
AddW	$\widehat{\alpha}$ = 0.06239	0.03659	1.7053	0.08814
	$\widehat{\gamma}$ = 7.434526	1.222366	6.0821	1.186e-09
	$\widehat{\beta}$ = 0.018312	0.016336	1.1210	0.26230
	$\widehat{\theta}$ = 2.7718	1.3371	2.0730	0.03817*

The values of the test statistics measures and discrimination criteria methods of the fitted models are in Table 4.5. In Table 4.5, we compare the fit of the GWW model with some common lifetime models; WW, EW, W, EGWW, NEW and AddW. The result shows that GWW model provides a good fit to the data set. This result is confirmed from AIC and BIC values since GWW distribution has the minimum values. Considering a significance level of 5%, the GWW distribution was the only model in which p-values returned from the K-S test was greater than 0.05. The log-likelihood value also revealed that the GWW model assumes better fit to the data than the other candidates models. The lowest values of the Cramr-Von Mises (W*) statistics also shown, GWW fits the dataset better than the other fitted models.

Table 4.5: Measures AIC, BIC and Log-likelihood Test for Strength of 1.5cm Glass Fiber Dataset

Model	lnL	AIC	BIC	W*	K-S	p-value
GWW	-12.73	32.07	40.63	0.1057	0.0999	0.5546
WW	-15.21	36.41	42.84	0.2373	0.1523	0.1075
EW	-14.68	35.35	41.78	0.2013	0.1469	0.1315
W	-15.21	34.41	38.69	0.2373	0.1522	0.1077
EGWW	-14.68	39.35	50.07	19.37	1	2.2×10^{-16}
NEW	-88.63	181.66	185.94	0.5702	0.418	5.49×10^{-10}
AddW	-13.75	35.47	44.04	0.2002	1.7016	2.2×10^{-16}

Figure 4.4 shows the plot of the empirical density and the fitted densities of the distributions. From the plot, we observe that the GWW distribution has a superior fit among the chosen model as shown in a visual correlation of the histogram of the information data with the fitted densities.

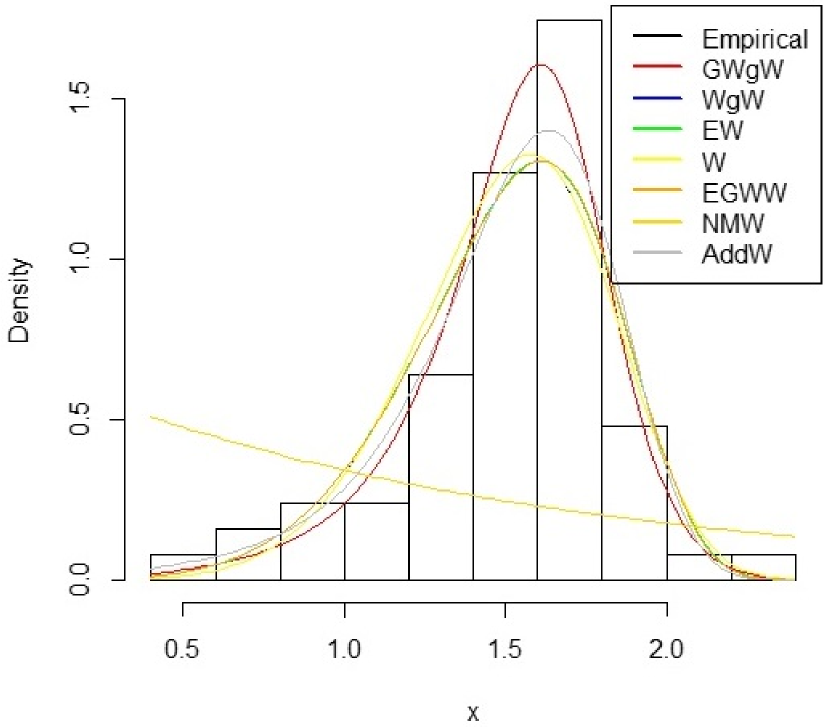

Figure 4.4: Empirical and Fitted Densities Plot for Strength of 1.5cm Glass Fibers

4.7.2 Application 2: Failure and Running Times Dataset

To investigate the merit of the proposed model, we fit the GWW, weighted Weibull (WW), new modified Weibull (NMW), and Additive Weibull (AddW) distributions to the failure and running dataset in Table 3.2. It very well seen from Table 4.6 that the data is negatively skewed having coefficient of skewness being -0.299 and a variance as 1.32.

Table 4.6: Summary of Data on Failure and Running Times of Devices

Min	Max	Median	Mean	Variance	Skewness	Kurtosis
0.02	3	1.97	1.77	1.32	-0.299	-1.61

The TTT transform curve of the data set exhibits a convex shape and then followed by an increasing concave shape above 45 degree line as shown in Figure 4.5. Thus, the failure rate function of the data set has a bathtub shape.

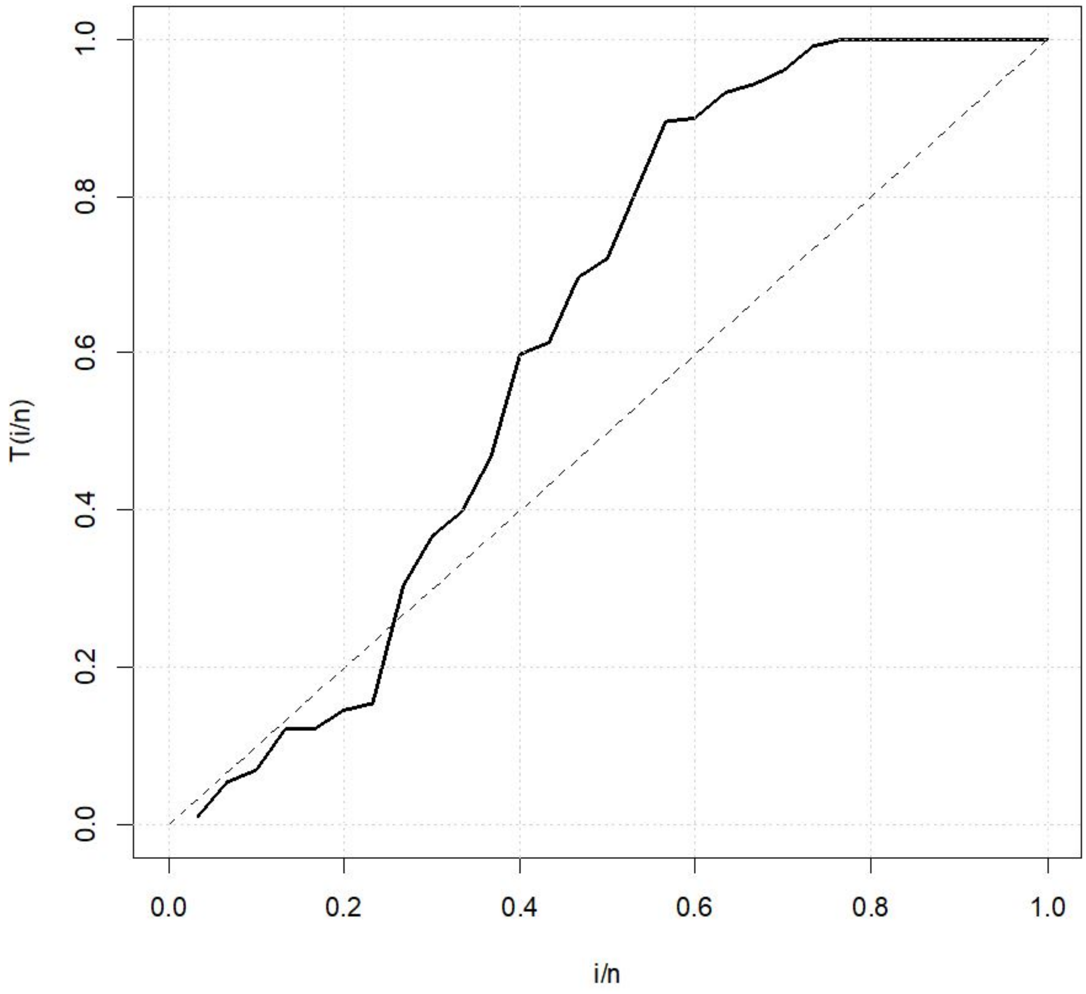

Figure 4.5: TTT-transform plot for the Failure and Running Times of Devices

Estimates of the competitive models with their corresponding standard errors and z-value are provided in Table 4.7. The significance of the parameters was verified by using the standard error test at a 5% significance level. This shows that $\hat{\theta}$ and $\hat{\gamma}$ estimates are statistically significant for GWW, for WW MLEs, the only $\hat{\theta}$ estimate was significant. For AddW model all the four parameters were statistically significant.

Table 4.7: The MLEs, Standard Errors and p-value for Failure and Running Data

Distribution	**Estimate**	**Std. error**	**Z-value**	***P*-value**
GWW	$\widehat{\alpha} = 0.56673$	18.61055	0.0305	0.975706
	$\widehat{\theta}$ =0.81885	0.18567	4.4103	1.032e-05*
	$\widehat{\gamma}$ =0.95637	0.26682	3.5843	0.000338*
	$\widehat{\lambda}$ =1.04138	70.15006	0.0148	0.988156
WW	$\widehat{\alpha} = 0.43633$	50.581440	0.0086	0.9931
	$\widehat{\lambda} = 0.06396$	195.769309	0.0003	0.9997
	$\widehat{\theta} = 1.26504$	0.204429	6.1882	6.086e-10*
NMW	$\widehat{\alpha} = 0.28488$	0.0509610	5.5903	2.267e-08*
	$\widehat{\lambda} = 0.98277$	0.0073221	134.2191	< 2.2e-16*
AddW	$\widehat{\alpha} = 0.65424$	0.057829	11.3135	< 2.2e-16*
	$\widehat{\gamma}$ =1.26506	0.411818	3.0719	0.002127*
	$\widehat{\beta}$ =-0.2044	0.057826	-3.5356	0.0004069*
	$\widehat{\theta} = 1.26505$	0.232964	5.4303	5.627e-08*

*: means significant at 5% significance level

Appropriate discrimination criteria and test statistics were computed for each fitted model to the data. Statistics for comparing the fitted models are provided in Table 4.8. Table 4.8 reveals GWW distribution was the best model for the dataset. It assumed highest log-likelihood value of -44.6 and the smallest values for AIC. In order to check the appropriateness of the model, Kolmogorov Smirnov statistics was considered at the value of 0.1871 that returned the highest p-value of 0.244, which demonstrated that the Geometric weighted Weibull (GWW) fits well than the Weighted Weibull (WW) as in Nasiru (2015) and other sub-models.

Table 4.8: Comparing Models with Discrimination Criteria and Goodness-of-Fit

Model	**lnL**	**AIC**	**BIC**	**W***	**K-S**	***p*-*value***
GWW	-44.6	97.19	102.79	0.2639	0.1871	0.244
WW	-44.8	98.31	102.52	0.3034	0.2194	0.111
NWE	-47.1	98.27	101.07	0.3215	0.2160	0.121
AddW	-46.2	100.32	105.92	0.3401	0.2088	0.191

4.8 Summary

The main structural properties of the GWW distribution are investigated and derived. The GWW distribution exhibits a wide range of shapes with varying skewness and assuming all possible forms of the hazard function. MLE is employed to evaluate the parameters and numerical examples are also provided. GWW model assumes a superior performance among the compared distributions as evidenced by the AIC and the K-S values.

CHAPTER 5

EXPONENTIATED WEIGHTED WEIBULL DISTRIBUTION

5.1 Introduction

The Weibull distribution provides a good alternative to exponential, gamma, and lognormal distribution in biological studies and life testing (Nasiru et al., 2018; Almalki and Yuan , 2013; Zhang and Xie , 2007). However, certain lifetime data; business life cycle, human mortality records, and graduate unemployment demand non-monotonic shapes like the modified unimodal, bathtub shape, and unimodal. Modified weighted Weibull distributions often fit the first and middle phases of the hazard functions. However, it lacks the credibility to fit the last phase of the bathtub and the modified unimodal shapes.

5.2 Exponentiated Weighted Weibull Distribution

Suppose the random variable W follows the exponentiated weighted Weibull (EWW) distribution. Then the cumulative distribution function (CDF) is:

$$Q(w) = \left(1 - e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}\right)^{\beta}, \alpha > 0, \lambda > 0, \gamma > 0, w > 0. \tag{5.1}$$

The corresponding probability density function (PDF) is obtained by differentiating equation (5.1):

$$q(w) = \alpha\beta\gamma(1 + \lambda^{\gamma})w^{\gamma-1}e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}\left(1 - e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}\right)^{\beta-1}, \ w > 0. \tag{5.2}$$

Figure 5.1 displays the PDF plots of the EWW that can be symmetric, left skewed, right skewed, J-shape, reverse J-shape and unimodal.

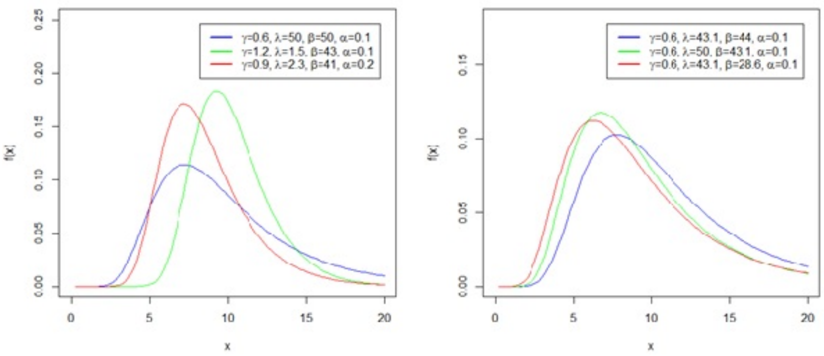

Figure 5.1: Combine Plots of Exponentiated Weighted Weibull Density Curves for some Parameters Values

The survival and hazard rate of the EWW model are respectively:

$$S(w) = 1 - \left(1 - e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}\right)^{\beta}, \; w > 0 \tag{5.3}$$

and

$$h(w) = \frac{\alpha\beta\gamma(1+\lambda^{\gamma})w^{\gamma-1}e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}\left(1 - e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}\right)^{\beta-1}}{1 - \left(1 - e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}\right)^{\beta}}, w > 0 \tag{5.4}$$

The plots of the hazard function display various attractive shapes such as monotonically decreasing, monotonically increasing and bathtub shape for different combinations of the parameter values. The patterns of the different shapes of the EWW make it possible to model monotonic and non-monotonic failure rates.

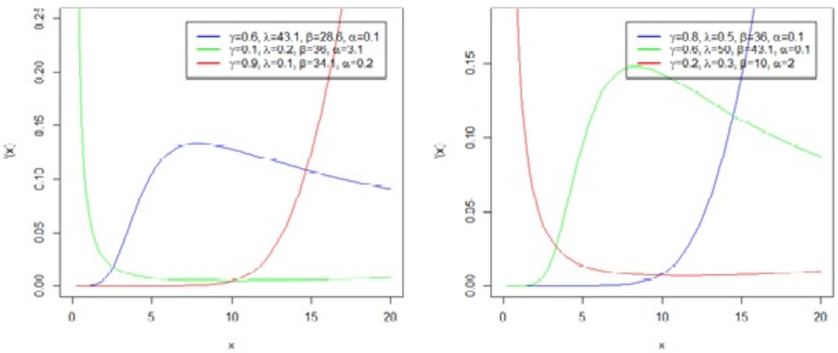

Figure 5.2: Combine Plots of the Hazard Rate for EWW at different Parameters Values.

5.3 Mixture Representation

Proposition 5.3.1. *The density of the EWW model can be expressed as*

$$q(w) = \alpha\beta\gamma(1+\lambda^{\gamma})w^{\gamma-1}\sum_{i=0}^{\infty}(-1)^{i}\begin{pmatrix}\beta-1\\ i\end{pmatrix}e^{-\alpha(1+\lambda^{\gamma})(1+i)w^{\gamma}}. \tag{5.5}$$

Proof. By definition, using Equation (5.2)

Using the power series expansion

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty}(-1)^{i}\begin{pmatrix}\beta-1\\ i\end{pmatrix}z^{i},\ |z|<1.$$

Let $z = e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}$

By substituting, we obtain

$$q(w) = \alpha\beta\gamma(1+\lambda^{\gamma})w^{\gamma-1}e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}\sum_{i=0}^{\infty}(-1)^{i}\begin{pmatrix}\beta-1\\ i\end{pmatrix}e^{-i\alpha w^{\gamma}(1+\lambda^{\gamma})}.$$

Hence,

$$q(w) = \alpha\beta\gamma(1+\lambda^{\gamma})w^{\gamma-1}\sum_{i=0}^{\infty}(-1)^{i}\begin{pmatrix}\beta-1\\ i\end{pmatrix}e^{-\alpha(1+\lambda^{\gamma})(1+i)w^{\gamma}}.$$

□

5.4 Statistical Properties

The researcher discussed quantile, Rényi entropy, moments, moment generating function, incomplete moment, reliability, and stochastic ordering as some of the statistical properties of the model.

5.4.1 Quantile

Quantile is the inverse cumulative distribution function used to generate random samples for the probability distribution. It can also serve as an alternative to the probability density function.

Proposition 5.4.1. *The quantile function w of the Exponentiated WW distribution can be as:*

$$w = \left[-\frac{\log(1-u^{\frac{1}{\beta}})}{\alpha(1+\lambda^{\gamma})} \right]^{\frac{1}{\gamma}}, u \in (0,1). \tag{5.6}$$

Proof. The quantile function is obtained by solving the equation

$$(1-e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})})^{\beta} = u$$

Hence,

$$w^{\gamma} = \left[-\frac{\log(1-u^{\frac{1}{\beta}})}{\alpha(1+\lambda^{\gamma})} \right],$$

$$w = \left[-\frac{\log(1-u^{\frac{1}{\beta}})}{\alpha(1+\lambda^{\gamma})} \right]^{\frac{1}{\gamma}}$$

completes the proof. □

5.4.2 Moments

Moments are used to compute various measures of central tendency, variation, kurtosis, and skewness among others.

Proposition 5.4.2. *The moment of a random variable W of an exponentiated weighted Weibull distribution is given by*

$$\mu_s^{\prime} = \beta \sum_{i=0}^{\infty} (-1)^i \binom{\beta - 1}{i} \frac{\Gamma\left(\frac{s}{\gamma}+1\right)}{[\alpha(1+\lambda^{\gamma})]^{\frac{s}{\gamma}}(1+i)^{\frac{s}{\gamma}+1}}. \tag{5.7}$$

Proof. By definition

$$\mu_s^{\prime} = \int_0^{\infty} w^s q(w) dw,$$

where $f(x)$ represent the PDF in equation (5.5).

This implies

$$\mu_s' = \int_0^\infty w^s.\alpha\beta\gamma(1+\lambda^\gamma)w^{\gamma-1}\sum_{i=0}^\infty (-1)^i \begin{pmatrix} \beta-1 \\ i \end{pmatrix} e^{-\alpha(1+\lambda^\gamma)(1+i)w^\gamma}dw$$

$$\mu_s' = \alpha\beta\gamma(1+\lambda^\gamma)\sum_{i=0}^\infty (-1)^i \begin{pmatrix} \beta-1 \\ i \end{pmatrix} \int_0^\infty w^s.w^{\gamma-1}e^{-\alpha(1+\lambda^\gamma)(1+i)w^\gamma}dw.$$

Let $y = \alpha(1+\lambda^\gamma)(1+i)w^\gamma, w \to 0, y \to 0, w \to \infty, y \to \infty$

$$w = \left[\frac{y}{\alpha(1+\lambda^\gamma)(1+i)}\right]^{\frac{1}{\gamma}}$$

$$\frac{dy}{dw} = \alpha\gamma(1+\lambda^\gamma)(1+i)w^{\gamma-1}$$

$$dw = \frac{dy}{\alpha\gamma(1+\lambda^\gamma)(1+i)w^{\gamma-1}}.$$

By substituting w and dw , we obtain,

$$\mu_s' = \alpha\beta\gamma(1+\lambda^\gamma)\sum_{i=0}^\infty (-1)^i \begin{pmatrix} \beta-1 \\ i \end{pmatrix} \int_0^\infty w^{s+\gamma-1}e^{-\alpha(1+\lambda^\gamma)(1+i)w^\gamma}\frac{dy}{\alpha\gamma(1+\lambda^\gamma)(1+i)w^{\gamma-1}}.$$

simplifying, we have

$$\mu_s' = \beta\sum_{i=0}^\infty (-1)^i \begin{pmatrix} \beta-1 \\ i \end{pmatrix} \frac{1}{[\alpha(1+\lambda^\gamma)]^{\frac{s}{\gamma}}(1+i)^{\frac{s}{\gamma}+1}} \int_0^\infty y^{\frac{s}{\gamma}+1-1}e^{-y}dy$$

Recall,

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt.$$

$$\mu'_s = \beta\sum_{i=0}^\infty (-1)^i \begin{pmatrix} \beta-1 \\ i \end{pmatrix} \frac{\Gamma\left(\frac{s}{\gamma}+1\right)}{[\alpha(1+\lambda^\gamma)]^{\frac{s}{\gamma}}(1+i)^{\frac{s}{\gamma}+1}]}.$$

This completes the proof.

For the first moment, $s = 1$ yields

$$E(w) = \beta \sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} \beta - 1 \\ i \end{pmatrix} \frac{\Gamma\left(\frac{1}{\gamma} + 1\right)}{[\alpha(1+\lambda^{\gamma})]^{\frac{1}{\gamma}}(1+i)^{\frac{1}{\gamma}+1}}.$$

□

5.5 Moment Generating Function

The moment generating function of the EWW distribution has been derived.

Proposition 5.5.1. *The MGF of W have an exponentiated weighted Weibull distribution denoted by $M_w(t)$ can be expressed as*

$$M_w(t) = \beta \sum_{s=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^i t^s}{\gamma!} \begin{pmatrix} \beta - 1 \\ i \end{pmatrix} \frac{\Gamma(\frac{s}{\gamma} + 1)}{[\alpha(1+\lambda^{\gamma})]^{\frac{s}{\gamma}}(1+i)^{\frac{s}{\gamma}+1}}. \tag{5.8}$$

Proof. By definition

$$M_w(t) = E(e^{tw}) = \int_{-\infty}^{\infty} e^{tw} q(w) dw,$$

Using Taylor series

$$M_w(t) = \left(\int_0^{\infty} 1 + \frac{tw}{1!} + \frac{t^2 w^2}{2!} + \ldots + \frac{t^n w^n}{n!} + \ldots \right) q(w) dw$$

$$\mathrm{M}_w(t) = E\left[\sum_{s=0}^{\infty} \frac{t^s W^s}{s!} \right]$$

$$\mathrm{M}_w(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!} E\left[W^s\right]$$

$$\mathrm{M}_w(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!} \mu'_s.$$

Where μ'_s is defined in equation (5.7)

$$M_w(t) = \beta \sum_{s=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^i t^s}{\gamma!} \begin{pmatrix} \beta - 1 \\ i \end{pmatrix} \frac{\Gamma\left(\frac{s}{\gamma} + 1\right)}{[\alpha(1+\lambda^{\gamma})]^{\frac{s}{\gamma}}(1+i)^{\frac{s}{\gamma}+1}}.$$

This completes the proof. □

5.6 Reliability

The reliability of a component plays a significant role in the stress-strength analysis of a model.

Proposition 5.6.1. *If X is the strength and is the stress, then the reliability of the component R can be expressed as*

$$R = \sum_{i=0}^{\infty} (-1)^i \binom{2\beta-1}{i} \frac{\Gamma\left(\frac{s}{\gamma}+1\right)}{[\alpha(1+\lambda^{\gamma})]^{\frac{s}{\gamma}}(1+i)^{\frac{s}{\gamma}+1}]}. \tag{5.9}$$

Proof. By definition

$$R = \int_0^{\infty} q(w)Q(w)dw = 1 - \int_0^{\infty} q(w)\bar{Q}(w)dw.$$

This implies

$$q(w)Q(w) = \alpha\beta\gamma(1+\lambda^{\gamma})w^{\gamma-1}e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}(1-e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})})^{2\beta-1},$$

$$R = \int_0^{\infty} \alpha\beta\gamma(1+\lambda^{\gamma})w^{\gamma-1}e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}\left(1-e^{-\alpha w^{\gamma}(1+\lambda^{\gamma})}\right)^{2\beta-1}dw.$$

Using the power series expansion, we obtain

$$R = \sum_{i=0}^{\infty} (-1)^i \binom{2\beta-1}{i} \int_0^{\infty} \alpha\beta\gamma(1+\lambda^{\gamma})w^{\gamma-1}e^{-\alpha(1+\lambda^{\gamma})(1+i)w^{\gamma}}dw.$$

Let $y = \alpha(1+\lambda^{\gamma})(1+i)w^{\gamma}, w \to 0, y \to 0, w \to \infty, y \to \infty$

$$w = \left[\frac{y}{\alpha(1+\lambda^{\gamma})(1+i)}\right]^{\frac{1}{\gamma}}$$

$$\frac{dy}{dw} = \alpha\gamma(1+\lambda^{\gamma})(1+i)w^{\gamma-1}$$

$$dw = \frac{dy}{\alpha\gamma(1+\lambda^{\gamma})(1+i)w^{\gamma-1}}.$$

By w substituting dw we obtain

$$R=\sum_{i=0}^{\infty}(-1)^i\begin{pmatrix}2\beta-1\\ i\end{pmatrix}\int_0^{\infty}\alpha\beta\gamma(1+\lambda^{\gamma})w^{\gamma-1}e^{-y}\frac{dy}{\alpha\gamma(1+\lambda^{\gamma})(1+i)w^{\gamma-1}}.$$

Canceling out the same variables,

$$R=\sum_{i=0}^{\infty}(-1)^i\begin{pmatrix}2\beta-1\\ i\end{pmatrix}\frac{1}{[\alpha(1+\lambda^{\gamma})]^{\frac{s}{\gamma}}(1+i)^{\frac{s}{\gamma}+1}}\int_0^{\infty}y^{\frac{s}{\gamma}+1-1}e^{-y}dy$$

$$\Gamma(a)=\int_0^{\infty}t^{a-1}e^{-t}dt$$

$$R=\sum_{i=0}^{\infty}(-1)^i\begin{pmatrix}2\beta-1\\ i\end{pmatrix}\frac{\Gamma\left(\frac{s}{\gamma}+1\right)}{[\alpha(1+\lambda^{\gamma})]^{\frac{s}{\gamma}}(1+i)^{\frac{s}{\gamma}+1}]}.$$

This completes the proof. □

5.6.1 Stochastic Ordering

Proposition 5.6.2. *The stochastic ordering of a random variables* $X_1 \sim EWW(\alpha,\lambda,\gamma,\beta_1)andX_2 \sim EWW(\alpha,\lambda,\gamma,\beta_2)$ *can be expressed as*

$$\frac{d}{dx}\log\frac{f(x_1)}{f(x_2)}=\frac{(\beta_1-\beta_2)\alpha x^{\gamma}(1+\lambda^{\gamma})}{\left(1-e^{-\alpha x^{\gamma}(1+\lambda^{\gamma})}\right)}. \tag{5.10}$$

Proof. By definition

$$f(x_1)=\alpha\beta_1\gamma(1+\lambda^{\gamma})x^{\gamma-1}e^{-\alpha x^{\gamma}(1+\lambda^{\gamma})}(1-e^{-\alpha x^{\gamma}(1+\lambda^{\gamma})})^{\beta_1-1}$$

$$f(x_2)=\alpha\beta_2\gamma(1+\lambda^{\gamma})x^{\gamma-1}e^{-\alpha x^{\gamma}(1+\lambda^{\gamma})}(1-e^{-\alpha x^{\gamma}(1+\lambda^{\gamma})})^{\beta_2-1}$$

$$\frac{f(x_1)}{f(x_2)}=\left(\frac{\beta_1}{\beta_2}\right)\left(1-e^{-\alpha x^{\gamma}(1+\lambda^{\gamma})}\right)^{\beta_1-\beta_2}.$$

Taking logarithm of both sides

$$\log\frac{f(x_1)}{f(x_2)}=\log\left(\frac{\beta_1}{\beta_2}\right)+(\beta_1-\beta_2)log\left(1-e^{-\alpha x^{\gamma}(1+\lambda^{\gamma})}\right).$$

By differentiating

$$\frac{d}{dx}\log\frac{f(x_1)}{f(x_2)} = \frac{(\beta_1-\beta_2)\alpha x^{\gamma}(1+\lambda^{\gamma})}{\left(1-e^{-\alpha x^{\gamma}(1+\lambda^{\gamma})}\right)},$$

If $\beta_1 > \beta_2$, then

$$\frac{d}{dx}\log\left(\frac{f(x_1)}{f(x_2)}\right) < 0, X_1 \leq X_2.$$

This completes the proof. □

5.6.2 Rényi Entropy

To examine the uncertainty of the random variable Rényi entropy can be employed.

Proposition 5.6.3. *Rényi entropy for a random variable X having an exponentiated weighted Weibull distribution can be expressed as*

$$I_R(\delta) = \frac{1}{1-\delta}\left[A\sum_{0}^{\infty}(-1)^i\begin{pmatrix}\delta\beta\\ i\end{pmatrix}\frac{1}{[\alpha\delta(1+\lambda^{\gamma})(1+i)]^{\delta(1-\frac{1}{\gamma})-\gamma+2}}\Gamma\left(\delta(1-\tfrac{1}{\gamma})-\gamma+2\right)\right], \quad (5.11)$$

where $\delta > 0, \delta \neq 1.$

Proof. By definition

$$I_R = \frac{1}{1-\delta}\log\left[\int_0^{\infty} f(x)dx\right]^{\delta}, \ \delta > 0, \delta \neq 1,$$

$$[f(x)]^{\delta} = \left(\alpha\beta\gamma(1+\lambda^{\gamma})\right)^{\delta} x^{\delta(\gamma-1)}e^{-\alpha\delta x^{\gamma}(1+\lambda^{\gamma})}(1-e^{-\alpha x^{\gamma}(1+\lambda^{\gamma})})^{\delta\beta}.$$

After some simplification

$$I_R = \frac{1}{1-\delta}\log\left[\left(\alpha\beta\gamma(1+\lambda^{\gamma})\right)^{\delta}\int_0^{\infty} x^{\delta(\gamma-1)}e^{-\alpha\delta x^{\gamma}(1+\lambda^{\gamma})}(1-e^{-\alpha x^{\gamma}(1+\lambda^{\gamma})})^{\delta\beta}dx\right],$$

Using the binomial expansion,

$$I_R = \frac{1}{1-\delta}\log\left[\left(\alpha\beta\gamma(1+\lambda^{\gamma})\right)^{\delta}\int_0^{\infty}\sum_{0}^{\infty}(-1)^i\begin{pmatrix}\delta\beta\\ i\end{pmatrix}x^{\delta(\gamma-1)}e^{-\alpha\delta(1+\lambda^{\gamma})x^{\gamma}}dx\right.$$

Further

$$I_R = \frac{1}{1-\delta}\log\left[A\sum_{0}^{\infty}(-1)^i\binom{\delta\beta}{i}\int_0^\infty x^{\delta(\gamma-1)}e^{-\alpha\delta(1+\lambda^\gamma)(i+1)x^\gamma}dx\right].$$

Let $y = \alpha\delta(1+\lambda^\gamma)(1+i)x^\gamma,\ x\to 0,\ y\to 0,\ x\to\infty,\ y\to\infty$

$$x = \left[\frac{y}{\alpha\delta(1+\lambda^\gamma)(1+i)}\right]^{\frac{1}{\gamma}}$$

$$\frac{dy}{dx} = \alpha\delta(1+\lambda^\gamma)(1+i)x^{\gamma-1}$$

$$dx = \frac{dy}{\alpha\delta(1+\lambda^\gamma)(1+i)x^{\gamma-1}}$$

We write

$$I_R(\delta) = \frac{1}{1-\delta}\left[A\sum_{0}^{\infty}(-1)^i\binom{\delta\beta}{i}\int_0^\infty\left[\frac{y}{\alpha\delta(1+\lambda^\gamma)(1+i)}\right]^{\frac{\delta(\gamma-1)}{\gamma}}e^{-y}.\frac{dy}{\alpha\delta(1+\lambda^\gamma)(1+i)x^{\gamma-1}}\right]$$

$$I_R(\delta) = \frac{1}{1-\delta}\left[A\sum_{0}^{\infty}(-1)^i\binom{\delta\beta}{i}\int_0^\infty\left[\frac{y}{\alpha\delta(1+\lambda^\gamma)(1+i)}\right]^{\frac{\delta(\gamma-1)}{\gamma}}\left[\frac{y}{\alpha\delta(1+i)(1+\lambda^\gamma)}\right]^{1-\gamma}B\right],$$

where $B = \frac{dy}{\alpha\delta(1+i)(1+\lambda^\gamma)}$

$$I_R(\delta) = \frac{1}{1-\delta}\left[A\sum_{0}^{\infty}(-1)^i\binom{\delta\beta}{i}\int_0^\infty\left[\frac{y}{\alpha\delta(1+\lambda^\gamma)(1+i)}\right]^{\delta(1-\frac{1}{\gamma})-\gamma+1}\left[\frac{e^{-y}}{\alpha\delta(1+i)(1+\lambda^\gamma)}\right]dy\right]$$

$$I_R(\delta) = \frac{1}{1-\delta}\left[A\sum_{0}^{\infty}(-1)^i\binom{\delta\beta}{i}\int_0^\infty\left[\frac{y^{\delta(1-\frac{1}{\gamma})-\gamma+1}}{[\alpha\delta(1+\lambda^\gamma)(1+i)]^{\delta(1-\frac{1}{\gamma})-\gamma+2}}\right]^{\frac{\delta(\gamma-1)}{\gamma}}e^{-y}dy\right],$$

$$I_R(\delta) = \frac{1}{1-\delta}\left[A\sum_{0}^{\infty}(-1)^i\begin{pmatrix}\delta\beta\\ i\end{pmatrix}\frac{\Gamma(\delta(1-\frac{1}{\gamma})-\gamma+2)}{[\alpha\delta(1+\lambda^\gamma)(1+i)]^{\delta(1-\frac{1}{\gamma})-\gamma+2}}\right].$$

This completes the proof. □

5.6.3 Incomplete Moment

The incomplete moments are also known as the conditional moment that describes the shape of many distributions.

Proposition 5.6.4. *The incomplete moment of Y can be expressed as*

$$\rho_r(x) = \sum_{i=0}^{\infty}(-1)^i\begin{pmatrix}\beta-1\\ i\end{pmatrix}\frac{1}{[\alpha(i+1)(1+\lambda^\gamma)]^{\frac{\gamma}{\gamma_1}}}\int_0^{\alpha(i+1)(1+\lambda^\gamma)x^\gamma}[z]^{\frac{\gamma}{\gamma_1}}e^{-z}\frac{dz}{(i+1)} \tag{5.12}$$

Proof. By definition

$$\rho_r(x) = \int_0^x y^r f(y)dy$$

$$\rho_r(x) = \int_0^x y^r\alpha\beta\gamma(1+\lambda^\gamma)y^{\gamma-1}e^{-\alpha y^\gamma(1+\lambda^\gamma)}(1-e^{-\alpha y^\gamma(1+\lambda^\gamma)})^{\beta-1}dy$$

$$\rho_r(x) = \int_0^x y^r\alpha\beta\gamma(1+\lambda^\gamma)y^{\gamma-1}\sum_{i=0}^{\infty}(-1)^i\left(\beta-1i\right)e^{-\alpha(i+1)(1+\lambda^\gamma)y^\gamma}y^{\beta-1}dy$$

$$\rho_r(x) = \alpha\beta\gamma(1+\lambda^\gamma)\sum_{i=0}^{\infty}(-1)^i\left(\beta-1i\right)\int_0^x y^r.y^{\gamma-1}.e^{-\alpha(i+1)(1+\lambda^\gamma)y^\gamma}y^{\beta-1}dy$$

Let $z = \alpha(i+1)(1+\lambda^\gamma)y^\gamma$,

$y \to 0, z \to 0, y = x, z \to \alpha(i+1)(1+\lambda^\gamma)x^\gamma$

$$dy = \frac{dz}{\alpha\gamma(i+1)(1+\lambda^\gamma)y^{\gamma-1}},$$

$$y=\left[\frac{z}{\alpha(i+1)(1+\lambda^{\gamma})}\right]^{\frac{1}{\gamma}},$$

$$\rho_r(x)=\sum_{i=0}^{\infty}(-1)^i\begin{pmatrix}\beta-1\\ i\end{pmatrix}\int_0^{\alpha(i+1)(1+\lambda^{\gamma})x^{\gamma}}\left[\frac{z}{\alpha(i+1)(1+\lambda^{\gamma})}\right]^{\frac{\gamma}{\gamma_1}}e^{-z}\frac{dz}{(i+1)},$$

$$\rho_r(x)=\sum_{i=0}^{\infty}(-1)^i\begin{pmatrix}\beta-1\\ i\end{pmatrix}\frac{1}{[\alpha(i+1)(1+\lambda^{\gamma})]^{\frac{\gamma}{\gamma_1}}}\int_0^{\alpha(i+1)(1+\lambda^{\gamma})x^{\gamma}}[z]^{\frac{\gamma}{\gamma_1}}e^{-z}\frac{dz}{(i+1)}.$$

This completes the proof. □

5.6.4 Order of Statistics

These statistics have been applied in statistical theory and practice, in order to explore the behavior of the outliers Let $y_1...,y_n$ be a random sample of size n from the GWW $(\theta,\alpha,\beta,\lambda)$ distribution and also let $Y_{1:n},Y_{2:n},...Y_{n:n}$ represent the corresponding order statistics drawn from this sample, has the probability density function of the K^{th} order statistics as follows:

$$f_{k:n}(x)=\frac{n!}{(k-1)!(n-k)!}f(y)F(y)^{k-1}[1-F(y)]^{n-1}, \tag{5.13}$$

where the function $F(x)$ and $f(x)$ are the CDF and PDF of the EWW model.

Using the binomial expansion

$$[1-F(y)]^{n-1}=\sum_{r=0}^{n-1}\begin{pmatrix}n-k\\ r\end{pmatrix}(-1)^rF(x)^r \tag{5.14}$$

Substituting equation (5.14) into equation (5.13), we obtain the expression as

$$f_{k:n}(y)=\frac{n!}{(k-1)!r!(n-k)!}f(y)F(y)^{k-1}\sum_{r=0}^{n-1}\begin{pmatrix}n-k\\ r\end{pmatrix}(-1)^rF(y)^r \tag{5.15}$$

Simplifying Equation (5.15), we have

$$f_{k:n}(y) = \sum_{r=0}^{n-k} \frac{n!}{(k-1)!r!(n-k)!}(-1)^r f(y)F(y)^{k+r-1} \tag{5.16}$$

Therefore, the PDF of the smallest t order statistics denotes of the proposed model is

$$f_{1:n}(y) = \sum_{r=0}^{n-1} \frac{n!}{r!(n-1-r)!}(-1)^r \alpha\beta\gamma(1+\lambda^\gamma)y^{\gamma-1}e^{-\alpha y^\gamma(1+\lambda^\gamma)}\left(1-e^{-\alpha y^\gamma(1+\lambda^\gamma)}\right)^{\beta(1-r)-1} \tag{5.17}$$

The PDF of the largest order statistics denotes $X_{(n)}$ of the proposed model is given by:

$$f_{n:n}(y) = \sum_{r=0}^{n-1} \frac{n!}{r!(n-1-r)!}(-1)^r \alpha\beta\gamma(1+\lambda^\gamma)y^{\gamma-1}e^{-\alpha y^\gamma(1+\lambda^\gamma)}\left(1-e^{-\alpha y^\gamma(1+\lambda^\gamma)}\right)^{\beta(1-r)-1} \tag{5.18}$$

5.7 Parameter Estimation

Assume $x_1 ..., x_n$ be a random sample of size n from the EWW $(\gamma, \alpha, \beta, \lambda)$ distribution. The log-likelihood function for the vector of parameters $\xi = (\gamma, \alpha, \beta, \lambda)^T$ becomes

$$L = \prod_{i=1}^{n} \alpha\beta\gamma(1+\lambda^\gamma)x^{\gamma-1}e^{-\alpha x^\gamma(1+\lambda^\gamma)}\left(1-e^{-\alpha x^\gamma(1+\lambda^\gamma)}\right)^{\beta-1} \tag{5.19}$$

$$L = \alpha^n\beta^n\gamma^n(1+\lambda^\gamma)^n\left(\prod_{i=1}^{n} x^{\gamma-1}\right)\left(\prod_{i-1}^{n} e^{-\alpha x^\gamma(1+\lambda^\gamma)}\right)\left(\prod_{i-1}^{n}\left[1-e^{-\alpha x^\gamma(1+\lambda^\gamma)}\right]\right)^{\beta-1}. \tag{5.20}$$

Taking the natural logarithm of L

$$L = n\ln\alpha + n\ln\beta + n\ln\gamma + n\ln(1+\lambda^\gamma) + (\gamma-1)\sum_{i=1}^{n}\ln x_i \times H, \tag{5.21}$$

where $H = -\alpha(1+\lambda^\gamma)\sum_{i=1}^{n} x_i{}^\gamma + (\beta-1)\sum_{i=1}^{n}\ln\left(1-e^{-\alpha x_i{}^\gamma(1+\lambda^\gamma)}\right)$.

Components of the score vector $U(\xi)$ are obtained by partially differentiating the log-likelihood function.

The associated components of the score function $U(\xi) = \left[\frac{\partial L}{\partial\alpha}, \frac{\partial L}{\partial\beta}, \frac{\partial L}{\partial\lambda}, \frac{\partial L}{\partial\gamma},\right]^T$

are

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left(1+\lambda^{\gamma}\right)x_i^{\gamma}+(-1+\beta)\sum_{i=1}^{n}\frac{e^{-\alpha\left(1+\lambda^{\gamma}\right)x_i^{\gamma}}(1+\lambda^{\gamma})\log[e]\,x_i^{\gamma}}{1-e^{-\alpha(1+\lambda^{\gamma})x_i^{\gamma}}} \tag{5.22}$$

$$\frac{\partial L}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left[1-e^{-\alpha\left(1+\lambda^{\gamma}\right)x_i^{\gamma}}\right] \tag{5.23}$$

$$\frac{\partial L}{\partial \lambda}=\frac{n\gamma\lambda^{-1+\gamma}}{1+\lambda^{\gamma}}-\sum_{i=1}^{n}\alpha\gamma\lambda^{-1+\gamma}x_i^{\gamma}+(-1+\beta)\sum_{i=1}^{n}\frac{e^{-\alpha\left(1+\lambda^{\gamma}\right)x_i^{\gamma}}\alpha\gamma\lambda^{-1+\gamma}\log[e]\,x_i^{\gamma}}{1-e^{-\alpha(1+\lambda^{\gamma})x_i^{\gamma}}} \tag{5.24}$$

$$\frac{\partial L}{\partial \gamma}=\frac{n}{\gamma}+\frac{n\lambda^{\gamma}\log[\lambda]}{1+\lambda^{\gamma}}+\sum_{i=1}^{n}\log[x_i]-\sum_{i=1}^{n}\left(\alpha\lambda^{\gamma}\log[\lambda]\,x_i^{\gamma}+\alpha\left(1+\lambda^{\gamma}\right)\log[x_i]\,x_i^{\gamma}\right)E, \tag{5.25}$$

where

$$E=(-1+\beta)\sum_{i=1}^{n}-\frac{e^{-\alpha\left(1+\lambda^{\gamma}\right)x_i^{\gamma}}\log[e]\left(-\alpha\lambda^{\gamma}\log[\lambda]\,x_i^{\gamma}-\alpha(1+\lambda^{\gamma})\log[x_i]\,x_i^{\gamma}\right)}{1-e^{-\alpha(1+\lambda^{\gamma})x_i^{\gamma}}}.$$

A good set of initial values is essential. The numerical result is derived directly with the application of lifetime data set using statistical software R, see appendix B2 for the appropriate code.

5.8 Numerical Illustration

In this section, random numbers are generated and two real datasets are considered.

5.8.1 Simulation Study

In this section, we study the performance of the MLE estimators of the unknown parameters of the EWW distribution by implementing the Monte Carlo simulation method. A random sample of sizes $n = 50, 100, 150$ and 200 is generated from the quantile function given in Equation (5.6) of the distribution. Comparisons of different methods are made using the estimated values of parameters, the average Bias, and the mean square error MSE for 1000 replicates. The processes were performed using the R-Software *(see the appropriate code provided in appendix B1)*.

The purpose of generating random numbers is to know if:

- There exists any discrepancy between the average estimates and the true values, whether it would be small or not.
- The MLE would converge to the true value in all cases when the sample size increases.
- The standard errors of the MLEs would decrease as the sample size increases.

The results of this simulation study are in Table 5.1. Table 5.1 reveals the average BIAS and Root Mean Square Error (in the parentheses) for the maximum likelihood estimator $(\gamma, \alpha, \beta, \lambda) = (7.5, 0.3, 0.2, 7.1)$ for $n = 50, 100, 150, 200$. ABIAS decreases for estimators of some parameters as the sample size increases while for others it fluctuates. The RMSE for the estimators on average decreases for all parameters as the sample size increases. The process is repeated, in this case, the parameter values are fixed at $(\gamma, \alpha, \beta, \lambda) = (7.6, 0.3, 0.2, 7.1)$ for $n = 50, 100, 150, 200$. The ABIAS for some estimators of the parameters also decreases as the sample size increases while it fluctuates others. The RMSE for the estimators of all the parameters showed a decreasing pattern.

Table 5.1: MLEs, ABIAS and RMSE (in parentheses) for EWW Simulation Study for different Parameters Values

Parameter Val				***n***	**ABIAS/RMSE**			
α	**γ**	**β**	**Λ**		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$
0.3	**7.5**	**0.2**	**7.1**	***50***	0.1775	0.1860	-1.6118	0.3770
					(0.7908)	(0.4882)	(2.3212)	(0.8755)
				100	0.07704	0.1132	-1.3071	0.2462
					(0.1765)	(0.1548)	(1.7839)	(0.5015)
				150	0.05215	0.0948	-1.1773	0.2015
					(0.1426)	(0.1236)	(1.5848)	(0.4073)
				200	0.0313	0.0802	-1.0019	0.1704
					(0.1223)	(0.10267)	(1.3933)	(0.3540)
α	**γ**	**β**	**Λ**	***n***	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$
0.3	**7.6**	**0.2**	**7.1**	***50***	0.1442	0.1959	-1.555	0.4233
					(0.3274)	(0.3584)	(2.3174)	(1.2247)
				100	0.0557	0.1174	-1.2376	0.2706
					(0.1729)	(0.1682)	(1.8304)	(0.6123)
				150	0.0227	0.0928	-1.0549	0.1994
					(0.1235)	(0.1196)	(1.4962)	(0.4173)
				200	0.0069	0.0809	-0.9762	0.1788
					(0.0959)	(0.1005)	(1.3248)	(0.3661)

Table 5.2: MLEs, ABIAS and RMSE (in parentheses) for EWW Simulation Study for different Parameters Values

Parameter Val				***n***	**ABIAS/RMSE**			
α	**γ**	**β**	**Λ**	***N***	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$
0.3	**7.4**	**0.4**	**7.2**	***50***	0.1313	0.2003	-1.5599	0.3209
					(0.3492)	(0.4759)	(2.3102)	(1.5249)
				100	0.0397	0.1231	-1.2249	0.1651
					(0.2048)	(0.1716)	(1.8098)	0.4325
				150	0.0001	0.0933	-1.030	0.1156
					(0.1484)	(0.1217)	(1.4946)	(0.3180)
				200	-0.0249	0.0811	-0.9257	0.1004
					(0.1251)	(0.1037)	(1.3096)	(0.2902)
α	**γ**	**β**	**Λ**		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$
0.3	**7.4**	**0.4**	**7.2**	***50***	0.1040	0.2143	-0.1727	0.0795
					(0.3697)	(0.5994)	(2.4955)	(0.6254)
				100	0.015	0.0741	0.1276	-0.0137
					(0.1505)	(0.2150)	(1.8968)	(0.3261)
				150	-0.0058	0.0452	0.1672	-0.0249
					(0.1058)	(0.1478)	(1.5942)	(0.2777)
				200	-0.0073	0.0452	0.0636	-0.0108
					(0.0907)	(0.1294)	(1.4041)	(0.2414)

5.8.2 Applications

In order to investigate the advantage of the EWW proposed model, we apply two lifetime datasets. We consider criteria like the Kolmogorov-Smirnov test statistics (K-S), Log-likelihood Ratio Test (lnL), Akaike Information Criteria (AIC), and Bayesian Information Criteria (BIC). The better distribution corresponds to the smallest (W*), (K-S), (lnL), (AIC), and (BIC) values. The computations were done using R-Software *(see appendix B3)*.

5.8.3 Application 3: Electronic Components Failure Rate Data

The study fit the exponentiated weighted Weibull, weighted Weibull, exponentiated general weighted Weibull, and exponentiated Weibull distribution to real data of 20 electronic components Nasiru (2015). The data is shown in Table 3.3. Table 5.3 gives a descriptive summary of the 20 electronic components of lifetime data. Table 5.3 shows the average electronic components failure rate at 1.94 and having a coefficient of skewness of 0.60 and kurtosis of the value of 2.72 of the data which have positive skewness and kurtosis.

Table 5.3: Descriptive statistics of electronic components life data

Minimum	Mean	Maximum	Variance	Skewness	Kurtosis
0.03	1.94	5.09	2.06	0.60	2.72

The dataset shows a bathtub failure rate since the TTT-transform plot is first concave above 45 degree line and then followed by convex shape as shown in Figure 5.3.

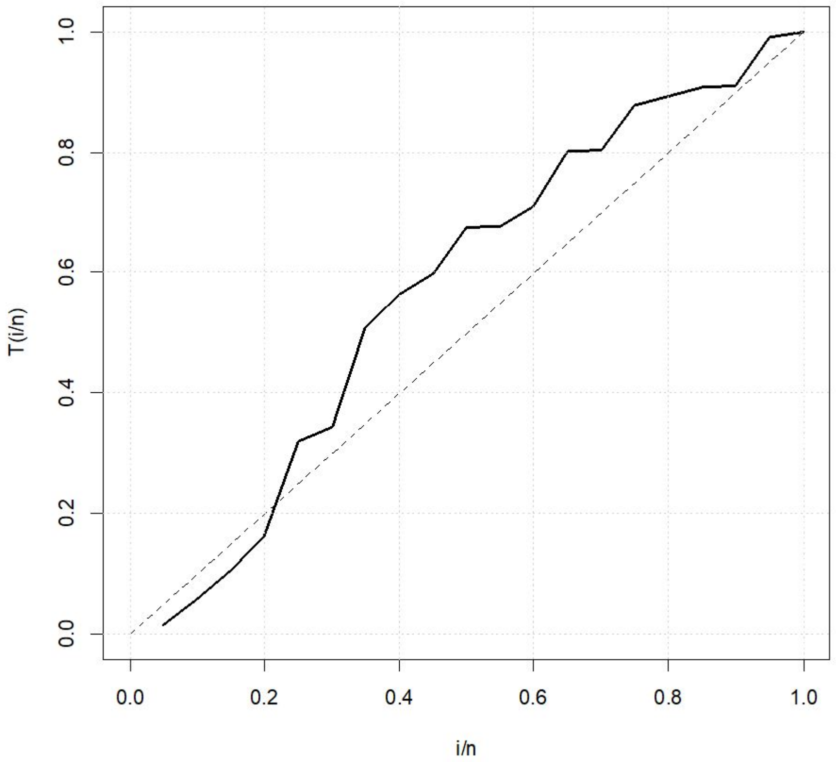

Figure 5.3: TTT-Transform plot for the Lifetime of Electronic Component

The accuracy of the approximation of the standard error of the MLEs in Table 5.4 was verified by using the standard error test which states that for a parameter to be significant at the 5% significance level the standard error should be less than half the parameter values. This output shows that MLEs for the parameters for EWW was not statistically significant. For the WW model only θ was significant and for the EGWW, the parameters γ and a were significant.

Table 5.4: Maximum likelihood estimates and the standard errors for electronic components

Distribution	Estimate	Std error	*z-value*	*p-value*
EWW	$\widehat{\alpha} = 0.0082$	0.07476	0.1093	0.9130
	$\widehat{\lambda} = 0.5743$	5.2515049	0.1094	0.9129
	$\widehat{\gamma} = 3.4149$	3.8971	0.8763	0.3809
	$\widehat{\beta} = 0.2365$	0.3281	0.7207	0.4711
WW	$\widehat{\alpha} = 0.1221$	4.9928	0.0245	0.9805
	$\widehat{\lambda} = 2.1456$	102.766	0.0209	0.9833
	$\widehat{\theta} = 1.1961$	0.2248	5.3197	1.04×10^{-7}*
EGWW	$\widehat{\alpha} = 0.0037$	9.9347e-03	0.3700	0.7114
	$\widehat{\lambda} = 0.0371$	6.2953e-04	58.9293	$<2\times10^{-16}$*
	$\widehat{\theta} = 2.5948$	1.5907e+0	1.6312	0.1028
	$\widehat{a} = 10.1457$	1.9592e-03	5178.4089	$<2\times10^{-16}$*
	$\widehat{b} = 0.3294$	2.5999e-01	1.2669	0.2052

*: means significant at 5% significance level

Appropriate information criteria and goodness-of-fit statistics computed for each fitted model to the data. Statistics for comparing the fitted models are presented in Table 5.5. Table 5.5 reveals that EWW distribution was the best model for the dataset since it has the highest log-likelihood value of -31.47 and the smallest values for AIC, BIC and CAIC. To check the appropriateness of the model, Kolmogorov Smirnov statistics was considered at the value of 0.1309. The

Anderson-Darling (A*) statistics at 0.214 also demonstrated that the exponentiated weighted Weibull (EWW) fits the data set better than the Weighted Weibull (WW) Nasiru (2015) and other sub-models.

Table 5.5: Goodness - of - Fit Statistics

Model	Log-likelihood	AIC	BIC	K-S	p-value	W*
EWW	-31.47	70.9	74.2	0.1039	0.9667	0.0323
WW	-32.79	71.6	74.6	0.1271	0.8638	0.0712
EGWW	-31.57	73.1	78.1	0.1052	0.9630	0.0344

In order to gain more insight into the EWW distribution, a visual correlation of the histogram of the information data with the fitted density functions was developed. Similarly, it is clear in Figure 5.3 that the EWW fits the left and right peaks of the histogram better.

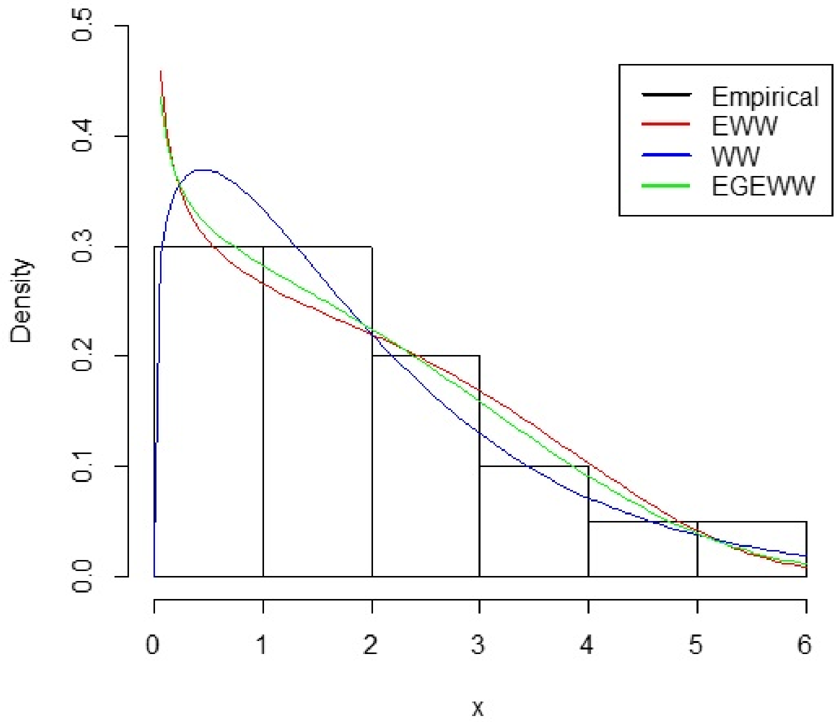

Figure 5.4: Plots of empirical density and densities of the fitted distribution for the electronic components data.

5.8.4 Application 4: Weight of the Diamond Stones in Carat

Illustration of the performance of the proposed model with data obtained from Singfat (1996) which represents the weight of the diamond stones in carat. To show the applicability and the assessment of the merit of the proposed model, we use the data set indicated in Table 3.4.

Table 5.6 gives a descriptive summary of the data which have positive skewness and kurtosis.

Table 5.6: Descriptive Statistics for weight of diamond stones dataset

Mean	Median	SD	Variance	Skewness	Kurtosis
0.659	0.40	0.765	0.584	1.896	2.950

The dataset exhibits a bathtub failure rate since the TTT-transform plot is first convex below the 45 degree line and then followed by concave shape above it as shown in Figure 5.5.

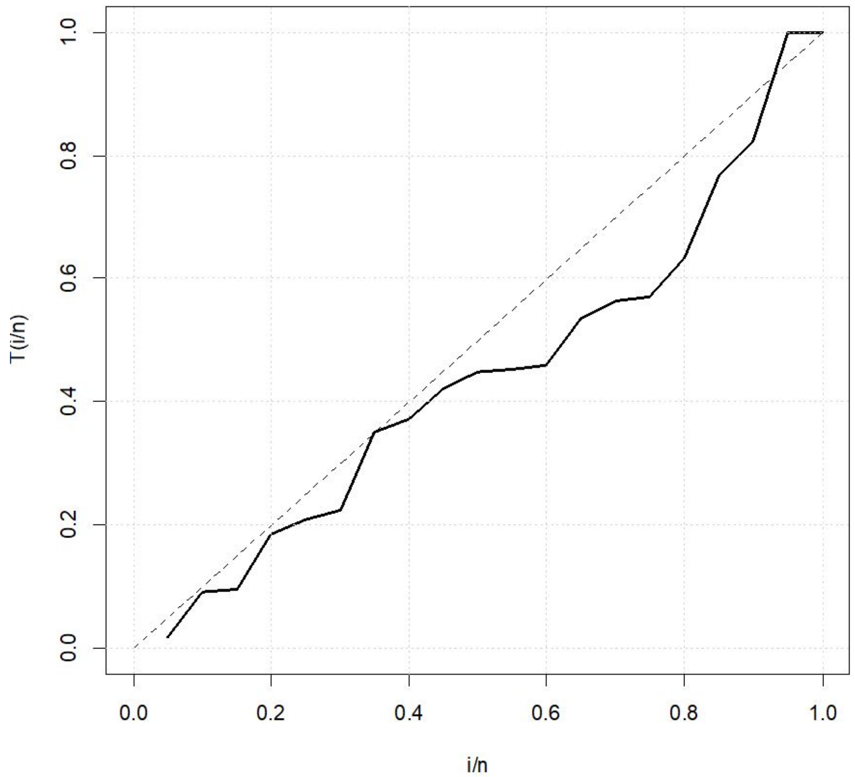

Figure 5.5: TTT-Transform plot for the weight of the Diamond Stones

Table 5.7 shows the MLEs of the parameters, their standard errors, z-values and the p-values for the fitted EWW, WW, EGWW and AddW distributions. The individual contribution to EWW is not statistically significant, since the p-values are above the threshold of 0.05. Only theta that is significantly accounted for WW. Similarly, theta and alpha are also significant due AddW.

Table 5.7: MLEs, Standard error and p-value for weight of the diamond stones dataset

Distribution	**Estimate**	**Std error**	***z-value***	***p-value***
EWW	$\widehat{\alpha} = 0.7552$	28.8227	0.0262	0.9791
	$\widehat{\lambda} = 2.8497$	254.1984	0.0112	0.9911
	$\widehat{\gamma} = 0.6453$	0.3928	1.6429	0.1004
	$\widehat{\beta} = 2.0213$	2.4774	0.8159	0.4146
WW	$\widehat{\alpha} = 0.96319$	101.17830	0.0095	0.9924
	$\widehat{\lambda} = 0.55677$	172.29280	0.0032	0.9974
	$\widehat{\theta} = 0.9218$	0.15698	5.8726	$4.29\times10^{-9*}$
EGWW	$\widehat{\alpha} = 1.11822$	39.45855	0.0283	0.9774
	$\widehat{\lambda} = 1.34105$	109.33363	0.0123	0.9902
	$\widehat{\theta} = 0.64531$	0.39281	1.6428	0.1004
	$\widehat{a} = 0.90702$	32.00593	0.0283	0.9774
	$\widehat{b} = 2.02172$	2.47817	0.8158	0.4146
AddW	$\widehat{\alpha} = 1.21714$	0.17049	7.1392	9.385e-13
	$\widehat{\beta} = 0.30743$	0.17049	1.8033	0.07134
	$\widehat{\theta} = 0.92188$	0.23580	3.9095	9.248e-05
	$\widehat{\gamma} = 0.92190$	0.71410	1.2910	0.19671

* means significant at 5% level

Table 5.8 reveals that the two-information criterion employed for the fitted model to the data, the EWW is more flexible than EGWW and AddW but unfavorable to Weighted Weibull distribution. According to Oguntunde et al. (2016), the lowest information criteria are considered to be the best fit, more flexible, and capable than the other models. The result is also similar to Obayelu et al. (2014) which concluded that the distribution can be more tractable but could not perform better than competitive distributions. This result is in accordance with what was obtainable in other notable researches. Obayelu et al. (2014), with respect to the probability characteristic of the distribution. For example, according to Oguntunde et al. (2016), the characteristics of the model could either be flexible or tractable. Hence, The EWW distribution is strongly viewed to be more tractable and capable than the Weighted Weibull distribution defined by Nasiru (2015) and more flexible as compared to EGWW and AddW to the Weight of the Diamond Stones in the Carat dataset.

Table 5.8: Comparing models with Information criteria and Goodness-of-Fit for diamond dataset

Model	lnL	AIC	BIC	W*	K-S	*p-value*
GWW	-11.35	30.7	34.68	0.040	0.111	0.943
WW	-11.54	29.1	32.07	0.329	0.135	0.8165
EGWW	-11.36	32.7	37.68	6.556	0.999	$2.2*10^{-16}$
AddW	-11.55	31.1	35.07	0.052	0.134	0.8164

5.9 Summary

Modification of the exponentiated weighted Weibull (EWW) distribution is derived and its probability characteristics were obtained. The shapes of its probability density function and failure rate are investigated. Statistical properties of EWW are derived. The method of maximum likelihood estimation is used to estimate the unknown parameters. The performance of the MLE estimators of the unknown parameters of the distribution by implementing the Monte Carlo simulation method. We provide applications to two datasets to prove the flexibility of the model. The EWW distribution is considered to be more tractable and capable than the weighted Weibull distribution.

CHAPTER 6

CONCLUSION AND RECOMMENDATIONS

6.1 Introduction

In this chapter, the conclusion, recommendations, and directions for further studies are presented.

6.2 Conclusion

In modeling real-life events with respect to probability theory, two particular characteristics are considered, either the probability distribution is Flexible or the distribution is Tractable. Statistically, to retain the originality of the data, appropriate probability distribution needs to be employed rather than to transform the existing dataset. The knowledge of the appropriate distribution of real datasets greatly improves the sensitivity, power, and efficiency of the statistical test associated with the data sets.

New compound distributions based on the weighted Weibull distribution have been proposed having an interest in statistical theory and its structural statistical properties studied . The proposed models provide a better flexibility extension and tractability of the mathematical concepts of the weighted Weibull (WW) distribution.

The idea is to model the systems connected in parallel setting, so that the hazard function is either monotonic or non-monotonic. The study consider N systems function independently and producing a certain product at a given time. Failure of device often occurs due to the present of an unknown number of initial defects in the system so that the hazard function is motononically increasing, monotonically decreasing, bathtub and unimodal. By using the proposed model, the distributions have appropriately improved flexibility.

In this study, the proposed modification of the GWW distribution and its cumulative distribution function (CDF) and probability density function (PDF) were derived. We have studied structural statistical properties of GWW distribution and several analytical properties and shown that it is a tractable distribution. We have also shown that the GWW provides excellent fits to two data set. Estimation of the unknown parameters was obtained by adopting the method of maximum likelihood estimation (MLE). Monte Carlo simulation experiment was carried out to examine the properties of the maximum likelihood estimator for the parameters of the GWW distribution. For each data set, the proposed modification was shown to give better fit than several other competitors including the weighted weibull distribution .

Finally, we introduced a new modification of the Exponentiated Weighted Weibull (EWW) distribution and its characteristics functions were obtained. Statistical properties of EWW were derived. We estimated the four unknown parameters by maximum likelihood estimation method. We studied the performance of the MLE estimators of the unknown parameters of the EWW distribution by implementing the Monte Carlo simulation method. We fitted the exponentiated weighted Weibull, weighted Weibull, exponentiated generalised weighted Weibull and exponentiated Weibull distribution to a real data of 20 electronic components. It revealed that EWW fits the left and right peaks of the histogram well of the electronic components data. Also, we fit weight of the diamond stones dataset to the EWW distribution and other models. The results shown that the information criteria of AIC and BIC for EWW distribution is higher than the WW distribution, the EWW distribution did not provide a better fit than the WW distribution for weight of the diamond stones dataset. The study revealed that the EWW distribution goodness-of-fit returns lower statistics than WW. Hence, the EWW distribution is more tractable and capable than the weighted Weibull distribution.

In conclusion, the proposed models have a better representation of the datasets than the other competitive models.

6.3 Recommendations

This study recommends the following:

1. The study can further be extended based on the derived distributions to construct autoregressive processes.

2. Further study can be done by employing different estimation techniques to estimate unknown parameters.

3. In this study, the stochastic representation $X = \max(X_1, X_2, ..., X_N)$ was adopted to propose the geometric weighted Weibull distribution. Further study can be adopted by considering stochastic representation $X = \min(X_1, X_2, ..., X_N)$ to construct for the case of systems connected in series.

6.4 Contribution to knowledge

- The extensions of the weighted Weibull distribution have been developed.

- Algorithms were developed in R programming as a code to achieve the results in the study.

- The study has shown that the extended weighted Weibull distributions fit the dataset better than the standard distributions.

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R-CODE FOR SIMULATION OF GWW DISTRIBUTION

```
######␣Simulation␣code␣for␣GWW##########

##############␣Quantile␣function␣##############

quantile<-function(alpha,beta,lambda,theta,u){

A<-(1-(theta*(u-1))-u)

B<-(1+(theta*(u-1)))

C<-log(A/B)

D<-(-1/(alpha*(1+(lambda^beta))))

quant<-(D*C)^(1/beta)

return(quant)

}

########################␣Negative␣Log-likelihood␣#############

LLa<-function(par){

alpha=par[1]

beta=par[2]

lambda=par[3]

theta=par[4]

A<-(alpha*(x^beta)*(1+(lambda^beta)))

B<-(1-theta)*(1-exp(-A))

C<-(1-theta*(1-exp(-A)))

##############␣GWW_PDF␣###########

p=(1+(lambda^beta))*alpha*beta*(x^(beta-1))

q=(1-theta)*p*exp(-A)

r=(1-(theta*(1-exp(-A))))^2

GWW_PDF<-q/r

NLLa<--sum(log(GWW_PDF))

return(NLLa)

}

###Algorithm␣for␣Monte␣Carlo␣Simulation␣Study

library(numDeriv)
```

```
library(Matrix)

alpha=0.1

beta=0.2

lambda=0.1

theta=0.3


n1=c(25,50,75,100)

for(j␣in␣1:length(n1)){

n=n1[j]

########################################

N=1000

#########################################

mle_alpha<-c(rep(0,N))

mle_beta<-c(rep(0,N))

mle_lambda<-c(rep(0,N))

mle_theta<-c(rep(0,N))

LC_alpha<-c(rep(0,N))

UC_alpha<-c(rep(0,N))

LC_beta<-c(rep(0,N))

UC_beta<-c(rep(0,N))

LC_lambda<-c(rep(0,N))

UC_lambda<-c(rep(0,N))

LC_theta<-c(rep(0,N))

UC_theta<-c(rep(0,N))

count_alpha=0

count_beta=0

count_lambda=0

count_theta=0

temp=1

#####␣if␣3␣parameters␣9,␣if␣4␣parameter␣16␣and␣so␣on,␣in␣the␣matrix␣space␣######
```

```
HH1<-matrix(c(rep(2,16)),nrow=4,ncol=4)
HH2<-matrix(c(rep(2,16)),nrow=4,ncol=4)
for(i␣in␣1:N)
{
print(i)
flush.console()
repeat{
x<-c(rep(0,n))
#␣Generate␣a␣random␣variable␣from␣uniform␣distribution
u<-0
u<-runif(n,min=0,max=1)
for(k␣in␣1:n){
x[k]<-quantile(alpha,beta,lambda,theta,u[k])
}
#Maximum␣likelihood␣estimation
mle.result<-nlminb(c(alpha,beta,lambda,theta),LLa,lower=c(0,0,0,0))
temp=mle.result$convergence
if(temp==0){
temp_alpha<-mle.result$par[1]
temp_beta<-mle.result$par[2]
temp_lambda<-mle.result$par[3]
temp_theta<-mle.result$par[4]
HH1<-hessian(LLa,␣c(temp_alpha,temp_beta,temp_lambda,temp_theta))
if(sum(is.nan(HH1))==0&(diag(HH1)[1]>0)&(diag(HH1)[2]>0)&(diag(HH1)[3]>0)&(diag(HI
HH2<-solve(HH1)
#print(det(HH1))
}
else{
temp=1}
}
```

```
if((temp==0)&(diag(HH2)[1]>0)&(diag(HH2)[2]>0)&(diag(HH2)[3]>0)&(diag(HH2)[4]>0)&(
break
}
else{
temp=1}
}
temp=1
mle_alpha[i]<-mle.result$par[1]
mle_beta[i]<-mle.result$par[2]
mle_lambda[i]<-mle.result$par[3]
mle_theta[i]<-mle.result$par[4]
HH<-hessian(LLa,c(mle_alpha[i],mle_beta[i],mle_lambda[i],mle_theta[i]))
H<-solve(HH)
LC_alpha[i]<-mle_alpha[i]-qnorm(0.975)*sqrt(diag(H)[1])
UC_alpha[i]<-mle_alpha[i]+qnorm(0.975)*sqrt(diag(H)[1])
if((LC_alpha[i]<=alpha)&(alpha<=UC_alpha[i])){
count_alpha=count_alpha+1
}
LC_beta[i]<-mle_beta[i]-qnorm(0.975)*sqrt(diag(H)[2])
UC_beta[i]<-mle_beta[i]+qnorm(0.975)*sqrt(diag(H)[2])
if((LC_beta[i]<=beta)&(beta<=UC_beta[i])){
count_beta=count_beta+1
}
LC_lambda[i]<-mle_lambda[i]-qnorm(0.975)*sqrt(diag(H)[3])
UC_lambda[i]<-mle_lambda[i]+qnorm(0.975)*sqrt(diag(H)[3])
if((LC_lambda[i]<=lambda)&(lambda<=UC_lambda[i])){
count_lambda=count_lambda+1
}
LC_theta[i]<-mle_theta[i]-qnorm(0.975)*sqrt(diag(H)[3])
UC_theta[i]<-mle_theta[i]+qnorm(0.975)*sqrt(diag(H)[3])
```

```
if((LC_theta[i]<=theta)&(theta<=UC_theta[i])){
count_theta=count_theta+1
}

}
#Calculate␣Average␣Bias
ABias_alpha<-sum(mle_alpha-alpha)/N
ABias_beta<-sum(mle_beta-beta)/N
ABias_lambda<-sum(mle_lambda-lambda)/N
ABias_theta<-sum(mle_theta-theta)/N
print(cbind(ABias_alpha,ABias_beta,ABias_lambda,ABias_theta))
#Calculate␣RMSE
RMSE_alpha<-sqrt(sum((alpha-mle_alpha)^2)/N)
RMSE_beta<-sqrt(sum((beta-mle_beta)^2)/N)
RMSE_lambda<-sqrt(sum((lambda-mle_lambda)^2)/N)
RMSE_theta<-sqrt(sum((theta-mle_theta)^2)/N)
print(cbind(RMSE_alpha,RMSE_beta,RMSE_lambda,RMSE_theta))

}
```



```
APPENDIX␣A2:␣R-code␣for␣Maximum␣likelihood␣estimates␣for␣Strength␣of␣1.5cm␣Glass␣
library(AdequacyModel)
library(pracma)
########␣␣LOAD␣DATA␣INTO␣R␣␣#############
library(bbmle)
xh<-c(0.55,␣0.93,␣1.25,␣1.36,␣1.49,␣1.52,␣1.58,␣1.61,␣1.64,␣1.68,␣1.73,␣1.81,␣2.0
#################␣␣␣␣GWgW␣␣␣␣#####################################
LLh<-function(alpha␣,theta,␣gamma,␣lambda){
A<-(alpha*(xh^gamma)*(1+(lambda^gamma))
```

```
p<-(1+(lambda^gamma))*alpha*gamma*(xh^(gamma-1))
q<-(1-theta)*p*exp(-A)
r<-(1-(theta*(1-exp(-A))))^2
GWW_PDF<-q/r
###CDF␣GWgW␣#####
##m<-(1-theta)*(1-exp(-A))
##n<-(1-(theta*(1-exp(-A))))
##cdf_GwgW<-m/n
NLLh<--sum(log(GWW_PDF))
return(NLLh)}

fith<-␣mle2(LLh,␣start=list(alpha=1.4,theta=0.8,␣gamma=1.3,␣lambda=),␣method="BFG
fith
summary(fith)
AIC(fith)
BIC(fith)
vcov(fith)

#################␣␣WEIGHTED␣WEIBULL␣␣␣################################
LLh<-function(alpha␣,lambda,␣theta){
A<-(alpha*(xh^theta))+(alpha*((lambda*xh)^theta))
B<-(1+(lambda^theta))*alpha*theta
C<-xh^(theta-1)
#WgW_CDF<-1-exp(-A)
WgW_PDFh<-B*C*exp(-A)
NLLh<--sum(log(WgW_PDFh))
return(NLLh)}
#alpha=0.6␣,lambda=1.2,␣theta=0.31
fith<-␣mle2(LLh,␣start=list(␣␣alpha=0.6␣,lambda=1.2,␣theta=0.31),␣method="BFGS",␣
fith
```

```
summary(fith)

AIC(fith)

BIC(fith)

vcov(fith)


#########␣EXPONENTIATED␣WEIBULL␣###########################################

LLh<-function(beta␣,gamma,␣lambda){

W_PDF<-beta*gamma*(xh^(beta-1))*exp(-gamma*(xh^beta))

W_CDF<-1-exp(-gamma*(xh^beta))

EW_PDF<-lambda*(W_PDF)*(W_CDF)^(lambda-1)

#EG_CDF<-(W_CDF)^lambda

NLLh<--sum(log(EW_PDF))

return(NLLh)}

fith<-␣mle2(LLh,␣start=list(␣beta=0.61␣,gamma=0.12,␣lambda=0.31),␣method="BFGS",␣

fith

summary(fith)

AIC(fith)

BIC(fith)

vcov(fith)


########################␣␣␣WEIBULL␣#########################

LLd<-function(␣beta,␣gamma){

WEIBULL_PDF<-␣beta*gamma*(xh^(beta-1))*exp(-gamma*(xh^beta))

###CDF_Weibull<-(1-exp(-gamma*(xh^beta)))

NLLd<--sum(log(WEIBULL_PDF))

return(NLLd)}

fitd<-␣mle2(LLd,␣start=list(beta=1.19␣,␣gamma=1.4),␣method="BFGS",␣data=list(xh))

fitd

summary(fitd)

AIC(fitd)
```

```
BIC(fitd)

vcov(fitd)


##Exponentiated␣Generalised␣exponential␣weighted␣weibull␣########

LLh<-function(a,␣b,␣alpha␣,lambda,␣theta){

A<-alpha*((xh)^theta)+alpha*(lambda*xh)^theta

B<-1+(lambda^theta)

#CDF_EGNWWD=(1-(1-exp(-A))^a)^b

#CDF_EGNWWD=

###␣the␣Probability␣density␣function␣#######

p<-a*b*alpha*theta*((xh)^(theta-1))*B

q<-(1-(1-exp(-A)))^(a-1)

r<-(1-(1-(1-exp(-A)))^a)^(b-1)

PDF_EGNWWD=p*q*r*exp(-A)

NLLh<--sum(log(PDF_EGNWWD))

return(NLLh)}

fith<-␣mle2(LLh,␣start=list(a=2.6,␣b=1.4,␣alpha=1.1,␣lambda=2.1,␣theta=0.3),␣meth

fith

summary(fith)

AIC(fith)

BIC(fith)

vcov(fith)


#####␣Oguntunde,␣the␣␣new␣weighted␣exponential␣distribution␣#####

LL<-function(alpha,␣lambda){

PDF<-(1+lambda)*alpha*(exp((1+lambda)*(-alpha*xh)))

NLL<--sum(log(PDF))

return(NLL)}

fit<-␣mle2(LL,␣start=list(␣alpha=1.2,lambda=1.13),␣method="BFGS",␣data=list(xh))

fit
```

```
summary(fit)
AIC(fit)
BIC(fit)
vcov(fit)

###␣exponentiated␣general␣####
LLh<-function(alpha␣,lambda,␣theta){
A<-lambda*alpha*theta*(xh^(theta-1))*(exp(-alpha*xh^theta))
B<-(1-(exp(-alpha*xh^theta)))^(lambda-1)
#WgW_CDF<-1-exp(-A)
WgW_PDFh<-A*B
NLLh<--sum(log(WgW_PDFh))
return(NLLh)}
fith<-␣mle2(LLh,␣start=list(␣alpha=1.21␣,␣lambda=1.2,␣theta=1.3),␣method="BFGS",␣
fith
summary(fith)
AIC(fith)
BIC(fith)
vcov(fith)
#########␣additive␣weibull␣#####################
LLh<-function(alpha,␣gamma,␣beta,␣theta){
A<-alpha*theta*xh^(theta-1)
B<-beta*gamma*xh^(gamma-1)
C<-exp(-alpha*xh^theta-(beta*xh^gamma))
PDF<-(A+B)*C
NLL<--sum(log(PDF))
return(NLL)}

fit<-mle2(LLh,start=list(alpha=0.8,␣gamma=0.6,␣beta=1.1,␣theta=0.8),␣method="BFGS"
fit
```

```
summary(fit)

AIC(fit)

BIC(fit)

vcov(fit)
```

```
APPENDIX␣A3:␣R-Code␣for␣Analysing␣goodness-of-fit␣for␣Strength␣of␣1.5cm␣Glass␣Fibe
>␣library(AdequacyModel)
>␣library(pracma)
>␣xh<-c(0.55,␣0.93,␣1.25,␣1.36,␣1.49,␣1.52,␣1.58,␣1.61,␣1.64,␣1.68,␣1.73,␣1.81,␣2
>␣GWW_PDF<-function(par,xh){
+␣alpha=par[1]
+␣theta=par[2]
+␣gamma=par[3]
+␣lambda=par[4]
+␣A<-(alpha*(xh^gamma)*(1+(lambda^gamma)))
+␣p<-(1+(lambda^gamma))*alpha*gamma*(xh^(gamma-1))
+␣q<-(1-theta)*p*exp(-A)
+␣r<-(1-(theta*(1-exp(-A))))^2
+␣GWW_PDF<-q/r
+␣}
>␣###############␣GWW␣CDF␣##############################
>␣GWW_CDF<-function(par,xh){
+␣alpha=par[1]
+␣theta=par[2]
+␣gamma=par[3]
+␣lambda=par[4]
+
+␣A<-(alpha*(xh^gamma)*(1+(lambda^gamma)))
+␣m<-(1-theta)*(1-exp(-A))
```

```
+␣n<-(1-(theta*(1-exp(-A))))
+␣GWW_CDF<-m/n
+
+␣␣␣return(GWW_CDF)
+␣}
>␣#######␣Use␣the␣parameter␣values␣generated␣in␣the␣MLE␣results␣#########
>␣#␣␣alpha␣␣␣␣␣theta␣␣␣␣␣gamma␣␣␣␣lambda
>␣#0.3437477␣0.9397980␣3.2031520␣1.0058321
>␣fitg<-goodness.fit(pdf=GWW_PDF,cdf=GWW_CDF,data=xh,method="BFGS",starts=c(0.343
Warning␣message:
In␣ks.test(x␣=␣data,␣y␣=␣"cdf",␣par␣=␣as.vector(parameters))␣:
ties␣should␣not␣be␣present␣for␣the␣Kolmogorov-Smirnov␣test
>
>␣fitg
$W
[1]␣0.1056847
$A
[1]␣0.5912285
$KS
One-sample␣Kolmogorov-Smirnov␣test
data:␣␣data
D␣=␣0.099984,␣p-value␣=␣0.5546
alternative␣hypothesis:␣two-sided
$mle
[1]␣0.3437479␣0.9397974␣3.2031520␣1.0058321
$AIC
[1]␣32.06722
$`CAIC␣`
[1]␣32.75687
$BIC
```

```
[1]␣40.63976
$HQIC
[1]␣35.43884
$Erro
[1]␣4.44457904␣0.07409753␣0.93686727␣8.06151380
$Value
[1]␣12.03361
$Convergence
[1]␣0

>␣#################␣␣WEIGHTED␣WEIBULL
>␣WW_PDF<-function(par,xh){
+␣alpha=par[1]
+␣lambda=par[2]
+␣theta=par[3]
+
+␣A<-(alpha*(xh^theta))+(alpha*((lambda*xh)^theta))
+␣B<-(1+(lambda^theta))*alpha*theta
+␣C<-xh^(theta-1)
+␣WW_PDF<-(B*C*exp(-A))
+␣}
>
>␣#####␣WW␣cdf␣###
>␣WW_CDF<-function(par,xh){
+␣alpha=par[1]
+␣lambda=par[2]
+␣theta=par[3]
+␣K<-(alpha*(xh^theta))+(alpha*((lambda*xh)^theta))
+␣WW_CDF<-(1-exp(-K))
+␣}
```

```
>
> ####### Use the parameter values generated in the MLE results
>  ##alpha      lambda       theta
> ## 0.05978470 0.06885147 5.77971940
>
> fitg<-goodness.fit(pdf=WW_PDF,cdf=WW_CDF,data=xh,method="BFGS",starts=c(0.05978
Warning message:
In ks.test(x = data, y = "cdf", par = as.vector(parameters)) :
ties should not be present for the Kolmogorov-Smirnov test
>
> fitg
$W
[1] 0.2372656
$A
[1] 1.30385
$KS
One-sample Kolmogorov-Smirnov test
data:  data
D = 0.15232, p-value = 0.1075
alternative hypothesis: two-sided
$mle
[1] 0.05978470 0.06885147 5.77971940
$AIC
[1] 36.41368
$`CAIC `
[1] 36.82046
$BIC
[1] 42.84309
$HQIC
[1] 38.9424
```

```
$Erro
[1]␣␣␣0.02048009␣393.42498603␣␣␣0.57515587


$Value
[1]␣15.20684
$Convergence
[1]␣0
>␣#######␣EXPONENTIATED␣WEIBULL␣pdf#########
>␣EW_PDF<-function(par,xh){
+␣beta=par[1]
+␣gamma=par[2]
+␣lambda=par[3]
+
+␣W_PDF<-beta*gamma*(xh^(beta-1))*exp(-gamma*(xh^beta))
+␣W_CDF<-(1-exp(-(gamma)*(xh^beta)))
+␣EW_PDF<-lambda*(W_PDF)*(W_CDF)^(lambda-1)
+␣}
>
>␣##␣exponentiated␣weibull␣cdf###
>
>␣EG_CDF<-function(par,xh){
+␣beta=par[1]
+␣gamma=par[2]
+␣lambda=par[3]
+␣W_CDF<-(1-exp(-(gamma)*(xh^beta)))
+
+␣EG_CDF<-(W_CDF)^lambda
+
+␣}
>␣␣␣␣###␣␣beta␣␣␣␣␣␣gamma␣␣␣␣␣lambda
```

```
>␣###7.19678872␣0.02069777␣0.68379835

>

>fitg<-goodness.fit(pdf=EW_PDF,cdf=EG_CDF,data=xh,method="BFGS",starts=c(7.196788

Warning␣messages:

1:␣In␣goodness.fit(pdf␣=␣EW_PDF,␣cdf␣=␣EG_CDF,␣data␣=␣xh,␣method␣=␣"BFGS",␣␣:

$W

[1]␣0.2012911

$A

[1]␣1.118081

$KS

One-sample␣Kolmogorov-Smirnov␣test

data:␣␣data

D␣=␣0.14697,␣p-value␣=␣0.1315

alternative␣hypothesis:␣two-sided

$mle

[1]␣7.19678872␣0.02069777␣0.68379835

$AIC

[1]␣35.35377

$‘CAIC␣‘

[1]␣35.76055


$BIC

[1]␣41.78317

$HQIC

[1]␣37.88249

$Erro

[1]␣1.51167547␣0.02286453␣0.23091346

$Value

[1]␣14.67688

$Convergence
```

```
[1]␣0
>␣#######################␣␣␣WEIBULL␣PDF␣####
>␣W_PDF<-function(par,xh){
+␣beta=par[1]
+␣gamma=par[2]
+␣W_PDF<-␣beta*gamma*(xh^(beta-1))*exp(-(gamma)*(xh^beta))
+␣}
>␣#####␣weibull␣cdf␣####
>␣W_CDF<-function(par,xh){
+␣beta=par[1]
+␣gamma=par[2]
+␣W_CDF<-(1-exp(-(gamma)*(xh^beta)))
+␣}
>
>␣␣␣##beta␣␣␣␣␣␣gamma
>␣##5.78011790␣0.05976476
>␣fitg<-goodness.fit(pdf=W_PDF,cdf=W_CDF,data=xh,method="BFGS",starts=c(5.7801179
Warning␣message:
In␣ks.test(x␣=␣data,␣y␣=␣"cdf",␣par␣=␣as.vector(parameters))␣:
ties␣should␣not␣be␣present␣for␣the␣Kolmogorov-Smirnov␣test
>
>␣fitg
$W
[1]␣0.2372574
$A
[1]␣1.303806
$KS
One-sample␣Kolmogorov-Smirnov␣test
data:␣␣data
D␣=␣0.15227,␣p-value␣=␣0.1077
```

```
alternative␣hypothesis:␣two-sided
$mle
[1]␣5.78011793␣0.05976631
$AIC
[1]␣34.41368
$‘CAIC␣‘
[1]␣34.61368
$BIC
[1]␣38.69995
$HQIC
[1]␣36.09949
$Erro
[1]␣0.57503067␣0.02046513

$Value
[1]␣15.20684
$Convergence
[1]␣0
>␣##Exponentiated␣Generalised␣exponential␣weighted␣weibull␣PDF␣##
>␣PDF_EGNWWD<-function(par,xh){
+␣a=par[1]
+␣b=par[2]
+␣alpha=par[3]
+␣lambda=par[4]
+␣theta=par[5]
+␣A<-alpha*((xh)^theta)+alpha*(lambda*xh)^theta
+␣B<-1+(lambda^theta)
+␣p<-a*b*alpha*theta*((xh)^(theta-1))*B
+␣q<-(1-(1-exp(-(A))))^(a-1)
+␣r<-(1-(1-(1-exp(-(A))))^a)^(b-1)
```

```
+
+␣PDF_EGNWWD=p*q*r*exp(-(A))
+␣}
>␣###␣CDF_EGNWWD###
>␣␣CDF_EGNWWD<-function(par,xh){
+␣a=par[1]
+␣b=par[2]
+␣alpha=par[3]
+␣lambda=par[4]
+␣theta=par[5]
+
+␣H<-alpha*((xh)^theta)+alpha*(lambda*xh)^theta
+␣B<-1+(lambda^theta)
+
+␣CDF_EGNWWD=(1-(1-exp(-(H)))^a)^b
+
+␣}
>␣␣␣##a␣␣␣␣␣␣␣␣␣␣b␣␣␣␣␣␣alpha␣␣␣␣␣lambda␣␣␣␣␣␣theta
>␣##0.37950460␣0.67101465␣0.04390804␣0.77918145␣7.28690555
>␣fitg<-goodness.fit(pdf=PDF_EGNWWD,cdf=CDF_EGNWWD,data=xh,method="BFGS",starts=c
>␣fitg
$W
[1]␣19.36569
$A
[1]␣124.4331
$KS
One-sample␣Kolmogorov-Smirnov␣test
data:␣␣data
D␣=␣1,␣p-value␣<␣2.2e-16
alternative␣hypothesis:␣two-sided
```

```
$mle
[1]␣0.37950460␣0.67101465␣0.04390812␣0.77918145␣7.28690555
$AIC
[1]␣39.35105
$‘CAIC␣‘
[1]␣40.40368
$BIC
[1]␣50.06672
$HQIC
[1]␣43.56557
$Erro
[1]␣5.0177614␣0.2484242␣0.2096007␣9.4294440␣1.7040478
$Value
[1]␣14.67552
$Convergence
[1]␣0
>␣#####␣Oguntunde,␣the␣␣new␣weighted␣exponential␣distribution␣#####
>␣NMW_PDF<-function(par,xh){
+␣alpha=par[1]
+␣lambda=par[2]
+␣NMW_PDF<-(1+lambda)*alpha*(exp((1+lambda)*(-alpha*xh)))
+␣}
>␣###␣CDF_NMW###
>␣␣NMW_CDF<-function(par,xh){
+␣alpha=par[1]
+␣lambda=par[2]
+␣CDF_NMW<-(1-(exp((1+lambda)*(-(alpha)*xh))))
+␣}
>␣##alpha␣␣␣␣lambda
>␣##0.2496403␣1.6584020
```

```
>
>␣fitg<-goodness.fit(pdf=NMW_PDF,cdf=NMW_CDF,data=xh,method="BFGS",starts=c(0.249
>␣fitg
$W
[1]␣0.5702018
$A
[1]␣3.127042
$KS
One-sample␣Kolmogorov-Smirnov␣test
data:␣␣data
D␣=␣0.418,␣p-value␣=␣5.497e-10
alternative␣hypothesis:␣two-sided
$mle
[1]␣0.2496411␣1.6584020
$AIC
[1]␣181.6606
$‘CAIC␣‘
[1]␣181.8606
$BIC
[1]␣185.9469
$HQIC
[1]␣183.3464
$Erro
[1]␣␣5.37465␣57.23501

$Value
[1]␣88.83032
$Convergence
[1]␣0
>␣###␣exponentiated␣general␣PDF␣####
```

```
>␣EgW_PDFh<-function(par,xh){
+␣alpha=par[1]
+␣lambda=par[2]
+␣theta=par[3]
+
+␣A<-lambda*alpha*theta*(xh^(theta-1))*(exp(-(alpha)*xh^theta))
+␣B<-(1-(exp(-(alpha)*xh^theta)))^(lambda-1)
+␣EgW_PDFh<-A*B
+␣}
>
>␣###␣exponentiated␣general␣CDF␣####
>␣EgW_CDF<-function(par,xh){
+␣alpha=par[1]
+␣lambda=par[2]
+␣theta=par[3]
+␣K<-lambda*alpha*theta*(xh^(theta-1))*(exp(-(alpha)*xh^theta))
+␣EgW_CDF<-1-exp(-K)
+␣}
>␣##alpha␣␣␣␣lambda␣␣␣␣␣theta
>␣##0.0202602␣0.6793453␣7.2254723
>␣fitg<-goodness.fit(pdf=EgW_PDFh,cdf=EgW_CDF,data=xh,method="BFGS",starts=c(0.02
Error␣in␣if␣(cdf(par␣=␣starts,␣x␣=␣domain[2])␣!=␣1)␣warning("The␣cdf␣function␣inf
missing␣value␣where␣TRUE/FALSE␣needed
>␣fitg
$W
[1]␣0.5702018
$A
[1]␣3.127042
$KS
One-sample␣Kolmogorov-Smirnov␣test
```

```
data:␣␣data

D␣=␣0.418,␣p-value␣=␣5.497e-10

alternative␣hypothesis:␣two-sided

$mle

[1]␣0.2496411␣1.6584020

$AIC

[1]␣181.6606

$‘CAIC␣‘

[1]␣181.8606

$BIC

[1]␣185.9469


$HQIC

[1]␣183.3464

$Erro

[1]␣␣5.37465␣57.23501

$Value

[1]␣88.83032

$Convergence

[1]␣0

>␣library(AdequacyModel)

>␣library(pracma)

>␣###␣Additive␣weibull␣PDF␣####

>␣AddW_PDF<-function(par,xh){

+␣alpha=par[1]

+␣gamma=par[2]

+␣beta=par[3]

+␣theta=par[4]

+␣A<-alpha*theta*xh^(theta-1)

+␣B<-beta*gamma*xh^(gamma-1)
```

```
+␣C<-exp(-(alpha)*xh^theta-(beta*xh^gamma))
+␣AddW_PDF<-(A+B)*C
+␣}
>
>␣###␣additive␣weibull␣CDF␣####
>␣AddW_CDF<-function(par,xh){
+␣alpha=par[1]
+␣gamma=par[2]
+␣beta=par[3]
+␣theta=par[4]
+␣L<-exp(-(alpha)*xh^theta-(beta*xh^gamma))
+␣␣AddW_CDF<-(1-exp(L))
+␣}
>
>␣␣##alpha␣␣␣␣␣␣gamma␣␣␣␣␣␣␣beta␣␣␣␣␣␣theta
>␣##0.06239614␣7.43452591␣0.01831173␣2.7718383
alternative␣hypothesis:␣two-sided
$mle
[1]␣0.06239614␣7.43452591␣0.01831173␣2.77183830
$AIC
[1]␣35.46688
$‘CAIC␣‘
[1]␣36.15654
$BIC
[1]␣44.03942
$HQIC
[1]␣38.8385
$Erro
[1]␣0.03586973␣1.14553789␣0.01526985␣1.32158460
$Value
```

```
[1]␣13.73344
$Convergence
[1]␣0
```

```
APPENDIX␣A4:␣R-code␣for␣developing␣GWW␣Histogram␣for␣Strength␣of␣1.5cm␣Glass␣Fiber
library(AdequacyModel)
library(pracma)
########␣␣LOAD␣DATA␣INTO␣R␣␣#############
library(bbmle)
x<-c(0.55,␣0.93,␣1.25,␣1.36,␣1.49,␣1.52,␣1.58,␣1.61,␣1.64,␣1.68,␣1.73,␣1.81,␣2.0,␣
#################␣␣␣␣GWgW␣######
GWW_PDF<-function(x,␣alpha␣,theta,␣gamma,␣lambda){
A<-(alpha*(x^gamma)*(1+(lambda^gamma)))
p<-(1+(lambda^gamma))*alpha*gamma*(x^(gamma-1))
q<-(1-theta)*p*exp(-A)
r<-(1-(theta*(1-exp(-A))))^2
GWW_PDF<-q/r
return(GWW_PDF)
}

GWW_CDF<-function(x,␣alpha␣,theta,␣gamma,␣lambda){
A<-(alpha*(x^gamma)*(1+(lambda^gamma)))
m<-(1-theta)*(1-exp(-A))
n<-(1-(theta*(1-exp(-A))))
GWW_CDF<-m/n
return(GWW_CDF)
}

###############␣␣WgW␣␣␣############################
```

```
WgW_PDF<-function(x,␣alpha␣,lambda,␣theta){
A<-(alpha*(x^theta))+(alpha*((lambda*x)^theta))
B<-(1+(lambda^theta))*alpha*theta
C<-x^(theta-1)
WgW_PDF<-B*C*exp(-A)
return(WgW_PDF)
}
WgW_CDF<-function(x,␣alpha␣,lambda,␣theta){
A<-(alpha*(x^theta))+(alpha*((lambda*x)^theta))
WgW_CDF<-1-exp(-A)
return(WgW_CDF)


}


############␣␣EW␣␣#####
EW_PDF<-function(x,beta␣,gamma,␣lambda){
W_PDF<-beta*gamma*(x^(beta-1))*exp(-gamma*(x^beta))
W_CDF<-1-exp(-gamma*(x^beta))
EW_PDF<-lambda*(W_PDF)*(W_CDF)^(lambda-1)
return(EW_PDF)
}


EW_CDF<-function(x,beta␣,gamma,␣lambda){
W_CDF<-1-exp(-gamma*(x^beta))
EW_CDF<-(W_CDF)^lambda
return(EW_CDF)
}


################␣␣Weibull␣######
W_PDF<-function(x,␣beta,␣gamma){
```

```
W_PDF<-␣beta*gamma*(x^(beta-1))*exp(-gamma*(x^beta))
return(W_PDF)
}
W_CDF<-function(x,␣beta,␣gamma){
W_CDF<-(1-exp(-(gamma)*(x^beta)))
return(W_CDF)
}
##Exponentiated␣Generalised␣exponential␣weighted␣weibu##
EGWW_PDF<-function(x,␣a,␣b,␣alpha␣,lambda,␣theta){
A<-alpha*((x)^theta)+alpha*(lambda*x)^theta
B<-1+(lambda^theta)
p<-a*b*alpha*theta*((x)^(theta-1))*B
q<-(1-(1-exp(-A)))^(a-1)
r<-(1-(1-(1-exp(-A)))^a)^(b-1)
EGWW_PDF=p*q*r*exp(-A)
return(EGWW_PDF)
}
EGWW_CDF<-function(x,␣a,␣b,␣alpha␣,lambda,␣theta){
A<-alpha*((x)^theta)+alpha*(lambda*x)^theta

EGWW_CDF=(1-(1-exp(-A))^a)^b
return(EGWW_CDF)
}

#####␣Oguntunde,␣the␣␣new␣weighted␣exponential␣distribution###
NMW_PDF<-function(x,␣alpha,␣lambda){
NMW_PDF<-(1+lambda)*alpha*(exp((1+lambda)*(-(alpha)*x)))
return(NMW_PDF)
}
NMW_CDF<-function(x,␣alpha,␣lambda){
```

```
NMW_CDF<-(1-(exp((1+lambda)*(-(alpha)*x))))
return(NMW_CDF)
}

###␣Additive␣weibull␣PDF␣####
AddW_PDF<-function(x,␣alpha,gamma,␣beta,␣theta){
A<-alpha*theta*x^(theta-1)
B<-beta*gamma*x^(gamma-1)
C<-exp(-(alpha)*x^theta-(beta*x^gamma))
AddW_PDF<-(A+B)*C
return(AddW_PDF)
}
AddW_CDF<-function(x,␣alpha,gamma,␣beta,␣theta){
L<-exp(-(alpha)*x^theta-(beta*x^gamma))
AddW_CDF<-(1-exp(L))

return(AddW_CDF)
}

##############␣Histogram␣#
windows(height=10,width=10)
#par(mfrow=c(1,2))
hist(x,probability␣=T␣,main="",ylim=c(0,1.8))
curve(GWW_PDF(x,0.3437477,␣0.9397980,␣3.2031520,␣1.0058321),col="red",add=TRUE)
curve(WgW_PDF(x,0.05978470,␣0.06885147,␣5.77971940),col="blue",add=TRUE)
curve(EW_PDF(x,7.19678872,␣0.02069777,␣0.68379835),col="green",add=TRUE)
curve(W_PDF(x,5.78011790,␣0.05976476␣),col="yellow",add=TRUE)
curve(EGWW_PDF(x,0.37950460,␣0.67101465,␣0.04390804,␣0.77918145,␣7.28690555),col='
curve(NMW_PDF(x,0.2496403,␣1.6584020),col="gold",add=TRUE)
curve(AddW_PDF(x,0.06239614,␣7.43452591,␣0.01831173,␣2.77183831),col="gray",add=T
```

```
legend("topright",inset=c(0.001,0.01),cex=1.0,legend=c("Empirical","GWgW","WgW","I
windows(height=6,width=4)

plot(ecdf(x),ylab="CDF",main="")
curve(GWW_CDF(x,0.3437477,␣0.9397980,␣3.2031520,␣1.0058321),col="red",add=TRUE)
curve(WgW_CDF(x,0.05978470,␣0.06885147,␣5.77971940),col="blue",add=TRUE)
curve(EW_CDF(x,7.19678872,␣0.02069777,␣0.68379835),col="green",add=TRUE)
curve(W_CDF(x,5.78011790,␣0.05976476␣),col="yellow",add=TRUE)
curve(EGWW_CDF(x,0.37950460,␣0.67101465,␣0.04390804,␣0.77918145,␣7.28690555),col="
curve(NMW_CDF(x,␣0.2496403,␣1.6584020),col="gold",add=TRUE)
curve(AddW_CDF(x,0.06239614,␣7.43452591,␣0.01831173,␣2.77183831),col="gray",add=TI
legend("topright",inset=c(0.001,0.6),cex=1.0,legend=c("Empirical","GWgW","WgW","EI
```

APPENDIX␣A5:␣R-Code␣for␣Hazard␣rate␣of␣Geometric␣Weighted␣Weibull␣distribution

```
HAZARD_GWWD<-function(x,alpha,lambda,gamma,theta){
A<-(alpha*(x^gamma)*(1+(lambda^gamma)))
p<-(1+(lambda^gamma))*alpha*gamma*(x^(gamma-1))
q<-(1-theta)*p*exp(-A)
r<-(1-theta*(1-exp(-A)))^2
GWW_PDF<-q/r
CDF<-((1-theta)*(1-exp(-A)))/(1-(theta*(1-exp(-A))))
GWW_HAZARD<-(GWW_PDF/(1-CDF))
return(GWW_HAZARD)
}
windows(width=20,height=10)
par(mfrow=c(1,2))
curve(HAZARD_GWWD(x,0.1,1.4,1.4,0.9),0,10,col="blue",ylab=expression(paste('f',"(
curve(HAZARD_GWWD(x,1.1,0.1,0.3,0.6),0.10,col="green",add=TRUE,lty=1,lwd=1)
```

```
curve(HAZARD_GWWD(x,0.6,2.2,0.9,0.9),0,10,col="red",add=TRUE,lty=1,lwd=1)
legend("topright",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",0.1,"

curve(HAZARD_GWWD(x,1.2,0.5,0.5,0.9),0,10,col="blue",ylab=expression(paste('f',"(x
curve(HAZARD_GWWD(x,0.3,0.5,1.1,0.6),0.10,col="green",add=TRUE,lty=1,lwd=1)
curve(HAZARD_GWWD(x,0.8,0.7,0.6,0.9),0,10,col="red",add=TRUE,lty=1,lwd=1)
legend("topright",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",1.2,"
```

```
APPENDIX␣A6:␣R-Code␣for␣PDF␣plots␣for␣Geometric␣Weighted␣Weibull␣distribution␣for␣
PDF_GWWD<-function(x,alpha,lambda,gamma,theta){
A<-(alpha*(x^gamma)*(1+(lambda^gamma)))
p<-(1+(lambda^gamma))*alpha*gamma*(x^(gamma-1))
q<-(1-theta)*p*exp(-A)
r<-(1-theta*(1-exp(-A)))^2
GWW_PDF<-q/r
return(GWW_PDF)
}
windows(width=20,height=10)
par(mfrow=c(1,2))

curve(PDF_GWWD(x,0.1,0.6,1.6,0.9),0,10,col="blue",ylab=expression(paste('f',"(x)")
curve(PDF_GWWD(x,0.1,1.1,1.6,0.9),0.10,col="green",add=TRUE,lty=1,lwd=1)
curve(PDF_GWWD(x,0.1,3.0,1.1,0.9),0,10,col="red",add=TRUE,lty=1,lwd=1)
legend("topright",inset=c(0.03),cex=1.0,legend=c(expression(paste(alpha,"=",0.1,"

curve(PDF_GWWD(x,0.1,1.9,1.4,0.1),0,10,col="blue",ylab=expression(paste('f',"(x)")
curve(PDF_GWWD(x,0.1,0.3,2.2,0.9),0.10,col="green",add=TRUE,lty=1,lwd=1)
curve(PDF_GWWD(x,0.1,0.1,1.9,0.9),0,10,col="red",add=TRUE,lty=1,lwd=1)
legend("topright",inset=c(0.05),cex=1.0,legend=c(expression(paste(alpha,"=",0.1,"
```

```
APPENDIX␣A7:␣R-Code␣for␣analysing␣Maximum␣Likelihood␣Estimates␣of␣GWW␣for␣Failure␣
library(AdequacyModel)
library(pracma)
########␣␣LOAD␣DATA␣INTO␣R␣␣#############
library(bbmle)
xh<-c(2.75,␣0.13,␣1.47,␣0.23,␣1.81,␣0.30,␣0.65,␣0.10,␣3.00,1.73,␣1.06,␣3.00,␣3.00
#################␣␣␣␣GWgW␣␣␣␣####################################
LLh<-function(alpha␣,theta,␣gamma,␣lambda){
A<-(alpha*(xh^gamma)*(1+(lambda^gamma)))
p<-(1+(lambda^gamma))*alpha*gamma*(xh^(gamma-1))
q<-(1-theta)*p*exp(-A)
r<-(1-(theta*(1-exp(-A))))^2
GWW_PDF<-q/r
NLLh<--sum(log(GWW_PDF))
return(NLLh)}
fith<-␣mle2(LLh,␣start=list(alpha=1.21,theta=0.2,␣gamma=1.56,␣lambda=0.82),␣metho
fith
summary(fith)
AIC(fith)
BIC(fith)
vcov(fith)
#################␣WWW␣####################################################
LLh<-function(alpha␣,lambda,␣theta){
A<-(alpha*(xh^theta))+(alpha*((lambda*xh)^theta))
B<-(1+(lambda^theta))*alpha*theta
C<-xh^(theta-1)
#WgW_CDF<-1-exp(-A)
WgW_PDFh<-B*C*exp(-A)
NLLh<--sum(log(WgW_PDFh))
return(NLLh)}
```

```
fith<- mle2(LLh, start=list( alpha=1.8 ,lambda=1.3, theta=1.5), method="BFGS", da
fith
summary(fith)
AIC(fith)
BIC(fith)
vcov(fith)
##### Oguntunde, the  new weighted exponential distribution #####
LL<-function(alpha, lambda){
PDF<-(1+lambda)*alpha*(exp((1+lambda)*(-alpha*xh)))
NLL<--sum(log(PDF))
return(NLL)}
fit<- mle2(LL, start=list( alpha=1.2,lambda=1.13), method="BFGS", data=list(xh))
fit
summary(fit)
AIC(fit)
BIC(fit)
vcov(fit)
######### additive weibull #####################
LLh<-function(alpha, gamma, beta, theta){
A<-alpha*theta*xh^(theta-1)
B<-beta*gamma*xh^(gamma-1)
C<-exp(-alpha*xh^theta-(beta*xh^gamma))
PDF<-(A+B)*C
NLL<--sum(log(PDF))
return(NLL)}
fit<-mle2(LLh,start=list(alpha=2.2, gamma=2.1, beta=1.4, theta=0.1), method="BFGS
fit
summary(fit)
AIC(fit)
BIC(fit)
```

```
vcov(fit)



APPENDIX␣A8:␣R-Code␣for␣goodness-of-fit␣of␣GWW␣for␣Failure␣and␣Running␣Times␣Data

>␣library(AdequacyModel)

>␣library(pracma)

>␣xh<-c(2.75,␣0.13,␣1.47,␣0.23,␣1.81,␣0.30,␣0.65,␣0.10,␣3.00,1.73,␣1.06,␣3.00,␣3.

>␣################␣␣␣␣GWgW

>␣GWW_PDF<-function(par,xh){

+␣alpha=par[1]

+␣theta=par[2]

+␣gamma=par[3]

+␣lambda=par[4]

+␣A<-(alpha*(xh^gamma)*(1+(lambda^gamma)))

+␣p<-(1+(lambda^gamma))*alpha*gamma*(xh^(gamma-1))

+␣q<-(1-theta)*p*exp(-A)

+␣r<-(1-(theta*(1-exp(-A))))^2

+␣GWW_PDF<-q/r

+␣}


>␣###############␣GWW␣CDF␣#############################

>␣GWW_CDF<-function(par,xh){

+␣alpha=par[1]

+␣theta=par[2]

+␣gamma=par[3]

+␣lambda=par[4]

+␣A<-(alpha*(xh^gamma)*(1+(lambda^gamma)))

+␣m<-(1-theta)*(1-exp(-A))

+␣n<-(1-(theta*(1-exp(-A))))

+␣GWW_CDF<-m/n
```

```
+␣␣␣return(GWW_CDF)
+␣}
>
>␣#######␣Use␣the␣parameter␣values␣generated␣in␣the␣MLE␣results␣#########
>␣#######
>␣␣##␣alpha␣␣␣␣␣theta␣␣␣␣␣gamma␣␣␣␣lambda
>␣##0.5667310␣0.8188510␣0.9563676␣1.0413797
>␣fitg
$W
[1]␣0.2638773
$A
[1]␣1.63814
$KS
One-sample␣Kolmogorov-Smirnov␣test
data:␣␣data
D␣=␣0.18717,␣p-value␣=␣0.244
alternative␣hypothesis:␣two-sided
$mle
[1]␣0.5667324␣0.8188497␣0.9563681␣1.0413800
$AIC
[1]␣97.1928
$‘CAIC␣‘
[1]␣98.7928
$BIC
[1]␣102.7976
$HQIC
[1]␣98.98582
$Erro
[1]␣24.7117756␣␣0.1856016␣␣0.2667750␣93.1579972
$Value
```

```
[1]␣44.5964
$Convergence
[1]␣0
>␣#################␣␣WEIGHTED␣WEIBULL
>␣WW_PDF<-function(par,xh){
+␣alpha=par[1]
+␣lambda=par[2]
+␣theta=par[3]
+␣A<-(alpha*(xh^theta))+(alpha*((lambda*xh)^theta))
+␣B<-(1+(lambda^theta))*alpha*theta
+␣C<-xh^(theta-1)
+␣WW_PDF<-(B*C*exp(-A))
+␣}
>
>␣#####␣WW␣cdf␣###
>␣WW_CDF<-function(par,xh){
+␣alpha=par[1]
+␣lambda=par[2]
+␣theta=par[3]
+␣K<-(alpha*(xh^theta))+(alpha*((lambda*xh)^theta))
+␣WW_CDF<-(1-exp(-K))
+␣}
>␣#######␣Use␣the␣parameter␣values␣generated␣in␣the␣MLE␣results
>␣###␣␣alpha␣␣␣␣␣lambda␣␣␣␣␣␣theta
>␣###␣0.43633231␣0.06396642␣1.26504503
One-sample␣Kolmogorov-Smirnov␣test
data:␣␣data
D␣=␣0.21944,␣p-value␣=␣0.1112
alternative␣hypothesis:␣two-sided
$mle
```

```
[1]␣0.43633245␣0.06396637␣1.26504515
$AIC
[1]␣98.31747
$‘CAIC␣‘
[1]␣99.24054
$BIC
[1]␣102.5211
$HQIC
[1]␣99.66223
$Erro
[1]␣␣␣␣␣␣␣NaN␣␣␣␣␣␣␣NaN␣0.2044279
$Value
[1]␣46.15873
$Convergence
[1]␣0
>␣#####␣Oguntunde,␣the␣␣new␣weighted␣exponential␣distribution␣#####
>␣NMW_PDF<-function(par,xh){
+␣alpha=par[1]
+␣lambda=par[2]
+
+␣NMW_PDF<-(1+lambda)*alpha*(exp((1+lambda)*(-alpha*xh)))
+␣}
>␣###␣CDF_NMW###
>␣␣NMW_CDF<-function(par,xh){
+␣alpha=par[1]
+␣lambda=par[2]
+
+␣CDF_NMW<-(1-(exp((1+lambda)*(-(alpha)*xh))))
+
+␣}
```

```
>␣###␣alpha␣␣␣␣lambda
>␣###␣0.2848870␣0.9827704
>␣fitg
$W
[1]␣0.3215435
$A
[1]␣1.905793
$KS
One-sample␣Kolmogorov-Smirnov␣test
data:␣␣data
D␣=␣0.21607,␣p-value␣=␣0.1214
alternative␣hypothesis:␣two-sided
$mle
[1]␣0.2848870␣0.9827704
$AIC
[1]␣98.27007
$‘CAIC␣‘
[1]␣98.71452
$BIC
[1]␣101.0725
$HQIC
[1]␣99.16658
$Erro
[1]␣10.47275␣72.88965
$Value
[1]␣47.13504
$Convergence
[1]␣0
>␣####␣additive␣weibull␣###
>␣AddW_PDF<-function(par,xh){
```

```
+␣alpha=par[1]
+␣gamma=par[2]
+␣beta=par[3]
+␣theta=par[4]
+␣A<-alpha*theta*xh^(theta-1)
+␣B<-beta*gamma*xh^(gamma-1)
+␣C<-exp(-(alpha)*xh^theta-(beta*xh^gamma))
+␣AddW_PDF<-(A+B)*C
+␣}
>␣###␣additive␣weibull␣CDF␣####
>␣AddW_CDF<-function(par,xh){
+␣alpha=par[1]
+␣gamma=par[2]
+␣beta=par[3]
+␣theta=par[4]
+␣L<-exp(-(alpha)*xh^theta-(beta*xh^gamma))
+␣␣AddW_CDF<-(1-exp(L))
+␣}
>␣␣###␣alpha␣␣␣␣␣␣gamma␣␣␣␣␣␣␣beta␣␣␣␣␣␣theta
>␣␣###␣0.6542442␣␣1.2650600␣-0.2044464␣␣1.2650558
>␣fitg<-goodness.fit(pdf=AddW_PDF,cdf=AddW_CDF,data=xh,method="BFGS",starts=c(0.6
Error␣in␣c(0.6542442,␣1.26506,␣-0.2044464,␣1.2650558,␣)␣:
argument␣5␣is␣empty
>␣fitg
$W
[1]␣0.3215435
$A
[1]␣1.905793
$KS
One-sample␣Kolmogorov-Smirnov␣test
```

```
data:␣␣data
D␣=␣0.21607,␣p-value␣=␣0.1214
alternative␣hypothesis:␣two-sided
$mle
[1]␣0.2848870␣0.9827704
$AIC
[1]␣98.27007
$‘CAIC␣‘
[1]␣98.71452
$BIC
[1]␣101.0725
$HQIC
[1]␣99.16658
$Erro
[1]␣10.47275␣72.88965
$Value
[1]␣47.13504
$Convergence
[1]␣0
```