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QUASICONVEX FUNCTIONS ON TIME SCALES AND APPLICATIONS

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QUASICONVEX FUNCTIONS ON TIME SCALES AND APPLICATIONS

BY

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DECLARATION

Student

I hereby declare that this thesis is my original work and no part of it has been presented for any degree in this university or elsewhere.

…………………………................ Date: ……………...............

GREGORY ABE-I-KPENG

Supervisor

I hereby declare that the preparation and presentation of the thesis was supervised in accordance with the guidelines on supervision of thesis laid down by the University for Development Studies:

..................……………………….. Date: …………….........

DR. MOHAMMED MUNIRU IDDRISU

ABSTRACT

In this thesis, the notion of quasiconvex functions on time scales and some properties are established. The subdifferential for quasiconvex function on time scales is presented as well as some properties regarding quasiconvex function. Some Jensen's inequalities for quasiconvex functions on time scales are also given with some applications. The study again proves that Jensen's inequality holds for quasiconcave monetary utility function in conjunction with convex, concave, quasiconvex and quasiconcave functions. Jensen's inequality in addition holds for monetary utility functions with respect to quasiconvex and quasiconcave functions that are linear fractional functions.

ACKNOWLEDGEMENT

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Finally, my heartfelt appreciation goes to all my course mates and lecturers especially Mr. Kwara Nantomah who have contributed in diverse ways to my success. May God bless you all.

DEDICATION

This thesis is dedicated to my lovely mum Elizabeth Pengfo Kuzagh.

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CHAPTER ONE

INTRODUCTION

1.0 Introduction

This chapter gives an introduction and background to the study. It briefly presents the problem statement and outlines the objectives as well as the research questions of the study. The scope and the limitations of the study are stated and the chapter concluded with information on the organization of the thesis.

1.1Background of the study

Convexity is a concept which can be traced back to Archimedes in connection with his famous estimate of the value of π and has a great indirect impact on our everyday life through applications in industry, business, medicine, etc. (Niculescu and Persson, 2006).

Convexity plays a critical role in many areas of mathematics such as graph theory, partial differential equations, discrete mathematics, probability theory, and coding theory as well as in areas outside mathematics such as chemistry, physics, biology and other sciences (Dwilewicz, 2009).

Convex and quasiconvex analysis is the study of sets with some algebraic and topological properties. A quasiconvex function is a real-valued function defined on an interval or a convex subset of a real vector space such that the inverse image of the form (-∞, ∞) is a convex set ("Quasiconvex function", n.d.).

Quasiconvex programming is an aspect of optimization, introduced with the aim of curbing the weakness of convex programming and is applied to solving problems in meshing, scientific computing, information visualization, automated algorithm analysis and robust statistics (Eppstein, 2005).They are relevant in the study of optimization problems where they are differentiated by a number of suitable properties.

Dinu (2008) defined the notion of a convex function on time scales and established some results connecting this notion with the notion of functions on a classic interval and convex sequences.

Stefan Hilger in his PhD thesis introduced the concept of time scales in 1988 in order to hybridize continuous and discrete analysis (Bohner and Peterson, 2001). Many results of problems can easily be carried from the continuous case to the discrete case, but others seem to be completely impossible. The study on time scales exposes such discrepancies and helps us to understand the difference between the two cases. Thus, time scale calculus is a very important tool in many computational and numerical applications.

This time scale calculus has received a lot of attention in recent times and its applications are quite substantial. The most important ones among others include the dynamic equations, which involve both differential and difference equations, which are of great relevance in biology, mathematical modeling and engineering. Other applications are economics, neural networks, physics, optimization which have come lately (Dinu, 2008).

From the optimization perspective, it can be revealing to model a problem which incorporates decision space which has both continuous and discrete nature, namely, an arbitrary closed subset of reals. A natural question to ask is whether it is possible to provide a framework which allows us to get some understanding of the nature of the problem and their solutions.

The answer is yes especially for dynamic systems due to the recently developed theory of "dynamic systems in time scales" (Kaymakcalan et al, 1996).

Aulbach and Hilger initiated the development of time scales or measure chain (the union of disjoint closed intervals of R) with the aim of treating dynamic problems from the qualitative point of view (Bohner and Peterson, 2001). Later KaymakCalan et al (1996) extended this theory to a unified analysis of nonlinear systems from the point of view of qualitative and quantitative behavior of such systems. Most of the results are contained in the monograph written by KaymakCalan et al, which is the earliest text containing extensive coverage in the area of time scales. Recently, Bohner and Peterson presented new results in the area in their monographs, which give very detailed insight (Gray, 2007).

1.2 Problem statement

Bell intimated that a major task of mathematics is to harmonize the continuous and the discrete analysis to include them in a general mathematical framework in order to eliminate obscurity from both (Bohner and Peterson, 2001).

The theory of timescales was introduced by Hilger in1988 in order to unite continuous and discrete analysis. The theory has received a lot of attention with researchers investigating into areas such as dynamic equations, inequalities and some functions such as gamma and convex functions.

In the paper of Dinu (2008) on convex functions on time scales, a larger class of functions called quasiconvex functions was not investigated and this thesis seeks to extend the discussion to include such functions.

1.3 Objectives

1.3.1Main Objective

The main objective of this study is to explore the notion of quasi convex functions on time scales.

1.3.2 Specific Objectives

This study seeks specifically to:

- Present some definitions of quasiconvexity in the context of time scales together with some remarks.
- Establish results connecting the notion of quasi convex functions on time scales.
- Examine and present the sub-differential of a quasiconvex function on time scale.
- Present some Jensen inequalities for quasiconvex and quasiconcave functions on time scales with some applications.

1.4 Research questions

1. Can time scales calculus as a tool be used to characterize the concept of quasiconvex functions and to what extent can results connecting the notion of quasiconvex functions on time scales be established?
2. Is there a subdifferential for a quasiconvex function on time scale?
3. Can Jensen inequalities for quasiconvex and quasiconcave functions on time scales be established with some applications?

1.5 Scope of study

The scope of the study is limited to quasiconvex functions on time scales with emphasis on the set of integers and reals as well as some time scales. The study is also confined to single variable functions of uni-dimensional range.

1.6 Significance of the study

At the end of the study, this thesis will

- extend the frontiers of knowledge in the area of mathematical analysis.
- bring to fore some notions in quasi-convex functions on time scales and their applications.

1.7 Limitations of the study

One major limitation is the inability to access some other relevant literature from certain non-free journals due to financial constraint.

1.8 Organization of the study

The study is organized into five chapters. Chapter One provides an overview of the research undertaken in this study. Chapter Two entitled "Literature Review" summarizes some research work done in the area of time scale calculus and quasiconvexity. The methods and materials utilized to carry out this study are described in Chapter Three. Chapter Four considers the results and discussion of the study. Chapter Five is devoted to summary of findings, conclusions and recommendation.

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

This chapter reviews literature in the areas of time scales and quasiconvex analysis. In this regard, it begins with the groundbreaking exploits of Hilger and then to some related works in quasiconvexity.

Background concepts on the theory of time scales are taken from Bohner and Peterson (2001) and Gray (2007) and that of convex and quasiconvex analysis from Dinu (2008), Greenberg and Pierskalla (1970) and Crouzeix (2005). The review on time scales is categorized into three areas namely: dynamic equations, inequalities and functions.

2.1 Dynamic Equations on Time Scales

In 1988, Hilger introduced the theory of time scales calculus in order to hybridize continuous and discrete analysis. This concept has received considerable attention in recent times.

Agarwal et al (2002) in a survey of dynamic equations on time scales presented various properties of exponential function on an arbitrary time scales and used it to solve linear dynamic equations of the first order. They considered examples and applications especially the insect population model and used the exponential function to define hyperbolic and trigonometric functions to solve linear dynamic equations of second order with constant coefficients.

Stability and instability for dynamic equations on time scales has been studied by Hoffacker and Tisdell (2005). They used Lyapunov functions to develop an invariance principle regarding solutions of first order system of equations.

Jackson (2006) presented findings regarding partial dynamic equations on time scales and generalized existing ideas of the univariate case of time scales calculus to the bivariate case. He discovered that, in particular, solutions of the homogeneous and nonhomogeneous heat and wave operators are found when initial distributions are given in terms of elementary functions by means of the generalized Laplace Transform for the time scale setting.

In his PhD thesis, Jackson in 2007, studied general linear systems theory on time scales by considering Laplace transforms, stability, controllability, observability and realizability. He provided sufficient conditions for a given function to be transformable and an inverse formula for the transform. Also, he presented sufficient conditions for the inverse transform to exist and developed an analogue of the convolution theorem for arbitrary time scales as well as algebraic properties of the convolution. He investigated applications of the transform to linear time invariant systems and linear time varying systems.

Zaidi (2009) studied the existence and uniqueness of solutions to nonlinear first order dynamic equations in his PhD thesis. He presented a series of results regarding non-multiplicity, existence, uniqueness and successive approximations to solutions of first order dynamic equations on time scales that modeled nonlinear phenomena of hybrid stop-start nature.

Ucar et al (2012) worked on stability of dynamic equations on time scales via dichotomic maps to check the stability of ordinary differential equations and difference equations and extended the method to dynamic equations on time scales. Using dichotomic and strictly dichitomic maps they examined the stability and asymptotic stability to the trivial solution of the first order system of dynamic equations.

2.2 Inequalities on Time Scales

Bohner and Kaymakcalan (2001) worked on opial inequalities on time scales. They pointed out some of its applications to dynamic equations and offered various extensions of their inequality.

Time scale integral inequalities were studied by Anderson in 2005. He extended some recent and classical integral inequalities to the general time scale calculus including the inequalities of Steffesen, Iyenger, Cebysev and Hermite-Hadamard.

Li (2005) investigated certain new dynamic inequalities on time scales which provide explicit bounds on unknown functions. His results unify and extend some continuous inequalities and their corresponding analogues.

Under the supervision of B. Kaymakcalan, Gray (2007) studied opial's inequality on time scales and an application in his MSc. Thesis. He gave an example that concerns upper bound estimates of dynamic initial value problems and illustrated the usage of the developed dynamic opial inequality.

Agarwal et al (2007) presented a survey on inequalities on time scales. They gave time scales versions of the inequalities: Holder, Cauchy-Schwarz, Minkowski, Jensen, Gronwall, Bernoulli, Bihari, Opial, Wirtinger and Lyapunov. Ostrowski inequalities on time scales were studied by Bohner and Mathews in 2008. They applied their results to the quantum calculus case. Liu and Bohner (2010) presented Gronwall-Oulang-type integral inequalities on time scales. Their results contained continuous Gronwall-type inequalities and their discrete analogues for some special cases. They also extended the Gronwall-type inequalities to multiple integrals.

Xu et al (2010) worked on some integral inequalities on time scales and their applications. They established some new dynamic inequalities and observed that their results unified and extended some continuous inequalities and their discrete analogues. In 2010, Saker studied some nonlinear dynamic inequalities and applications. He gave some sufficient conditions for global existence and an estimate of the rate of decay of solutions obtained.

In their work on dynamic inequalities on time scales in permanence of predator-prey system, Hu and Wang (2012) provided conditions for permanence of predator-prey system incorporating a refuge on time scales. Numerical simulations were presented to illustrate the feasibility and effectiveness of their results.

Agarwal et al (2014), in their monograph discussed extensively dynamic inequalities on time scales. They established some fundamental inequalities on time scales such as Young's inequality, Jensen's inequality, Holder's inequality, Minkowski's inequality, Steffensen's inequality, Cebysev's inequality, Opial's inequality, Lyapunov's inequality, Halanay's inequality and Wirtinger's inequality. In 2016, Pachpatte obtained the estimates on some dynamic integral inequalities in three variables which can be used to study certain dynamic equations.

2.3 Some Functions on Time Scales

Bohner and Guseinov (2005) presented an introduction to complex functions on products of two time scales. They studied the concept of analyticity for complex-valued functions of a complex time scale variable and derived a time scale counterpart of the classical Cauchy-Riemann equations. They introduced complex line delta and nabla integrals along time scales curves and obtained a time scale version of the classical Cauchy integral theorem.

Lyapunov functions for linear nonautonomous dynamical equations on time scales were investigated by Kloeden and Zmorzynska in 2006. The existence of Lyapunov function was established following a method of Yoshizawa for the uniform exponential asymptotic stability of the zero solution of nonautonomous linear dynamic equation on a time scale with uniformly bounded graininess.

A survey on exponential functions on time scales was investigated by Bohner and Peterson in 2007. They gave several recent results concerning this important function and stated some relevant properties regarding it. They also use this function to solve first order linear dynamic equations and second order linear dynamic equations with constant coefficients. They also solved certain second order linear dynamic equations with variable coefficients using exponential functions and discussed Euler-Cauchy dynamic equations on time scales.

Kapcak (2007), in his MSc thesis, studied analytic functions on time scales. He worked on continuous, discrete and semi-discrete analytic functions and developed completely nabla differentiability, nabla analytic functions on the product of two time scales and Cauchy-Riemann equations for nabla case.

In 2008, Dinu defined the notion of convex functions on time scales. He presented some results connecting this notion with that of convex function on a classic interval and convex sequences. The subdiffrential of a convex function on time scale was defined and some properties regarding it presented. Bohner and Karpuz (2013) studied gamma function on time scales. They introduced the generalized gamma function on time scales and proved some of its properties which coincide with the ones known in continuous case. They also defined an appropriate factorial function for computing the values of the generalized gamma function in some special cases.

2.4 Quasiconvex Analysis

Many properties regarding convex functions have appeared in the literature since the pioneering work of Jensen. Some results have been obtained in recent times for a larger class of functions called quasiconvex functions. Greenberg and Pierskalla (1970) in their review summarized in condensed form results regarding quasiconvex functions and provided some refinements to gain further generality. In their work, they clarified the structure underlying quasiconvex functions by presenting analogues to properties of convex functions.

Continuity and differentiability of quasiconvex functions have been studied by Crouzeix (2005). He obtained a result that the convexity of the epigraph of a convex function induces important properties with respect to the continuity and differentiability of the function. Moreover, the function is locally Lipschitz in the interior of the domain of the function. Also, he stated an important property that quasiconvex functions are locally nondecreasing with respect to some positive cone.

Daniilidis et al (2002) introduced a subdifferential that is related to quasiconvex functions in a similar way that the Fenchel-Moreau subdifferential is related to the convex ones. They showed that this quasiconvex subdifferential is always a cyclically quasimonotone operator that coincides with the Fenchel-Moreau subdifferential whenever the function is convex.

In conclusion, time scale calculus, a relatively new theory has received a lot of attention from researchers and students establishing results that unify and extend continuous and discrete analysis. Based on this notion of time scales, there has been extensive development in areas of ordinary calculus such as dynamic equations, linear theory, uniqueness and existence of solutions, inequalities and functions.

Applications in insect population and prey-predator modeling via time scales have been noted. Deep insights can be gained in the field of fluid mechanics by modeling problems in the time scale setting. This is possible since some work has been done on partial differential equations on time scales.

It is also worth noting that much has not been done in employing time scale calculus in solving optimization problems and so this creates opportunities for further research in order to advance the development and applications of this great mathematical concept.

CHAPTER THREE

METHODOLOGY

3.0 Introduction

This chapter captures the methods and tools utilized in this study. These methods are theoretical and analytical in nature and limited to time scales calculus and quasiconvex analysis.

In this chapter we outline central concepts and definitions of the time scale calculus initiated by Hilger in 1988 under the supervision of Bernd Aulbach. Throughout this chapter the similarities and differences in considering the time scale as in the $\mathbb{R}$ and $\mathbb{Z}$ setup are remarked. Attention is given to the concepts such as continuity, Rd- Continuity, differentiability which are relevant in the analysis of hybrid continuous and discrete systems. Basic concepts of convex and quasiconvex functions are briefly discussed.

3.1 Time scale calculus

A time scale (which is a special case of a measure chain) is an arbitrary non empty closed subset of real numbers (together with the topology of subspace of $\mathbb{R}$). Thus the real numbers ($\mathbb{R}$), the integers ($\mathbb{Z}$), the natural numbers ($\mathbb{N}$) and the non-negative integers ($\mathbb{N}_0$) are examples of time scales as well as [0, 1] U [2, 3], [0, 1] U $\mathbb{N}$, and the cantor set. The rational number ($\mathbb{Q}$), the irrational number ($\mathbb{R}/\mathbb{Q}$), the complex number ($\mathbb{C}$) and the open interval between 0 and 1 are not time scales. The calculus of time scales was initiated by Stefan Hilger in his PhD thesis (Bohner and Peterson, 2001) in order to create a theory that can unite discrete and continuous analysis. Introducing the delta derivative f^{Δ} for a function f defined on $\mathbb{T}$, and it turns out that;

(i) $f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and

(ii) $f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

In this section we introduce the basic notion connected with time scales and differentiability of functions on them and consider the above two cases as examples. The general theory is applicable to many more time scales $\mathbb{T}$. The definitions of the forward and backward jump operators are given.

Definition 3.1 Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, the mapping $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, such that

$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}$ and

$\rho(t) = \sup\{s \in \mathbb{T}: s < t\}$

are called the forward and backward jump operators respectively. In this definition we put

$\inf\emptyset = \sup \mathbb{T}\ (\sigma(t) = t \text{ if } \mathbb{T} \text{ has a maximum } t)$ and

$\sup\emptyset = \inf \mathbb{T}\ (\text{ie, } \rho(t) = t \text{ if } \mathbb{T} \text{ has a minimum } t)$, where $\emptyset$ denotes the null set.

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are right scattered and left scattered at the same time are isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense (Bohner and Peterson, 2001).

Throughout this thesis, time scale is denoted by $\mathbb{T}$ and for any interval $\mathbb{I}$ of $\mathbb{R}$, $\mathbb{I}_{\mathbb{T}} = \mathbb{I} \cap \mathbb{T}$, is called a time scale interval.

Let $\mathbb{T}_k = \mathbb{T} | \{m\}$ if $\mathbb{T}$ has a right-scattered minimum m; otherwise $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, then define $\mathbb{T}^k = \mathbb{T} | \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

These jump operators enable us to classify the points {t} of a time scale as right-dense and left- scattered depending on whether $\sigma(t) = t, \sigma(t) > t, \rho(t) = t$ and $\rho(t) < t$, respectively for any $t \in \mathbb{T}$ (see Table 1.1 and Figure 1.1)

Table 1.1 Classifications of Points

t right − scattered (rs)	$t < \sigma(t)$
t right − dense (rd)	$t = \sigma(t)$
t left − scattered (ls)	$\rho(t) < t$
t left − dense (ld)	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

Source: Bohner and Peterson, 2001

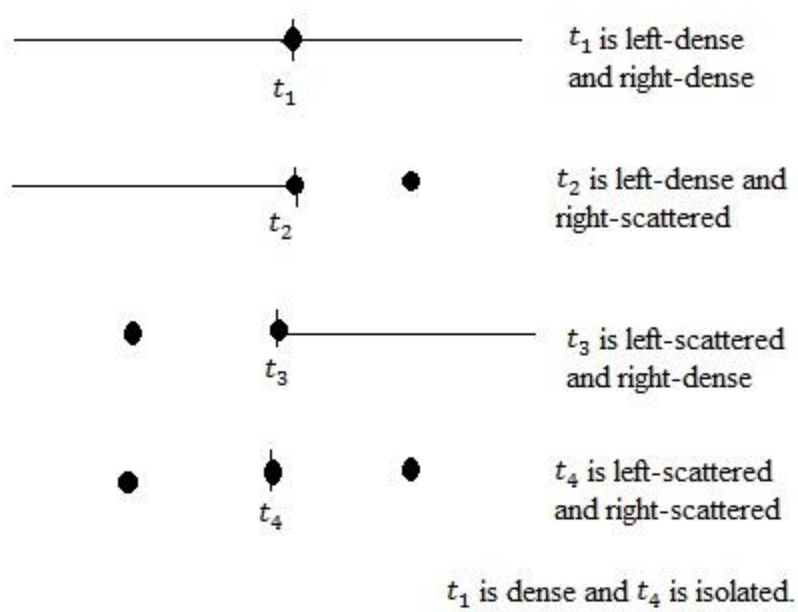


Figure 1.1 Classifications of points (Source: Bohner and Peterson, 2001)

From the definition above, both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. This is because of the assumption that $\mathbb{T}$ is a closed subset of $\mathbb{R}$.

To illustrate the classification of points in a time scale, consider the following example

$\mathbb{T} := \{s \in \mathbb{R}: -1 \leq s \leq 0\} \cup \left\{\frac{1}{s}: s \in \mathbb{N}\right\} \cup \{s \in \mathbb{R}: 2 \leq s \leq 3\} \cup \{4\}.$

The points which are:

- Right dense and left dense: all $s \in [-1,0] \cup [2,3]$.
- Right dense and left scattered: 2,4
- Right scattered and left dense: 3
- Right scattered and left scattered: all $\frac{1}{s}: s \in \mathbb{N}$.

Here,-1 is a minimal point and 4 is a maximal point respectively. Hence, $\rho(-1) = -1$, thereby implying -1 to be a left dense point and $\sigma(4) = 4$, implying that 4 is also a right dense point.

Table 1.2 Examples of Time Scales

$\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$\mathbb{Z}$	$t+1$	$t-1$	1
$h\mathbb{Z}$	$t+h$	$t-h$	h
$q^{\mathbb{N}}$	qt	$\frac{t}{q}$	$(q-1)t$
$2^{\mathbb{N}}$	$2t$	t	t
$\mathbb{N}_0^2$	$(\sqrt{t}+1)^2$	$(\sqrt{t}-1)^2$	$2\sqrt{t}+1$

Source: Bohner and Peterson, 2001

Definition 3.2 The mapping $\mu: \mathbb{T} \to \mathbb{R}^+$ such that $\mu(t) = \sigma(t) - t$ is called graininess.

When $\mathbb{T} = \mathbb{R}, \mu(t) \equiv 0$ and for $\mathbb{T} = \mathbb{Z}, \mu(t) \equiv 1$.

Definition 3.3 The mapping $v\colon \mathbb{T}_k \to \mathbb{R}_0^+$ such that $v(t) = t - \rho(t)$ is called backwards graininess.

Remark 3.1

(1) The direction in a time scale has not been used in any symmetric manner (both in positive and negative directions), thus, we will consider the direction for a time-scale $\mathbb{T}$ to be in the sense of increasing values of t, for $t \in \mathbb{T}$.

(2) If a time-scale $\mathbb{T}$ has a maximal element, which is moreover left-scattered, then this point plays a particular role in several respects and therefore we call it degenerate. All other elements of $\mathbb{T}$ are called non-degenerate and the subset of non-degenerate points of $\mathbb{T}$ is denotes by $\mathbb{T}^k$. Since each closed subset of A of time scale $\mathbb{T}$ is also time scale, it is possible that A^k can be formed. Naturally $A^k = A$ is possible as long as A does not have a left-scattered maximum. Thus, $\mathbb{T}_k$ is defined as the set

$\mathbb{T}_k = \mathbb{T}[\inf \mathbb{T}, \sigma(\inf \mathbb{T})]$ if $\inf \mathbb{T} < \infty$, and

$\mathbb{T} = \mathbb{T}$ if $\inf \mathbb{T} \equiv -\infty$.

Likewise $\mathbb{T}^k$ is defined as the set

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \mid [\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if} \sup \mathbb{T} < \infty \\ \mathbb{T}, & \text{if} \sup \mathbb{T} = \infty \end{cases}$$

If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$ otherwise $\mathbb{T}^k = \mathbb{T}$.

Finally, if $f\colon \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma\colon \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f^{\sigma(t)}$ for all $t \in \mathbb{T}$ i.e, $f^\sigma = fo\sigma$, where, o signifies some arbitrary binary operation.

Example 3.1 Consider the following three examples

$\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = h\mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

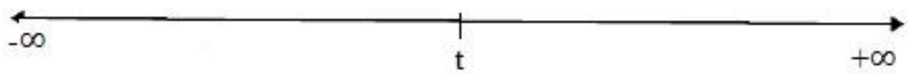
$\sigma(t) = \inf\{s \in \mathbb{R}: s > t\} = \inf(t, \infty) = t.$

Similarly

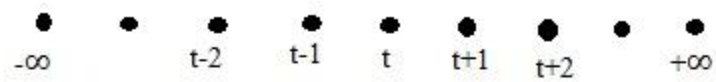
$\rho(t) = \sup\{s \in \mathbb{R}: s < t\} = \sup(-\infty, t) = t$

Hence, every point $t \in \mathbb{R}$ is dense. The graininess function μ is

$\mu(t) \equiv \sigma(t) - t \equiv t - t \equiv 0$ for all $t \in \mathbb{T}$.



(ii) if $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$



$\sigma(t) = \inf\{s \in \mathbb{Z}: s > t\} = \inf\{t + 1, t + 2, t + 3, \dots\} = t + 1$

Similarly $\rho(t) = \sup\{\dots t - 3, t - 2, t - 1\} = t - 1$

Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ in this case is

$\mu(t) \equiv t + 1 - t \equiv 1$ for all $t \in \mathbb{T}$.

(iii) If $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, then we have for any $t \in h\mathbb{Z}$

$\sigma(t) = \inf\{s \in t: s > t\} = \inf\{t + h, t + 2h, t + 3h, \dots\} = \inf\{t + nh: n \in \mathbb{N}\}$

$= t + h$

And similarly $\rho(t) = \sup\{s \in \mathbb{T}: s < t\} = \sup\{t - nh: n \in \mathbb{N}\} = t - h.$

Thus every point $t \in \mathbb{T}$ is isolated and

$\mathbb{N}(t) = \sigma(t) - t = t + h - t \equiv h$ for all $t \in \mathbb{T}$.

From the three cases discussed in Example 3.1, the graininess function is a constant function. The graininess function plays a crucial role in the analysis on time scale. For the general case, many formulas will have some term containing the factor $\mu(t)$. In various cases this fact is the reason for certain differences between the continuous and the discrete case. One example that illustrates this is the so-called Scalar Riccati equation on a general time scale $\mathbb{T}$ (Bohner and Peterson, 2001).

$$z^{\Delta} + q(t) + \frac{Z^2}{P(t)+\mu(t)Z} = 0.$$

Note that if $\mathbb{T} = \mathbb{R}$, then we get the well-known Riccati differential equation (Bohner and Peterson, 2001)

$$\frac{dz}{dt} + q(t)\frac{1}{p(t)}z^2 = 0,$$

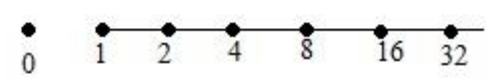
and if $\mathbb{T} = \mathbb{Z}$, then we get the Riccati difference equation (Bohner and Peterson, 2001)

$$\Delta z + q(t) + \frac{z^2}{p(t)+z} = 0.$$

For the general time scale, the graininess function might become a function of $t \in \mathbb{T}$.

Example 3.2 For each of the following time scales $\mathbb{T}$, we can find σ, ρ and μ, and classify each point $t \in \mathbb{T}$ as left-dense, less-scattered, right-dense, or right-scattered:

(i) $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$.



$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} = \inf\{2^n : n \in [m+1, \infty)\}$$

$$= 2^{m+1} = 2^m . 2^1 = 2.2^m = 2t$$

for all $t \in \mathbb{T}$ if $2^m = t$.

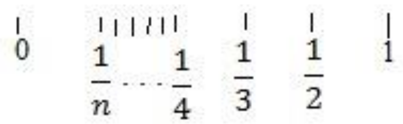
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} = \sup\{2^n : n \in (-\infty, m-1]\}$$

$$= 2^{m-1} = \frac{2^m}{2} = \frac{t}{2} \text{ for all } t \in \mathbb{T}$$

$$\mu(t) = \sigma(t) - t = 2t - t = t$$

Thus, $\frac{t}{2} < t < 2t$. Hence t is both right-scattered and left-scattered and thus it is isolated. $\mathbb{T}$ is dense if $t = 0$ and isolated otherwise.

(ii) $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$



- At the point $t = 0$

  $\sigma(0) = \inf\{s \in \mathbb{T} : s > 0\} = \inf\left\{1, \frac{1}{2}, \frac{1}{3} \ldots\right\}$

  And $\rho(0) = \sup\{s \in \mathbb{T} : s < 0\} = \sup\emptyset = \inf \mathbb{T} = 0$

- Since $\sigma(1) = \inf\emptyset = \sup \mathbb{T} = 1$, we obtain $\sigma(1) = 1$

- $t \in \left\{\frac{1}{n}\right\}_{n=2}^{\infty}$, $\sigma(t) = \inf\left\{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-1}\right\} = \frac{1}{(n-1)} = \frac{1}{\frac{1-t}{t}} = \frac{t}{1-t}$

  if $n = \frac{1}{t}$ for all $t \in \mathbb{T}$

  For $t \in \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$, $\rho(t) = \sup\left\{\frac{1}{n+1}, \frac{1}{n+2}, \ldots\right\}$

  $\rho(t) = \frac{1}{(n+1)} = \frac{1}{\frac{1}{t}+1} = \frac{1}{\frac{t+1}{t}} = \frac{t}{1+t}$, if $n = \frac{1}{t}$ for all $t \in \mathbb{T}$.

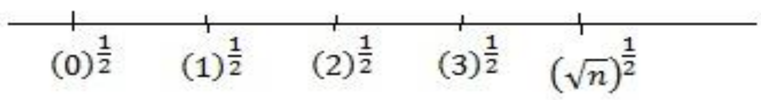
It is clear that,

$$\sigma(t) = \begin{cases} 0 & t = 0 \\ \dfrac{t}{t-1} & t \leftarrow \left\{\dfrac{1}{n}\right\}_{n=2}^{\infty} \\ 1 & t = 1 \end{cases}$$

Similarly,

$$\rho(t) = \begin{cases} 0, & t = 0 \\ \dfrac{t}{1+t}, & t \in \left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \dots\right\} \end{cases}$$

(iii). $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$.



$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} = \inf\{\sqrt{n} : n \in [m+1,)\} = \sqrt{m+1} = \sqrt{t+1}$$

for all $t \in \mathbb{T}$ if $t = m$.

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} = \sup\{\sqrt{n} : n \in (0, m-1]\} = \sqrt{m-1} = \sqrt{t-1}$$

for all $t \in \mathbb{T}$

$$\mu(t) = \sqrt{t+1} - t$$

Clearly,

$$\sigma(t) = \begin{cases} 1, & t = 0 \\ \sqrt{t+1}, & t \in \{\sqrt{n}\}_{n=1}^{\infty} \end{cases}$$

$$\rho(t) = \begin{cases} 0, & t = 0 \\ \sqrt{t-1}, & t \in \{\sqrt{n}\}_{n=2}^{\infty} \end{cases}$$

Theorem 3.1 (Induction Principle)

Let $t \in \mathbb{T}$ and assume that

$\{s(t): t \in [t_0, t)\}$ is a family of statements satisfying:

I. The statement $s(t_0)$ is true.

II. If $t \in [t_0, \infty)\}$ is right-scattered and $s(t)$ is true, then $S(\sigma(t))$ is also true.

III. If $t \in [t_0, \infty)$ is right-dense and $s(t)$ is true, then there is a neighborhood $\cup$ of t such that $S(s)$ is true for all $s \in \cup \cap (t, \infty)$.

IV. If $t \in (t_0, \infty)$ is left-dense and $S(s)$ is true for all $s[t_0, t)$, then $S(t)$ is true for all $t \in [t_0, \infty)$.

Proof:

Let $S^* = \{t \in [t_0, \infty): s(t) \text{is not true}\}$ we want to show $S^* = \emptyset$. To achieve a contradiction we assume $S^* = \emptyset$. But since S^* is closed and non-empty, we have

$\inf S^* = t^* \in \mathbb{T}$.

We claim that $S(t^*)$ is true. If $t^* = t_0$, then $S(t^*)$ is true from (i); if $t^* \neq t_0$ and $p(t^*) = t^*$, then $S(t^*)$ is true from (ii). Hence, in any case, $t^* \notin S^*$. Thus, t^* cannot be right-scattered, and $t^* \neq \max\mathbb{T}$ either. Hence t^* is right-dense. But now (iii) leads to a contradiction.

Theorem 3.2 Let $t_0 \in \mathbb{T}$ and assume that $\{s(t): t \in (-\infty, t_0]$ is a family of statements satisfying:

I. The statement $s(t_0)$ is true.

II. If $t \in (-\infty, t_0]$ is left-scattered and $s(t)$ is true, then $S(p(t))$ is also true.

III. If $t \in (-\infty, t_0]$ is left-dense and $s(t)$ is true, then there is a neighborhood $\cup$ of t such that $S(r)$ is true for all $r \in \mathcal{U} \cap (-\infty, t]$.

IV. If $t \in (-\infty, t_0]$ is right-dense and $S(r)$ is true for all $r \in (-\infty, t_0]$, then $S(t)$ is true.

Proof:

Let $S^* = \{t \in (-\infty, t_0]: s(t) \text{is not true}\}$. We want to show $S^* \neq \emptyset$. to achieve a contradiction we assume $S^* \neq \emptyset$. But since S^* is closed and non-empty, we have

Sup $S^* = t_0 \in \mathbb{T}$.

We claim that $S(t_0)$ is true. If $t^* = t_0$, then $S(t^*)$ is true from (i) if $t^* \neq t_0$ and $\rho(t_0) = t_0$, then $S(t_0)$ is true from (iv). Finally if $\rho(t_0) < t_0$, then $S(t_0)$ is true from (ii). Hence, in any case, $t_0 \notin S^*$. Thus,t_0 cannot be left-scattered, and $t_0 \neq \min\mathbb{T}$ either. Hence t_0 is left-dense. But (iii) leads to a contradiction. See (Bohner and Peterson, 2001) for comprehensive and detailed discussion of the theory of time scale calculus.

3.2 Order and Topological Structure

As subsets of $\mathbb{R}$, time scales carry an order structure in a canonical way. A time scale $\mathbb{T}$ may be bounded below or above. As a consequence of $\mathbb{T}$ being embedded in $\mathbb{R}$, all other theoretical notions such as bounds, least upper bounds, greatest lower bounds and intervals are available in $\mathbb{T}$ as they are in $\mathbb{R}$.

The order and topological structure of any time scale $\mathbb{T}$ is induced by that of $\mathbb{R}$. On time scales, there exist primarily three order structure, namely least upper bounds, greatest lower bounds and interval. (i.e. Time scale interval).

As a consequence of the definition of a time scale $\mathbb{T}$ being a closed subset of $\mathbb{R}$, topological structure of $\mathbb{T}$, especially from the openness point of view has several features. Clearly any subset B of $\mathbb{T}$ which is open in $\mathbb{R}$, is also open in $\mathbb{T}$.

The reverse is generally not true, though as the simple example $\mathbb{T} = \mathbb{Z}$ shows where any subset in the induced topology is open in $\mathbb{T}$ but not open in $\mathbb{R}$. This is taken care of by distinguishing between $\mathbb{R}$-openness and $\mathbb{T}$-openness. In order to investigate the details of the notion of openness in time scales, we define the concept of neighborhood. We give two different versions of neighborhood definitions, differentiating between the concepts of $\mathbb{R}$-neighborhood and $\mathbb{T}$-neighborhood, giving way to $\mathbb{R}$-openness and $\mathbb{T}$-openness.

Given a time scale$\mathbb{T}$, $t \in \mathbb{T}$ and $\delta > 0$, we denote

$\mathbb{R}_\delta(t) \coloneqq \{y \in \mathbb{R}: t - \delta < y < t + \delta\}$

$\mathbb{T}_\delta(t) \coloneqq \{y \in \mathbb{R}: t - \delta < y < t + \delta\}$

as the δ-neighborhoods of t in $\mathbb{R}$ and $\mathbb{T}$ respecyively. An interval, in the time scale context, is always understood as the intersection of a real interval with a given time scale. For any interval $\mathbb{I}$ of $\mathbb{R}$ (open or closed), $\mathbb{I}_\mathbb{T} = \mathbb{I} \cap \mathbb{T}$ a time scale interval. The following definition will lead to the concept of openness. For detailed study of the order and topological structure of time scales refer to Atasever (2011) and Gray (2007).

Definition 3.5 Let $\mathbb{T}$ be a time scale and $t \in \mathbb{T}$. The set of $\subseteq \mathbb{R}$ is called an $\mathbb{R}$-neighborhood of t provided that there is $\delta > 0$ with $\mathbb{R}_\delta(t) \subseteq \mathcal{U}$. The set $\mathcal{V} \in \mathbb{T}$ is called a $\mathbb{T}$-neighborhood of t, provided that there is $\delta > 0$ with $\mathbb{T}_\delta(t) \subseteq V$.

Neighborhood concepts give rise to further topological notions.

Theorem 3.3 Let $\mathbb{T} = \mathbb{R}$. Every neighborhood is an open set.

Proof:

Consider a neighborhood $\mathbb{T} = \mathbb{N}_r(t)$ and let s be any point of $\mathbb{T}$. Then there is a positive real number h such that

$d(t, s) = r - h$

For all points u such that $d(s, u) < h$, we have

$d(t, u) \leq d(t, s) + d(s, u) < r - h + h = r$

So that $u \in \mathbb{T}$. Thus s is an interior point of $\mathbb{T}$.

Definition 3.6 A subset A of time scale $\mathbb{T}$ is open in $\mathbb{T}$ if for each $t \in A$ there is a $\delta > 0$ such that $\mathbb{T}_\delta(t) \subseteq A$.

Remark 3.2 For any time scale $\mathbb{T}$, $\emptyset$ and $\mathbb{T}$ are open in $\mathbb{T}$.

Example 3.4 For the time scale $\mathbb{T} \coloneqq \{s \in \mathbb{R}: -1 \leq s \leq 0\} \cup \left\{\frac{1}{n}: n \in \mathbb{N}\right\}$, any $a \in \mathbb{T}$,

$a > 0$ is of the form $a = \frac{1}{n}$, for some $n \in \mathbb{N}$ and since

$\mathbb{T}_\delta(a) = \{x: \mathbb{T}: a - \delta < s < a + \delta] = \{a\} \subseteq \{a\}$, for all $a > 0, \{a\}$ is a $\mathbb{T}$-open set.

Similarly: for $-1 \leq b \leq c \leq 1; (b, c)$ is a $\mathbb{T} -$ open set, and since

$\mathbb{T}_\delta(-1) = \{s: \mathbb{T}: -1 - \delta \leq s < -1 + \delta\} \subseteq [-1, d], d > 0$ and

$\mathbb{T}_\delta(d) = \{s \in \mathbb{T}: d - \delta \leq s < d + \delta\} = \{d\} \subseteq [-1, d]$

We observe that $[-1, d]$ is a $\mathbb{T} -$ open set for $d > 0$. Likewise, $[-1, d)$ is a $\mathbb{T} -$ open for all $d < 0$. For all $e \in \mathbb{T}, e > 0, \mathbb{T} \setminus \{e\}$ is $\mathbb{T} -$ open, since for any

$t \in \mathbb{T}\setminus\{e\}, (t \neq e)$,

$\mathbb{T}_\delta(t) = \{s \in \mathbb{T}: t - \delta < s < t + \delta\} \subseteq \mathbb{T}\setminus\{e\}$

On the other hand, for $a \in \mathbb{T}, a \leq 0, \{a\}$ is not $\mathbb{T} -$open, since

$\mathbb{T}_\delta(a) = \{x \in \mathbb{T}: a - \delta < x < a + \delta\} \nsubseteq \{a\}$.

Example 3.5 For the time scale $\mathbb{T} \coloneqq \{s \in \mathbb{R}: -1 \leq s \leq 0\} \cup \left\{\frac{1}{n}: n \in \mathbb{N}\right\}$ the set $A = \{\frac{1}{n}: n \in \mathbb{N}\}$ is open in $\mathbb{T}$. This can easily be verified by observing that A is an arbitrary union of sets.

The next result assists us to observe a connection between the concepts of the form $\left\{\frac{1}{n}\right\}$, which are shown to be $\mathbb{T} -$ open for each $n \in \mathbb{N}$ by example 3.4. Hence A is also $\mathbb{T} -$ open.

Example 3.6 Let $\mathbb{T} \coloneqq \{s \in \mathbb{R}: -1 \leq s \leq 0\} \cup \left\{\frac{1}{n}: n \in \mathbb{N}\right\}$ be a time scale. We see from example 3.5 that the set $A = \{\frac{1}{n}: n \in \mathbb{N}\}$ is $\mathbb{T} -$ open. Thus, by Theorem 3.4 there should be a set B, open in $\mathbb{R}$, such that $A = B \cap \mathbb{T}$. Clearly, the set $B = \cup I_n$, where $I_n = \left\{s \in \mathbb{R}: \frac{1}{2n} \leq s \leq \frac{3}{2n}\right\}$ for each $n \in \mathbb{N}$ being an $\mathbb{R}$-open interval with this property: $B \cap \mathbb{T} = \left\{\frac{1}{n}: n \in \mathbb{N}\right\} = A$ and B is $\mathbb{R}$-open.

Definition 3.6 A subset $\mathbb{T}$ is called closed in $\mathbb{T}$ provided that $\mathbb{T}/_A$ is open in $\mathbb{T}$.

Definition 3.7 A subset A of a time scale $\mathbb{T}$ is called compact in $\mathbb{T}$ provided that A is bounded and closed in $\mathbb{T}$.

Next, we consider the concept of connectedness for time scale in the example below:

Example 3.7 Consider the time scale $\mathbb{T} = \{s \in \mathbb{R}: -1 \leq s \leq 0\} \cup \left\{\frac{1}{n}: n \in \mathbb{N}\right\}$. It is clear that $\{1\}$ is both open and closed in $\mathbb{T}$ resulting from Examples 3.5 and 3.6. Thus, $\mathbb{T}$ can be written as a disjoint union of non-empty $\mathbb{T}$-open sets, giving way to the disconnectedness of this particular time scale $\mathbb{T}$.

However, there exists obviously a connected time scale such as $\mathbb{T} = \mathbb{R}$. There is no single notion that applies to all time scales and thus we can say that a time scale $\mathbb{T}$ may or may not be connected. The concept of jump operators is employed to deal with this topological deficiency.

Remark 3.3

(1) Time scale can be used in both directions (positive and negative) in a symmetric manner. However, it is not necessary to do that; hence we will consider the direction for time scale $\mathbb{T}$ to be in the sense of increasing or decreasing values of $t \in \mathbb{T}$.

(2) If a time scale $\mathbb{T}$ has a maximal element which is left scattered, then this point plays a particular role in several respects and therefore is referred to as degenerate. All other elements of $\mathbb{T}$ are called non-degenerate and the subset of non-degenerate points of $\mathbb{T}$ is denote by $\mathbb{T}^k$. Since each closed subset A of a time scale $\mathbb{T}$ is also a time scale it is possible that A^k can be formed. Naturally, $A^k = A$ is possible as long as A does not have left scattered maximum.

3.3 Continuity, Rd-Continuity and Ld-Continuity

In order to describe and introduce classes of functions that are integrable, the notion related to the approximation of continuous functions by step functions is relevant.

Definition 3.7 If a function is defined on a compact interval $[t_a, t_b]$ of a time scale $\mathbb{T}$ and if there is a finite number of elements $t_0, t_1, \ldots, t_n$ of $\mathbb{T}$ with $t_a = t_0 < t_1 < \cdots < t_n = t_b$ such that $f: [t_a, t_b] \to \mathbb{R}$ is constant on $[t_i, t_{i+1}]$, for $i = 1,2, \ldots, n-1$, then f is called a step function.

It is possible to define the continuity concept for $\mathbb{R}^n$-valued functions on time scales. The continuity definition can be adopted from Real analysis without any major changes.

Definition 3.8 The function $f: \mathbb{T} \rightarrow \mathbb{R}$ is said to be continuous at $t_0 \in \mathbb{T}$ for all $\epsilon > 0$, if there exists a neighbourhood $N_\epsilon(t_0)$ such that

$|f(t) - f(t_0)| < \epsilon$ for all $t \in N_\epsilon(t_0)$.

Points of discontinuity are usually given by jump points graphically since time scales are not generally connected, a similar analysis is not necessary. In order to pave way for the concept of integration, we first have to obtain an appropriate class of functions having anti-derivatives. For this reason, the following notions are defined.

Definition 3.9 Let X be an arbitrary topological space and $\mathbb{T}$ a time scale. The mapping $g: \mathbb{T} \rightarrow X$ is said to be regulated if at each left dense $t \in \mathbb{T}$, $g(t^-) = \lim_{s \to t^-} g(s)$ exists and at each right dense point $t \in \mathbb{T}$, $g(t^+) = \lim_{s \to t^+} g(s)$ exists.

Definition 3.10 The mapping $g: \mathbb{T} \rightarrow X$ is called rd-continuous if

(i) It is continuous at each left dense or maximal $t \in \mathbb{T}$.

(ii) At each left dense point, left sided limit $g(t^-)$ exists.

We denote by $C_{rd}[\mathbb{T}, X]$ the set of rd-continuous mappings from $\mathbb{T}$ to X. The class of rd-continuous functions turns out to a "natural" class within the context of time scale calculus. The function $u: \mathbb{T} \rightarrow X$ in the case of $\mathbb{T} = [0,1] \cup \mathbb{N}$, for example is rd-continuous but not continuous at 1.

The following are implications from Definitions 3.8, 3.9 and 3.10.

Continuous ⇒ rd-continuous⇒ regulate.

If $\mathbb{T}$ contains left dense and right scattered points, then the first implication is not invertible. However, on a discrete time scale all three notions coincide.

As a generalization of Definition 3.10, the following is given:

Definition 3.11 The mapping $f: \mathbb{T}^k \times X \to X$ is called rd-continuous if

(i) It is continuous at each (t, x) with right dense or maximal t, and

(ii) The limits $f(t^-)$: $\lim_{(s,t)\to(t,x)} f(s, y), s < y$ and $\lim_{y\to x} f(t, y)$ exist at each (t, x) with left dense t.

Hence, in general for left dense t, the function $f(t,\cdot): \mathbb{T}^k \times X \to X$ is no way a continuous continuation of the mapping $f: (-\infty, t) \times X \to X$ to the point t.

Example 3.8 Given an rd-continuous function $g: \mathbb{T} \to X_1$, which is in the sense of Definition 3.10, let $h: X_2 \to X_3$ and $h: X_1 \times X_2 \to X_3$ be continuous functions, then the composite function $f(g(\cdot), h(\cdot))$ is rd-continuous in the sense of Definition 3.11.

This section is concluded by introducing a tool which is useful with some qualitative properties and relevant definition.

Definition 3.12 Consider the mapping $f^\tau: (-\infty, \tau] \times X \to X$ which is defined for a fixed $\tau \in \mathbb{T}^k$ as:

$$f^\tau(t, x) = \begin{cases} f(t, x) & \text{if } (t, x) \in (-\infty, \tau) \times X \\ f(\tau^-, x) & \text{if } (t, x) \in \{\tau\} \times X \end{cases}$$

Here f is assumed to be rd-continuous on $\mathbb{T} \times X$. f^τ does not necessarily coincide with f on $(-\infty, \tau]$ if τ is a left dense right scattered (ldrs) point, otherwise it does.

Definition 3.13 Let $f: \mathbb{T} \to \mathbb{R}$ be a function. f is ld-continuous at each left dense point in $\mathbb{T}$ and $\lim_{x\to t^+} f(x)$ exists as a finite number for all right dense points $t \in \mathbb{T}$.

3.4 Delta Derivative

Considering functions which are defined on a time scale $\mathbb{T}$ and taking their values in a topological space X, the concept of continuity arises due the embedding of $\mathbb{T}$ in $\mathbb{R}$. For that of differentiation, however, the topological structure of $\mathbb{T}$ plays an important role. The lack of openness of $\mathbb{T}$ generally requires a procedure which leads to special cases of differential calculus and difference calculus (Gray, 2007).

Definition 3.14 For $f: \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of f in t, to be the number denoted by $f^\Delta(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood V of t such that

$$\left|\left[f(\sigma(t)) - f(s)\right] - f^\Delta(t)[\sigma(t) - s]\right| < \varepsilon|\sigma(t) - s|$$

for all $s \in V_{\mathbb{T}}$.

$f: \mathbb{T} \to X$, whose X is any Banach space, is called delta differentiable if f is differentiable for each $t \in \mathbb{T}$.

Remark 3.4 In the two special cases $\mathbb{T}$ and $\mathbb{Z}$ the delta derivative is uniquely determined. In fact, one gets $a = \frac{df(t)}{dt}$ and $a = f(t+1) - f(t)$ respectively.

Theorem 3.4 Let $f: \mathbb{T} \to \mathbb{R}$ be delta differentiable and $t \in \mathbb{T}^k$, then the following properties arise:

(i). If f is delta differentiable at t, then f is continuous at t.

(ii). If f is left continuous at t, and t is right-scattered, then f is delta differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t) - f(t)}{\mu(t)}$$

(iii). If t is right-dense, then f is delta differentiable at t, if and only if the limit

$\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^\Delta(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$

(iv). If f is delta differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$.

Remark 3.5 Sometimes we may need the generalized delta derivatives corresponding to Dini derivatives from the right, in which case we write

$$f(\sigma(t)) - f(s) - (\sigma(t) - s)a < \varepsilon(\sigma(t) - s)$$

for all $s \in U$; U being a right-neighbourhood of $t \in \mathbb{T}$. This is denoted by $a = D^{+}f^{\Delta}(t)$. Note that if t is right-scattered, then $D^{+}f^{\Delta}(t)$ is the same as the $f^{\Delta}(t)$ given above. In this case, we write

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

For $\mathbb{T} = \mathbb{R}$, we have

$f^{\Delta}(t) = f^{1}(t)$ (Usual derivative) and

$f^{\Delta}(t) = \Delta f$ (Forward difference operator) if $\mathbb{T} = \mathbb{Z}$.

Example 3.8 Let $\mathbb{T} = \{s \in \mathbb{R}: -2 \leq s \leq 0\} \cup \left\{\frac{1}{n}: n \in \mathbb{N}\right\}$ and $f(s) = |s|$,

$f_n(s) = \left|s - \frac{1}{n}\right|$, $n \in \mathbb{N}$, for all $t \in \mathbb{T}$.

Then all functions f_n's are delta differentiable for all points of $\mathbb{T}$, but the limit function f has no delta derivative at 0, hence it is not delta differentiable.

Example 3.9 For $\mathbb{T} = \{0\} \cup \bigcup_{n\in\mathbb{N}} \left\{s \in \mathbb{R}: \frac{1}{2n} \leq s \leq \frac{1}{2n-1}\right\}$

and $f(s) = \begin{cases} 0, & \text{if } s = 0 \\ \frac{1}{2n}, & \text{if } \frac{1}{2n} \leq s \leq \frac{1}{2n-1} \end{cases}$

which gives, $f^{\Delta}(0) = 1$.

3.5 Nabla Derivative

Following the development of delta dynamic equations, the corresponding theory for nabla derivatives was extensively studied. See Atasever (2011).

Definition 3.15 For $f\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$, we define the nabla derivative of f in t, to be the number denoted by $f^{\nabla}(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|\left[f\big(\rho(t)\big) - f(s)\right] - f^{\nabla}(t)[\rho(t) - s]\right| < \varepsilon|\rho(t) - s|$$

for all $s \in U_{\mathbb{T}}$.

Theorem 3.5 Suppose that $f\colon \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}_k$, then the following properties hold:

(i). If f is nabla differentiable at t, then f is continuous at t.

(ii). If f is right continuous at t, and t is left-scattered, then f is nabla differentiable at t with

$$f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{v(t)}$$

(iii). If t is left-dense, then f is delta differentiable at t, if and only if the limit

$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$ exists as a finite number. In this case, $f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$

(iv). If f is nabla differentiable at t, then $f(\rho(t)) = f(t) - v(t)f^{\nabla}(t)$

Theorem 3.6 Assume that $f, g\colon \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_k$, then:

(i). The sum $f + g\colon \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with

$$(f + g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t).$$

(ii). The product $fg\colon \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t and we get the product rule

$$(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f^{\rho}(t)g^{\nabla}(t) = f(t)g^{\nabla}(t) + f^{\nabla}g^{\rho}(t)$$

(iii). If $g(t)g^{\rho}(t) \neq 0$, then $\frac{f}{g}$ is nabla differentiable at t, and we get the quotient rule:

$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{g(t)f^{\nabla}(t) - f(t)g^{\nabla}(t)}{g(t)g^{\nabla}(t)}$$

3.6 Antiderivative and Integral

Having discussed the development of the concepts of time scale analysis up to delta and nabla differentiability, we consider the concepts of delta and nabla antiderivative and integration. For this purpose, we restrict ourselves to the class of differentiable functions and consider the definition of antidifferentiation.

Once the main theorem which guarantees the existence for rd-continuous (ld-continuous for nabla derivative) functions is established, the concept of Cauchy-Integral can be introduced.

$\mathbb{T}$ is a time scale and $\mathbb{T}^1$ is a subinterval of $\mathbb{T}$ in the discourse below.

Definition 3.16 Let $f: \mathbb{T}^1 \to \mathbb{R}$ be a delta differentiable function. The function

$$f^{\Delta} = \begin{cases} (\mathbb{T})^k \to \mathbb{R} \\ t \to f^{\nabla}(t) \end{cases}$$

is called the delta differentiable of f on $\mathbb{T}^1$. In case $\mathbb{T} = \mathbb{T}^1$, the statement "on $\mathbb{T}^1$" disappears.

Remark 3.6

(i) From Theorem 3.6, it is clear that a mapping which is delta differentiable on $\mathbb{T}^1$ is continuous.

(ii) If $\mathbb{T}^1 \in \mathbb{R}$ is an interval which is open in $\mathbb{R}$; then the above concept coincides with the usual differentiation.

Definition 3.17 A function $f: \mathbb{T}^k \to \mathbb{R}$ is called a delta antiderivative of g on $\mathbb{T}^1$ and for all $t \in \mathbb{T}^k$ the condition $f^{\nabla}(t) = g(t)$ is satisfied.

For each rd-continuous function on time scale, there corresponds a delta antiderivative as shown in the following theorem.

Theorem 3.7 For any rd-continuous mapping $g: \mathbb{T}^k \to \mathbb{R}$, there exists a delta antiderivative function $f: \mathbb{T} \to \mathbb{R}$, such that

$$f: t \to \int_s^t g(s)\Delta s,\ s, t \in \mathbb{T}^k.$$

Definition 3.18 Suppose that the function $g: \mathbb{T}^k \to \mathbb{R}$ has a delta antiderivative function f on $[r, s] \in \mathbb{T}$, then

$$\int_r^s g(t)\Delta t = f(s) - f(r)$$

is called the Cauchy-Integral from r to s of the function g. For $\mathbb{T} = \mathbb{R}$, the Cauchy-Integral coincides with the Riemann integral.

For $\mathbb{T} = h\mathbb{Z}$, where $h > 0$, the identity

$$\int_r^s g(t)\Delta t = \begin{cases} \sum_{i=\frac{r}{n}}^{\frac{s}{n}-1} g(ih)h & \text{if } s > r \\ 0 & \text{if } s = r \\ -\sum_{i=\frac{r}{n}}^{\frac{s}{n}-1} g(ih)h & \text{if } s > r \end{cases}$$

can be shown.

Definition 3.19 A function $F: \mathbb{T} \to \mathbb{R}$ is called nabla antiderivative of $f: \mathbb{T} \to \mathbb{R}$ provided

$$F^{\nabla}(t) = f(t) \text{ holds for all } t \in \mathbb{T}_k.$$

We define the integral by

$$\int_a^t f(\tau)\nabla\tau = F(t) - F(a), \text{ for all } t \in \mathbb{T}.$$

Theorem 3.8 Suppose that f and f^{∇} are continuous, then

$$\left(\int_{a}^{t} f(t,s)\nabla s\right)^{\nabla} = f(\rho(t),t) + \int_{a}^{t} f^{\nabla}(t,s)\nabla s$$

Theorem 3.9 Assume that $f:\mathbb{T}\to\mathbb{R}$ is ld-continuous and $t\in\mathbb{T}_k$, then

$$\int_{\rho(t)}^{t} f(\tau)\nabla\tau = f(t)v(t).$$

Theorem 3.10 Let $a,b,c\in\mathbb{T}$, $\alpha\in\mathbb{R}$ and $f,g:\mathbb{T}\to\mathbb{R}$ be ld-continuous, then

(i) $\int_a^b [f(t)+g(t)\nabla t = \int_a^b f(t)\nabla t + \int_a^b g(t)\nabla t$

(ii) $\int_a^b \alpha f(t)\nabla t = \alpha\int_a^b f(t)\nabla t$

(iii) $\int_a^b f(t)\nabla t = -\int_a^b f(t)\nabla t$

(iv) $\int_a^b f(t)\nabla t = \int_a^c f(t)\nabla t + \int_c^b f(t)\nabla t$

(v) $\int_a^b f\big(\rho(t)\big)g^{\nabla}(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^{\nabla}(t)g(t)\nabla t$

(vi) $\int_a^b f\big(\rho(t)\big)g^{\nabla}(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^{\nabla}(t)g(\rho(t))\nabla t$

(vii) $\int_a^b f(t)\Delta t = 0$

The theorem following gives some relations between delta and nabla derivatives.

Theorem 3.11

(i). Assume that $f:\mathbb{T}\to\mathbb{R}$ is delta differentiable on $\mathbb{T}^k$. Then f is nabla differentiable at t and

$f^{\nabla}(t) = f^{\Delta}\big(\rho(t)\big)$ for $t\in\mathbb{T}^k$ such that $\sigma\big(\rho(t)\big) = t$. If, in addition, f^{Δ} is continuous on $\mathbb{T}^k$, then f is nabla differentiable at t and $f^{\nabla}(t) = f^{\Delta}\big(\rho(t)\big)$ holds for any $t\in\mathbb{T}_k$.

(ii). Assume that $f:\mathbb{T}\to\mathbb{R}$ is nabla differentiable on $\mathbb{T}_k$. Then f is delta differentiable at t and $f^{\Delta}(t) = f^{\nabla}\big(\sigma(t)\big)$ for $t\in\mathbb{T}_k$ such that $\rho\big(\sigma(t)\big) = t$. If in addition, f^{∇} is continuous on$\mathbb{T}_k$, then f is delta differentiable at t and $f^{\Delta}(t) = f^{\nabla}\big(\sigma(t)\big)$ hold for any $t\in\mathbb{T}^k$.

3.7 Some Time Scale Formulae

The following are formulas pertaining to time scales.

(i). $f^{\sigma} = f + \mu f^{\Delta}$

(ii). $f^{\rho} = f + v f^{\nabla}$

(iii). $(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta}$ (Product rule)

(iv). $(fg)^{\nabla} = f^{\nabla} g + f^{\rho} g^{\nabla}$ (Product rule)

(v). $\left(\frac{f}{g}\right)^{\Delta} = {(f^{\Delta} g - f g^{\Delta})}/{(g g^{\rho})}$ (Quotient rule)

(vi). $\left(\frac{f}{g}\right)^{\nabla} = {(f^{\nabla} g - f g^{\nabla})}/{(g g^{\rho})}$ (Quotient rule)

3.8 Convex and Quasiconvex functions

Some brief discussions on the properties of Convex and Quasiconvex functions are considered in this section.

Definition 3.20 A function $f: X \to \mathbb{R}$ defined on a convex subset of $\mathbb{R}^n$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for each $x, y \in X$ and $\lambda \in [0,1]$.

The function $f: X \to \mathbb{R}$ is called strictly convex if the above inequality is true as a strict inequality for each $x, y \in X$ and $\lambda \in [0,1]$.

Examples of Convex functions:

(i) Powers: $f(x) = x^p$; $p \geq 1$.

(ii) Exponential: $f(x) = e^{ax}$, for any $a \in \mathbb{R}$.

Definition 3.21 A real-valued function $f: X \to \mathbb{R}$ defined on a convex subset subset of $\mathbb{R}^n$ is said to be concave if

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$

for each $x, y \in X$ and $\lambda \in [0,1]$.

Examples of Concave functions:

(i). $f(x) = -x^2$

(ii). $f(x) = \sqrt{x}$

(iii). The function $f(x) = \sin x$ on $[0, \pi]$.

Definition 3.22 A function $f: X \to \mathbb{R}$ defined on a convex subset X of a real vector space is said to be quasiconvex for any $x, y \in X$ and $\lambda \in [0,1]$ if

$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$.

Furthermore if $f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}$ for any $x \neq y$ and $\lambda \in [0,1]$ then f is strictly quasiconvex. The function is said to be quasiconcave if –f is quasiconvex and a strictly quasiconcave function if a function whose negative is strictly quasiconvex. Equivalently, a function f is quasiconcave if

$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$ and strictly quasiconcave if

$f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\}$.

Examples of Quasiconvex functions:

(i). $\sqrt{|x|}$ is quasiconvex on $\mathbb{R}$.

(ii). $\log x$ is quasilinear (both quasiconvex and quasiconcave) on $\mathbb{R}_+$.

Definition 3.23 Given a sequence $\{a_n\}$ written as $\Delta a_n = a_n - a_{n+1}$, a sequence $\{a_n\}$ is said to be quasiconvex if

$\sum_1^\infty (n+1)|\Delta^2 a_n| < \infty$, where $n \in \mathbb{N}$ and

$$\Delta^2 a_n = \Delta(\Delta a_n) = \Delta(a_n - a_{n+1}) = a_{n+2} + a_n$$ (Mazhar, 1976).

Refer to Crouzeix (1999) and Pierskalla (1971) for detailed discussion on the properties of quasiconvex functions.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.0 Introduction

In this chapter some results regarding quasiconvex functions on time scales are established and discussed. Furthermore, definitions, lemmas, propositions and theorems are considered.

4.1 Quasiconvex functions on Time scale

In this section, quasiconvex function on time scale is defined and some properties are established.

Definition 4.1 A function $f:\mathbb{T}\to\mathbb{R}$ is called quasiconvex on $\mathbb{I}_{\mathbb{T}}$ if

$$f(\theta r+(1-\theta)t)\leq \max\{f(r),f(t)\} \tag{1}$$

for all $r,t\in\mathbb{I}_{\mathbb{T}}$ and $\theta\in(0,1)$.

Some examples of quasiconvex functions are:

$f(t)=\sqrt{|t|},\ \ f(t)=\frac{2t+5}{-2t+2},\ \ f(t)=\lfloor t\rfloor.$

Remark 4.2

The function f is strictly quasiconvex on $\mathbb{I}_{\mathbb{T}}$ if

$$f(\theta t+(1-\theta)r)< \max\{f(t),f(r)\} \tag{2}$$

for each $\theta\in(0,1)$ and each $t\in\mathbb{I}_{\mathbb{T}}$ such that $f(t)\neq f(r)$.

Definition 4.3 A function $f:\mathbb{T}\to\mathbb{R}$ is called quasiconcave on $\mathbb{I}_{\mathbb{T}}$ if

$$f(\theta r+(1-\theta)t)\geq \min\{f(r),f(t)\}. \tag{3}$$

for all $r,t\in\mathbb{I}_{\mathbb{T}}$ and $\theta\in(0,1)$.

Remark 4.4

(i). For $\mathbb{T} = S$ and $S \in \mathbb{R}$., Definition 4.1 is exactly the definition of a quasiconvex function.

(ii). For $\mathbb{T}$=S and S $\in\mathbb{N}$, Definition 4.1 gives the definition of quasiconvex sequences.

Some examples of quasiconcave functions are:

$$f(t) = -\sqrt{|t|}, \qquad f(t) = \frac{-2t-5}{-2t+2}, \qquad f(t) = -\lfloor t \rfloor.$$

Remark 4.5

The function f is strictly quasiconcave on $\mathbb{I}_{\mathbb{T}}$ if

$$f(\theta t + (1-\theta)r) > \min\{f(t), f(r)\} \text{ for each } \theta \in (0,1) \tag{4}$$

and for all $r, t \in \mathbb{I}_{\mathbb{T}}$ such that $f(t) \neq f(r)$

Remark 4.6

From Definition 4.1, a function f is called quasiconvex if

$$f(\theta r + (1-\theta)t) \leq f(r) \tag{5}$$

where $\max\{f(r), f(t))\} = f(r)$

at all convex combinations of t and r. Thus, f increases locally from its value at a point along the curve.

Definition 4.7 Let $f: \mathbb{T} \to \mathbb{R}$. The function f is quasiconvex on $\mathbb{I}_{\mathbb{T}}$ if the sublevel set of f

$S_\alpha = \{t \in \mathbb{T} : f(t) \leq \alpha\}$ is convex and

$\overline{S}_\alpha = \{t \in \mathbb{T} : f(t) < \alpha\}$ holds in the strict case, where $\alpha \in \mathbb{R}$.

Lemma 4.8

For any monotonically increasing or decreasing quasiconvex function, the inequality

$$f(\theta r + (1-\theta)t) \leq \theta f(r) + (1-\theta) f(t) \leq \max\{f(r), f(t)\}$$

holds for all $r, t \in \mathbb{I}_{\mathbb{T}}$ and $\theta \in (0,1)$.

We proof Lemma 4.8 geometrically.

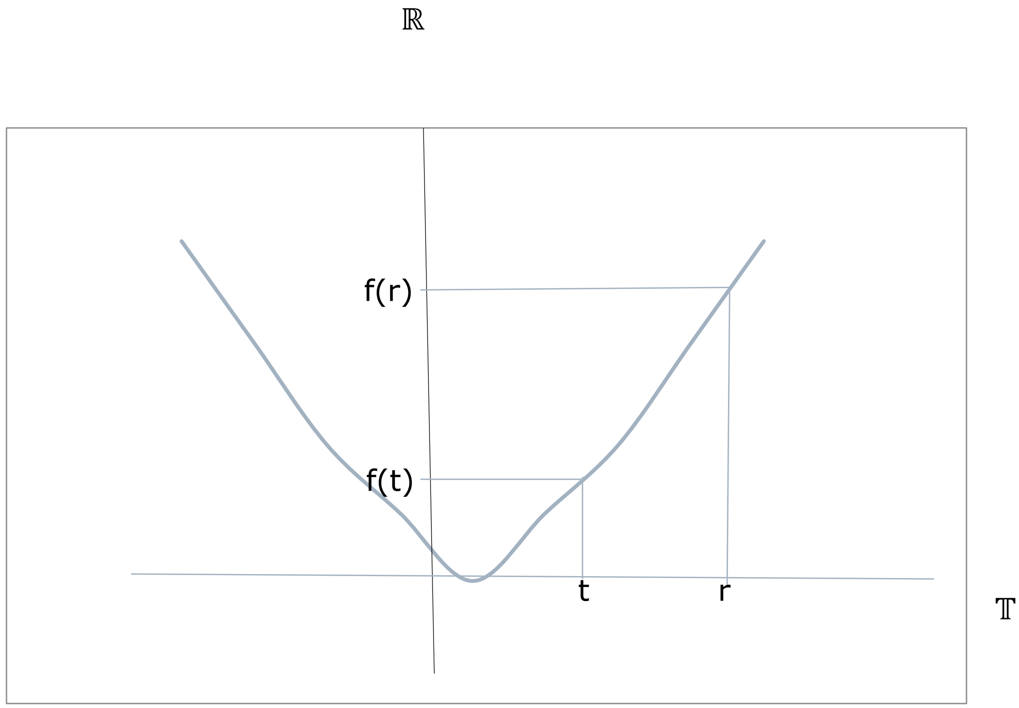


Figure 4.1 Geometrical Illustration of Lemma 4.8

Clearly, the first inequality is trivial because it defines convexity.

That is,

$f(\theta r + (1-\theta)t) \leq \theta f(r) + (1-\theta)f(t)$

Suppose $r, t \in \mathbb{I}_{\mathbb{T}}$ such that $r > t$ implies $f(r) > f(t)$ for increasing case.

For any $\theta \in (0,1)$, from the vertical axis, we have

$$\theta f(r) + (1-\theta)f(t) \leq \max\{f(r), f(t)\}. \quad (6)$$

This proofs the Lemma.

Proposition 4.9

Let $f: \mathbb{T} \to \mathbb{R}$ be a convex function that is increasing. Let $r, t \in \mathbb{I}_{\mathbb{T}}$ with $r \geq t$ and $s \in \mathbb{I}_{\mathbb{T}}$ such that $t \leq s \leq r$ and $s = \theta r + (1-\theta)t$ then f is quasiconvex if

$$(t-r)f(s)+(r-s)f(t)+(s-t)(r-t)f(r)\geq 0 \quad (7)$$

holds.

Proof

From $s=\theta r+(1-\theta)t$, we have

$$\theta=\frac{s-t}{r-t} \text{ and } 1-\theta=\frac{r-s}{r-t}. \quad (8)$$

Thus by convexity, we have

$$f(s)=f(\theta r+(1-\theta)t)\leq \theta f(r)+(1-\theta)f(t) \quad (9)$$

Using Lemma 4.8, the above inequality becomes

$$f(s)\leq \theta f(r)+(1-\theta)f(t)\leq \max\{f(r),f(t)\} \quad (10)$$

Since f is increasing $\max\{f(r),f(t)\}=f(r)$.

Substituting (8) into (10), we have

$$f(s)\leq \frac{s-t}{r-t}f(r)+\frac{r-s}{r-t}f(t)\leq f(r) \quad (11)$$

Rearranging (11), yields the required inequality (7).

Proposition 4.10

Let $f:\mathbb{T}\to\mathbb{R}$ be a convex function that is decreasing. Let $r,t\in \mathbb{I}_{\mathbb{T}}$ with $r\geq t$ and $s\in \mathbb{I}_{\mathbb{T}}$ such that $t\leq s\leq r$ and $s=\theta r+(1-\theta)t$ then f is quasiconvex if

$$(t-r)f(s)+(r-s)(r-t)f(t)+(s-t)f(r)\geq 0\,. \quad (12)$$

Proof

By convexity, we have

$$f(s)=f(\theta r+(1-\theta)t)\leq \theta f(r)+(1-\theta)f(t) \quad (13)$$

Using Lemma 4.8, the inequality (13) becomes

$$f(s)\leq \theta f(r)+(1-\theta)f(t)\leq \max\{f(r),f(t)\} \quad (14)$$

Since f is decreasing $\max\{f(r),f(t)\}=f(t)$.

Substituting (8) into (14), we have

$$f(s) \leq \frac{s-t}{r-t} f(r) + \frac{r-s}{r-t} f(t) \leq f(t) \tag{15}$$

Rearranging (15), yields the required inequality (12).

Lemma 4.11

Let f: $\mathbb{I}_{\mathbb{T}} \rightarrow \mathbb{R}$. f is quasiconvex if and only if the sublevel set

$S_{\propto}(f) = \{t \in \mathbb{T}: f(t) \leq \propto, \text{ for any } \propto \in \mathbb{R}\}$ is convex.

Proof

Let f be quasiconvex and suppose that $t \in \mathbb{I}_{\mathbb{T}}$ is isolated. Then there exist $t_1, t_2 \in S_\alpha \in \mathbb{I}_{\mathbb{T}}$.

Thus, $f(t_1) \leq \alpha$ and $f(t_2) \leq \alpha$.

Let $\theta \in [0,1]$ and $t = \theta t_1 + (1-\theta)t_2 \in S_\alpha \in \mathbb{I}_{\mathbb{T}}$.

Thus, $f(t) = f(\theta t_1 + (1-\theta)t_2) \leq \max\{f(t_1), f(t_2)$.

$f(t) \leq \max\{\alpha, \alpha\} = \alpha$.

Hence, $t \in S_\alpha$ and thus S_α is convex.

Conversely, suppose that $S_\alpha(f)$ is convex. Then there exists $t_1, t_2 \in S_\alpha \in \mathbb{I}_{\mathbb{T}}$ such that $\theta t_1 + (1-\theta)t_2 \in S_\alpha$ for any $\theta \in [0,1]$.

Therefore, $f(\theta t_1 + (1-\theta)t_2) \leq \max\{f(t_1), f(t_2)$.

Thus, f is quasiconvex.

Now consider the case where t is dense and f is quasiconvex, then there exists $[s, t]$ and $[t, t_1]$ in $I_{\mathbb{T}}$.

Let $f(s) \leq \alpha$ and $f(t_1) \leq \alpha$ such that $t = \theta s + (1-\theta)t_1$.

Thus, $f(t) = f(\theta s + (1-\theta)t_1) \leq \max\{f(s), f(t_1)$

$f(t) \leq \alpha$, therefore, $t \in S_\alpha$ is convex.

Conversely, assume S_α is convex, clearly f is quasiconvex.

Definition 4.12 The set $S \in \mathbb{I}_\mathbb{T}$ is closed if and only if for any convergent sequence of points $\{s_i\}$ contained in S the limit point $\bar{s} \in S$.

Definition 4.13 The set $S \in \mathbb{I}_\mathbb{T}$ is called closed if its closure (clS) is equal to S and open if its interior is equal to S.

Theorem 4.14 A quasiconvex function on $[a,b]_\mathbb{T} \in \mathbb{I}_\mathbb{T}$ is lower semi-continuous (lsc) if $S_\propto(f)$ is closed and upper semi-continuous (usc) if $\overline{S_\propto}(f)$ is open, $\forall \propto \in \mathbb{R}$.

Proof

1. Suppose that s is left scattered and right dense, then two cases arise:

(i). There exists $s_2 \in [a,b]_\mathbb{T} \in \mathbb{I}_\mathbb{T}$ such that $[s_1, s_2] \in [a,b]_\mathbb{T} \in \mathbb{I}_\mathbb{T}$.

Since $I_\mathbb{T}$ is convex, then for any $\theta \in [0,1]$,

$s = \theta s_1 + (1-\theta)s_2 \in S_\alpha \subseteq [a,b]_\mathbb{T} \subset \mathbb{I}_\mathbb{T}$.

Let $f(s_1) \leq \alpha$ and $f(s_2) \leq \alpha$.

Suppose that f is quasiconvex on$[a,b]_\mathbb{T} \in \mathbb{I}_\mathbb{T}$. Then from Definition 4.1, we have,

$f(s) = f(\theta s_1 + (1-\theta)s_2) \leq \max\{f(s_1), f(s_2)$.

$f(s) \leq \max\{\alpha, \alpha\} = \alpha$.

Therefore, $s \in S_\alpha$ and we conclude that S_α is convex.

Now, we show that S_α is closed:

Let $t \in clS_\alpha$ and $s \in S_\alpha$, for all $s, t \in \mathbb{I}_\mathbb{T}$.

Then for any $\theta \in (0,1)$, we get

$\theta t + (1-\theta)s \in S_\alpha$.

In the limit as $\theta \to 1$, $t \in S_\alpha$. Thus, $clS_\alpha = S_\alpha$.

From Definition 4.12 S_α is closed and therefore f is lower semicontinuous.

Considering the argument for the convexity of the strict sublevel set , $\bar{S}_\alpha$ follows the same argument for that of S_α. Thus, we show that it is open.

Let s be an interior point of $\mathrm{cl}\bar{S}_\alpha$ and $t \in \mathrm{int}\bar{S}_\alpha$.

Then,

$\theta s + (1-\theta)t \in \mathrm{int}\bar{S}_\alpha$ for all $\theta \in (0,1)$.

In the limit as $\theta \to 1$, $\theta s + (1-\theta)t \to s \in \mathrm{int}\bar{S}_\alpha$.

Therefore, $\mathrm{cl}\bar{S}_\alpha = \mathrm{int}\bar{S}_\alpha \Rightarrow \bar{S}_\alpha = \mathrm{int}\bar{S}_\alpha$ and $\bar{S}_\alpha$ is open. Hence f is upper semi continuous on $[a,b]_{\mathbb{T}} \in \mathbb{I}_{\mathbb{T}}$.

(ii). s is the limit of a decreasing sequence $s_1 > s_2 > \cdots > s$. Thus, we have s_i of isolated points.

For $i \in \mathbb{N}$ such that $\{s_i\}_{i=1}^{n}$ is qasiconvex sequence. Thus, by the Definition 3.23 there is a point $s \in \{s_i\}_{i=1}^{n}$ such that $\lim_{i\to\infty} s_i = s$ and therefore S is closed. Let $f(s_1) \le \alpha$ and $f(s_2) \le \alpha$ for all $\alpha \in \mathbb{R}$ such that $s = \theta s_1 + (1-\theta)s_2 \in S_\alpha \subseteq [a,b]_{\mathbb{T}}$ for all $\theta \in (0,1)$.

Since f is quasiconvex, we have

$f(s) = f(\theta s_1 + (1-\theta)s_2) \le \max\{f(s_1).f(s_2)\}$

$f(s) \le \max\{\alpha.\alpha\} = \alpha$.

Thus, $s \in S_\alpha \subseteq [a,b]_{\mathbb{T}}$ and therefore S_α is convex and closed and hence f is lower semicontinuous.

If the sublevel set is lower semicontinuous, then f is quasiconvex. The same argument goes for the the strict sublevel set.

2. If s is a dense point on $[a,b]_{\mathbb{T}}$, there are two quasimonotone sequences $\{s_i\}_{i=1}^{n}$ and $\{t_i\}_{i=1}^{n}$ such that $s_1 \le s_2 \le \cdots s_n \le \cdots \le s \le \cdots \le t_n \le \cdots \le t_1$ and s is the limit of both sequences.

Thus, taking arbitrarily $s_2, t_2 \in [a, b]_{\mathbb{T}}$. Let $f(s_2) \leq \alpha$ and $f(t_2) \leq \alpha$ with

$s = \theta s_2 + (1-\theta)t_2 \in S_\alpha \subseteq [a, b]_{\mathbb{T}}$, we have

$f(s) = \theta s_2 + (1-\theta)t_2 \leq \max\{f(s_2), f(t_2)$.

Thus, $f(s) \leq \alpha$, which implies that S_α is convex and closed. Hence f is lower semicontinuous.

The same argument goes for $\bar{S}_\alpha$ which is also convex and open and therefore f is upper semicontinuous.

3. Finally, for the case where s is isolated (that is both left and right scattered), the argument is the same for case 1(i). Therefore the assertion holds in both directions. Since all the cases are true, we conclude that a quasiconvex function on $[a, b]_{\mathbb{T}}$ is lower and upper semicontinuous.

Definition 4.15 A sequence $\{a_n\}$ is said to be quasimonotone if and only if $a_n \geq 0$ and $\Delta a_n \geq -\alpha n^{-1} a_n$ for some $\alpha \geq 0$ and $\Delta a_n = a_n - a_{n+1}$.

Remark 4.16 A set $S \subseteq \mathbb{T}$ is said to be evenly convex if it is the intersection of half spaces. A function f is said to be evenly quasiconvex if all $S_\propto(f)$ are convex. Lower semi-continuous functions and upper semi-continuous quasiconvex functions are evenly quasiconvex.

Let $cl(S_\propto(f)$,$co(S_\propto(f)$, $eco(S)$ and $\overline{co}(S)$ denote the closure, the convex hull, the evenly convex hull and closed convex hull of S respectively. Let $\bar{f}$, f_q, $f_e$and $f_{\bar{q}}$ be functions obtained respectively from the sets above and known respectively as the greatest semi-continuous, quasiconvex, evenly quasiconvex and lsc quasiconvex functions bounded above by f.

Proposition 4.17 Suppose that f is quasiconvex on $I_{\mathbb{T}}$ and $\mathrm{int}(S_{\propto}(f)) \neq \phi$. Then $\mathrm{Int}(\mathrm{cl}(S_{\propto}f) = \mathrm{Int}(S_{\propto}f)$.

Proof

Suppose t is right and left dense, then there exists s_1 and t_1 in $\mathbb{I}_{\mathbb{T}}$, such that $[s_1, t], [t, t_1]$ are all in $\mathbb{I}_{\mathbb{T}}$.

Thus for $[s_1, t] \in I_{\mathbb{T}}$, let $f(s_1) \leq \alpha$ and $f(t) \leq \alpha$ such that

$x = \theta s_1 + (1-\theta)t \in S_{\propto}(f)$ for all $x \in [s_1, t] \in \mathbb{I}_{\mathbb{T}}$ and $\alpha \in \mathbb{R}$.

Since f is quasiconvex we have,

$f(x) = f(\theta s_1 + (1-\theta)t) \leq \max\{f(s_1), f(t)\}$ for any $\theta \in (0,1)$

Thus $f(x) \leq \max\{f(s_1), f(t)\} = \max\{\alpha, \alpha\} = \alpha$

Therefore $x \in S_{\propto}(f)$ and $S_{\propto}(f)$ is convex.

Now, let $y \in \mathrm{Int}(\mathrm{cl}S_{\propto})$ and $x \in \mathrm{Int}S_{\propto}(f)$ and for any $\theta \in (0,1)$, we have

$\theta y + (1-\theta)x \in \mathrm{Int}S_{\propto}(f)$

In the limit as $\theta \to \infty$, $\theta y + (1-\theta)x \to y \in \mathrm{Int}S_{\propto}(f)$.

Therefore, $\mathrm{Int}(\mathrm{cl}(S_{\propto}f) = \mathrm{Int}(S_{\propto}f)$.

Consider the case where t is right and left scattered (isolated). Then there exist two quasimonotone sequences such that

$s_1 \leq s_2 \leq \cdots s_n \leq \cdots \leq t \leq \cdots \leq t_n \leq \cdots \leq t_1$

Thus, we have $t \in [s_1, t_1]$.

Let $f(s_1) \leq \alpha$ and $f(t_1) \leq \alpha$ such that $t = \theta s_1 + (1-\theta)t_1 \in S_{\propto}(f) \in [s_1, t_1]$.

Since f is quasiconvex, we get

$f(t) = f(\theta s_1 + (1-\theta)t_1) \leq \max\{f(s_1), f(t_1)\} = \max\{\alpha, \alpha\} = \alpha$

$f(t) \leq \alpha$ and therefore $t \in S_\propto(f)$. $S_\propto(f)$ is convex and closed since the sequences converge.

Thus, $clS_\propto(f) = S_\propto(f)$. This implies that $Int(cl(S_\propto f) = Int(S_\propto f)$.

Definition 4.18 A function f is said to be quasi-monotonic if and only if

$S_\propto = \{t : f(t) \leq\propto\}$ and $S_\propto = \{t : -f(t) \geq\propto\}$ are convex.

Theorem 4.19 Let $f : \mathbb{T} \to (-\infty, +\infty)$ be quasiconvex. Then f is lower semi-continuous at $t \in \mathbb{I}_{\mathbb{T}}$ if and only if $\bar{f}(t) = f(t)$.

Proof

Suppose that $\bar{f}(t) = f(t)$. Then their sublevel sets are the same. That is,

$S_\propto(f) = S_\propto(\bar{f}) = \{t \in \mathbb{I}_{\mathbb{T}} : f(t) = \bar{f}(t) \leq \alpha, \alpha \in \mathbb{R}\}$.

Now, consider that t is isolated. Then

$s_1 \leq s_2 \leq \cdots s_n \leq \cdots \leq t \leq \cdots \leq t_n \leq \cdots \leq t_1$

Taking any two arbitrary points s_1 and t_1 from the sequence , let $f(s_1) \leq \alpha$ and $f(t_1) \leq \alpha$.

For any $\theta \in (0,1)$, we have $t = \theta s_1 + (1-\theta)t_1 \in S_\propto(f) \in \mathbb{I}_{\mathbb{T}}$.

Since f is quasiconvex, then we have

$f(t) = f(\theta s_1 + (1-\theta)t_1) \leq \max\{f(s_1), f(t_1)\} = \max\{\alpha, \alpha\} = \alpha$.

Therefore, $t \in S_\propto(f)$ and $S_\propto(f)$ is convex.

From Definition 4.12, $S_\propto(f)$ is closed and hence f is lower semicontinuous.

Conversely, suppose f is lower semi continuous. That is to say $S_\propto(f)$ is closed. Then,

$S_\propto(f) = \{ t \in \mathbb{I}_{\mathbb{T}} : f(t) \leq \alpha, \alpha \in \mathbb{R}\}$.

$cl(S_\propto(f)) = \{ t^* \in \mathbb{T} : \bar{f}(t) \leq \alpha, \alpha \in \mathbb{R}\}$.

Now, we show that $S_\propto(f) = cl(S_\propto(f))$ and therefore $\bar{f}(t) = f(t)$.

Let $t_1, t^* \in cl(S_\propto(f))$ and $s_1 \in S_\propto(f)$, then we have

$\theta t^* + (1-\theta)t_1 \in S_\propto(f)$ for all $\theta \in (0,1)$.

Thus, suppose that $\lambda \in (0,1)$, then

$\lambda t_1 + (1-\lambda)[(\theta t^* + (1-\theta)t_1)] \in S_\propto(f)$.

In the limit as $\theta \to 1$, $\lambda t_1 + (1-\lambda)t^* \in S_\propto(f)$. This implies that $t_1, t^* \in S_\propto(f)$.

Therefore, $cl(S_\propto(f)) = S_\propto(f)$.

Thus, we can write $cl(S_\propto(f)) = S_\propto(f) = \{t \in \mathbb{I}_\mathbb{T} : \bar{f}(t) = f(t) \leq \alpha, \forall\, \alpha \in \mathbb{R}\}$.

Hence $\bar{f}(t) = f(t)$.

Consider the case where t is dense, then there are closed intervals $[s_1, t], [t, t_1]$ in $\mathbb{I}_\mathbb{T}$.

Let $\bar{f}(t) = f(t)$. For any arbitrary points s_1 and t_1, let $f(s_1) \leq \alpha$ and $f(t_1) \leq \alpha$ be such that

$t = \theta s_1 + (1-\theta)t_1 \in S_\propto(f)$ for any $\theta \in (0,1)$. Since f is quasiconvex, we have

$f(t) = f(\theta s_1 + (1-\theta)t_1) \leq \max\{f(s_1), f(t_1)\} = \max\{\alpha, \alpha\} = \alpha$

Thus, $t \in S_\propto(f)$ and $S_\propto(f)$ is convex and closed because it's within a closed interval $[s_1, t_1]$.

Therefore f is lower semicontinuous on $\mathbb{I}_\mathbb{T}$.

Conversely, suppose that f is lower semicontinuous. Then

$S_\propto(f) = \{ t \in I_\mathbb{T} : f(t) \leq \alpha, \alpha \in \mathbb{R}\}$ and

$cl(S_\propto(f)) = \left\{ t^* \in \mathbb{T} : \bar{f}(t) \leq \alpha, \alpha \in \mathbb{R}\right\}$.

Already $cl(S_\propto(f)) = S_\propto(f)$ is shown to be true. Therefore, we have

$cl(S_\propto(f)) = S_\propto(f) = \{t \in \mathbb{I}_\mathbb{T} : \bar{f}(t) = f(t) \leq \alpha, \forall\, \alpha \in \mathbb{R}\}$.

Hence, $\bar{f}(t) = f(t)$.

Definition 4.20 A quasiconvex function is called monotonically decreasing or nonincreasing if whenever $s \leq t$ then $f(s) \geq f(t)$.

Theorem 4.21 Let $f\colon \mathbb{I}_{\mathbb{T}} \to \mathbb{R}$ be quasiconvex. If the sub-level set

$S_{\propto}(f) = \{t \in \mathbb{I}_{\mathbb{T}} : f(t) \leq \propto \text{ any } \propto \in \mathbb{R}\}$ is convex, then

$S_{\beta}(-f) = \{t \in \mathbb{I}_{\mathbb{T}} : -f(t) \geq \beta, \text{for any } \beta \in \mathbb{R}\}$ is also convex.

Proof

Let f be quasiconvex over $\mathbb{I}_{\mathbb{T}}$ and the sublevelset

$S_{\propto}(f) = \{t \in \mathbb{T} : f(t) \leq \propto, \text{for any } \propto \in \mathbb{R})$ be convex. Therefore for every $s, t \in \mathbb{I}_{\mathbb{T}}$,

$f(\theta s + (1-\theta)t) \leq \max\{f(s), f(t)\}$ and

$\theta s + (1-\theta)t \in S_{\propto}(f)$ for the time scale interval $\mathbb{I}_{\mathbb{T}}$.

Thus for $-f$, we have

$-f(\theta s + (1-\theta)t) \geq -\max\{f(s), f(t)\}$. $-f$ is quasiconcave over the same interval and thus there is a $\beta \in \mathbb{R}$ such that

$S_{\beta}(-f) = \{t \in \mathbb{I}_{\mathbb{T}} : -f(t) \geq \beta, \text{for any } \beta \in \mathbb{R}\}$.

Thus for any $s, t \in S_{\beta}$, there exists $\theta s + (1-\theta)t \in S_{\beta}$ and therefore S_{β} is convex.

Theorem 4.22

Let $f\colon \mathbb{I}_{\mathbb{T}} \to \mathbb{R}$ be delta differentiable function on $\mathbb{I}_{\mathbb{T}}$. If f^{Δ} is quasimonotone on $\mathbb{I}_{\mathbb{T}}$, then f^{Δ} is quasiconvex on $\mathbb{I}_{\mathbb{T}}$.

Proof

First, we establish that the function is delta differentiable and show that it is quasimonotone and therefore quasiconvex.

Let $x < y < z \in \mathbb{I}_{\mathbb{T}}$. Then there exist the points $x_1, x_2 \in [x, y)$ and $y_1, y_2 \in [y, z)$ such that

$$f^{\Delta}(x_1) \leq \frac{f(y) - f(x)}{y - x} \leq f^{\Delta}(x_2) \text{ and}$$

$$f^{\Delta}(y_1) \leq \frac{f(z)-f(y)}{z-y} \leq f^{\Delta}(y_2) \tag{17}$$

Now for $x < x_2 < y_1$, equation (17) becomes

$$\frac{f(y)-f(x)}{y-x} \leq f^{\Delta}(x_2) \leq f^{\Delta}(y_1) \leq \frac{f(z)-f(y)}{z-y}, \text{ for nondecreasing } f^{\Delta} \tag{18}$$

If $x > x_2 > y$, then equation becomes

$$\frac{f(y)-f(x)}{y-x} \geq f^{\Delta}(x_2) \geq f^{\Delta}(y_1) \geq \frac{f(z)-f(y)}{z-y}, \text{ for nonincreasing } f^{\Delta} \tag{19}$$

Thus the delta function f^{Δ} exists. Next we establish that f^{Δ} is quasimonotone, that is to say the sublevel sets $S_{\propto}(f^{\Delta})$ and $S_{\beta}(-f^{\Delta})$ are convex.

Now, let $f^{\Delta}: \mathbb{I}_{\mathbb{T}} \to \mathbb{R}$ such that there exists

$S_{\propto}(f^{\Delta}) = \{t \in \mathbb{I}_{\mathbb{T}} : f^{\Delta}(t) \leq \propto, \forall \propto \in \mathbb{R}\}$.

Thus for any $s, t \in S_{\propto}$, there exists $\theta s + (1-\theta)t \in S_{\propto}$, since $\mathbb{I}_{\mathbb{T}}$ is convex.

Hence $S_{\propto}$ is convex. Theorem 4.23 already establishes that $S_{\beta}(-f^{\Delta})$ is convex if $S_{\propto}$ is convex.

Therefore, f^{Δ} is quasimonotone. From Definition 4.1 f^{Δ} is quasiconvex.

Theorem 4.23 Let $f: \mathbb{I}_{\mathbb{T}^k} \to \mathbb{R}$ be nabla differentiable function on $\mathbb{I}_{\mathbb{T}^k}$. If f^{∇} is quasimonotone on $\mathbb{I}_{\mathbb{T}_k}$, then f is quasiconvex on $\mathbb{I}_{\mathbb{T}_k}$.

Proof

Suppose $s > t > u \in \mathbb{I}_{\mathbb{T}_k}$ such that $s_1, s_2 \in (s, t]$ and $u_1, u_2 \in (t, u]$ such that

$$f^{\nabla}(s_1) \geq \frac{f(s)-f(t)}{s-t} \geq f^{\nabla}(s_2) \text{ and } f^{\nabla}(u_1) \geq \frac{f(t)-f(u)}{t-u} \geq f^{\nabla}(u_2) \tag{20}$$

Since $s > s_2 > u_1$, equation (20) becomes

$$\frac{f(s)-f(t)}{s-t} \geq f^{\nabla}(s_2) \geq f^{\nabla}(u_1) \geq \frac{f(t)-f(u)}{t-u} \text{ for nondecreasing } f^{\nabla}. \tag{21}$$

If $s < t < u$, then equation (21)becomes,

$$\frac{f(s)-f(t)}{s-t} \leq f^{\nabla}(s_2) \leq f^{\nabla}(u_1) \leq \frac{f(t)-f(u)}{t-u} \text{, for nonincreasing } f^{\nabla}. \quad (22)$$

Thus f^{∇} is quasimonotone. Therefore f^{∇} is quasiconvex from same arqument in Theorm 4.23.

Theorem 4.24 Let $\mathbb{T}^0$ be the relative interior of $\mathbb{T}$ and $f : \mathbb{T}^0 \to \mathbb{R}$ be quasiconvex function. Then f is continuous almost everywhere over $\mathbb{T}^0$.

Proof

Let $t \in \mathbb{T}^0$ be an interior point within the interval $I_{\mathbb{T}}$ such that $t \in [s, u]$and $f(t) \neq 0$.

Then $\exists$ a $\delta > 0$ such that f(x) has the same same sign as f(t), for every $x \in (t-\delta, t+\delta)$.

Since f is continuous at an interior point $t \in \mathbb{T}^0$ of $[s, u]$, therefore for any $\varepsilon > 0, \exists$ a $\delta > 0$ such that:

$$|f(x) - f(t)| < \varepsilon, \forall\, x \in (t-\delta, t+\delta) \text{ or } f(t) - \varepsilon < f(x) < f(t) + \varepsilon \quad (23)$$

When $f(t) > 0$, taking ε to be greater than f(t), we have

$f(x) > 0, \forall\, x \in (t-\delta, t+\delta)$.

Also, when $f(t) < 0$, taking ε to be less than f(t), we have

$f(x) < 0, \forall\, x \in (t-\delta, t+\delta)$.

Theorem 4.25 A function $f: \mathbb{I}_{\mathbb{T}} \to \mathbb{R}$ is quasiconvex on $\mathbb{I}_{\mathbb{T}} = \mathbb{I} \cap \mathbb{T}$ if and only if there exists a quasiconvex function $\bar{f} : \mathbb{I} \to \mathbb{R}$ such that $\bar{f}(t) = f(t)\ \forall\, t \in \mathbb{I}_{\mathbb{T}}$.

Proof

For the sufficient part, since if there exists a quasiconvex function$\bar{f}$ on $\mathbb{I}$ such that $\bar{f}(t) = f(t)$, then

$\bar{f}(\theta s) + (1-\theta)t \leq \theta f(s) + (1-\theta)f(t) \leq \max\{f(s)\}, f(t)\}$

for all $t \in \mathbb{I}_{\mathbb{T}}, s, t \in \mathbb{I}$ and all $\theta \in [0,1]$. When $t, s, \theta + (1-\theta)t \in \mathbb{T}, f(t)$, then we get inequality (1), which is the quasiconvexity on $\mathbb{I}_{\mathbb{T}}$.

Thus,

$$\bar{f}(t) = \left\{ \begin{array}{ll} f(t), & \text{if } t \in \mathbb{I}_{\mathbb{T}} \\ f(s) + \frac{f(\sigma(s)) - f(s)}{\mu(s)}(t-s), & \text{if } t \in (s, \sigma(s), s \in \mathbb{I}_{\mathbb{T}} \text{ a nd s is right scattered} \end{array} \right\} \quad (24)$$

For any $x, y \in \mathbb{I}_{\mathbb{T}}$ and $\theta \in [0,1]$, we have

$$\bar{f}(\theta x + (1-\theta)y) \leq \theta\bar{f}(x) + (1-\theta)\bar{f}(y) \quad (25)$$

For $x, y \in \mathbb{I}_{\mathbb{T}}$ and $y > \sigma(x)$, the chord joining $(x, f(x))$ and $(y, f(y))$ is above all points $(z, f(z))$, with $z \in \mathbb{I}_{\mathbb{T}}$. If $x \in \mathbb{I}_{\mathbb{T}}$ and $y \in \mathbb{I}/\mathbb{T}$ with $y \leq \sigma(x)$, then $(y, f(y))$ is on the chord from $(x, f(x))$ to $(\sigma(x), f(\sigma(x))$ and so are all the points

$(\theta x + (1-\theta)y, f(\theta x + (1-\theta)y)$.

If $y > \sigma(x)$, then we can find $z \in \mathbb{I}_{\mathbb{T}}$ such that $x < z$ and $z < y < \sigma(z)$ such that

$$\frac{f(x)-f(z)}{x-z} \leq \frac{f(x)-f(\sigma(x))}{x-\sigma(x)} \quad (26)$$

while associating $\bar{f}$ on $[z, \sigma(z))$ we have

$$\frac{f(x)-f(z)}{x-z} \leq \frac{f(x)-\bar{f}(y)}{x-y} \leq \frac{f(x)-f(\sigma(z))}{x-\sigma(z)} \quad (27)$$

Thus for $\theta \in [0,1]$, $\theta x + (1-\theta)y \in [x, z]$ such that

$\bar{f}(\theta x + (1-\theta)y) = f(\theta x + (1-\theta)y)$,

where $t = \theta x + (1-\theta)y \in \mathbb{I}_{\mathbb{T}}$. Therefore the conclusion holds.

Proposition 4.26 A quasiconvex function on $[q, r]_{\mathbb{T}}$ is lower semicontinuous on $(q, r)_{\mathbb{T}}$

Proof

Let $f: [q, r]_{\mathbb{T}} \to \mathbb{R}$ be quasiconvex and $t \in (q, r)_{\mathbb{T}}$.

We need to show that the sublevel

$S_\alpha(f) = \{t: f(t) \leq \alpha\,;\ \alpha \in \mathbb{R}\}$ is closed and $t \in [q, r]_{\mathbb{T}}$.

$S_\alpha(f)$ is closed within the interval $(q, r)_{\mathbb{T}}$ if and only if $\overline{S_\alpha}(f)$ is open.

Suppose $\overline{S_\alpha}(f)$ is the limit point of $S_\alpha(f)$. Then every neighbourhood of t contains a point of $S_\alpha(f)$, so that t is not in an interior of $\overline{S_\alpha}(f)$. Since $\overline{S_\alpha}(f)$ is open, this means that $t \in S_\alpha(f)$. It follows that $S_\alpha(f)$ is closed.

Next assume that $S_\alpha(f)$ is closed.

Let $t \in \overline{S_\alpha}(f)$. This means that t is not a limit point of $S_\alpha(f)$. Therefore the exists a neighbourhood N of t such that

$S_\alpha(f) \cap N_\varepsilon(t)(\varepsilon > 0)$ is empty, that is

$N_\varepsilon \subset \overline{S_\alpha}(f)$. Thus t is an interior point of $\overline{S_\alpha}(f)$ and hence $\overline{S_\alpha}(f)$ is open.

This concludes the proof that $S_\alpha(f)$ is closed and therefore f is lower semicontinuous at $t \in [q, r]_{\mathbb{T}}$, and hence lower semicontinuous on $(q, r)_{\mathbb{T}}$.

Proposition 4.27 A quasiconvex function is continuous at $t \in [a, b]$ if and only if it is upper and lower semicontinuous at $t \in (a, b)$.

Proof

If $a \leq s < t < u \leq b$, then

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$$

Let $t \in (a, b)$ and consider $a \leq t < t_n < s \leq b$, where (t_n) is such that

$t_n \to t$ as $n \to \infty$ and

$s \in (a, b)$ is fixed. Then we have

$t_n = \lambda_n t + (1 - \lambda_n)s$

where,

$$\lambda_n = \frac{t_n - s}{t - s} \to 1, \text{as } n \to \infty \qquad (28)$$

Taking the limit superior of both sides of (28), we consequently have

$$\lim_{n\to\infty} \sup f(t_n) = \lim_{n\to\infty} \sup f(\lambda_n t + (1-\lambda_n)s) \leq \lim_{n\to\infty} \sup(\lambda_n f(t) + (1-\lambda_n)f(s)$$

Thus $f(t) \geq \lim_{n\to\infty} \sup f(t_n)$

Hence f is upper semicontinuous at $t \in (a, b)$.

Similarly, suppose (t_n) is such that $t_n \to t$ as $n \to \infty$ and

$a \leq t_n < t < z \leq b$, we have

$$t = \mu_n t_n + (1-\mu_n)z$$

where

$$\mu_n = \frac{t - z}{t_n - z}$$

$$\mu_n{}^{-1} = \frac{t_n - z}{t - z} \to 1, \text{as } n \to \infty$$

Thus

$$f(t) = f(\mu_n t_n + (1-\mu_n)z) \leq \mu_n f(t_n) + (1-\mu_n)f(z) \qquad 29$$

Rewriting (29) gives

$$(\mu_n{}^{-1} f(t) \leq f(t_n) + 1 - \mu_n)\mu_n{}^{-1} f(z)$$

Taking the limit inferior of both sides, we have

$$\lim_{n\to\infty} \inf(\mu_n{}^{-1} f(t)) \leq \lim_{n\to\infty} \inf(f(t_n) + 1 - \mu_n)\mu_n{}^{-1} f(z))$$

Thus,

$$f(t) \leq \lim_{n\to\infty} \inf f(t_n)$$

Therefore f is lower semicontinuous at $t \in (a, b)$

Conversely, suppose that f is upper semicontinuous at $t_0 \in (a, b)$.

Then for every $\epsilon > 0$, there exists a neighbourhood $\mathcal{U}$ of t_0 such that

$f(t) \leq f(t_0) + \epsilon$

for all $t \in \mathcal{U}$ when $f(t_0) > -\infty$ and $f(t)$tends to $-\infty$

as t tends to when $f(t_0) = -\infty$.

This implies that

$|f(t) - f(t_0)| < \epsilon$, when $|t - t_0| < \delta$

where δ is a very small number.

Thus, f is continuous.

Similarly, if f is lower semicontinuous, then for every $\epsilon > 0$, there exists $\mathcal{U}$ of t_0 such that $f(t) \geq f(t_0) - \epsilon$ for all$t \in \mathcal{U}$ when $f(t_0) < +\infty$, $f(t)$ tends to $+\infty$ and t tends to $+\infty$ when $f(t_0) \equiv +\infty$.

Thus $\epsilon \geq f(t_0) - f(t)$ 30

Rewriting (30), we have

$| f(t_0) - f(t)| \geq \epsilon \Longrightarrow |-(f(t) - f(t_0))| \leq \epsilon$

Therefore,

$|f(t) - f(t_0) | < \epsilon$, when $|t - t_0| < \delta$

Hence f is continuous.

4.3 The Subdifferential

We briefly define the left nabla and right delta derivatives of a quasiconvex function before examining the quasiconvex subdiffrerential.

For a left − dense and right − dense point $t \in \mathbb{T}$,

$f^{\Delta}(t) = \lim_{r \to t+} \frac{f(t) - f(r)}{t - r}$, when the right limit exists.

It holds likewise for $f^{\nabla}(t)$ which is

$f^{\nabla}(t) = \lim_{r \to t-} \frac{f(t)-f(r)}{t-r}$, when the left limit exists.

The existence or otherwise of the limit may be caused by the lateral limits in the above relation.

Thus, for a dense point $t \in \mathbb{T}$, we define

$$f_{-}^{1}(t) = \lim_{r \to t, r<t} \frac{f(t) - f(r)}{t - r}$$

$$f_{+}^{1}(t) = \lim_{r \to t, r>t} \frac{f(t) - f(r)}{t - r}$$

For an arbitrary quasiconvex function $f : \mathbb{T} \to \mathbb{R}$ and a point $t \in \mathbb{T}^{k}$, such that $f^{\Delta}(t)$ or $f_{+}^{1}(t)$ exist, we define

$$f_{+}^{\Delta}(t) = \begin{cases} f^{\Delta}(t), & \text{if t is right scattered} \\ f_{+}^{1}(t), & \text{if t is dense} \end{cases}$$

Also, if $f^{\nabla}(t)$ or $f_{-}^{1}(t)$ for a point $\mathbb{T}_{k}$, we define

$$f_{-}^{\nabla}(t) = \begin{cases} f^{\nabla}(t), & \text{if t is left scattered} \\ f_{-}^{1}(t), & \text{if t is dense} \end{cases}$$

$f_{-}^{\nabla}(t)$ and $f_{+}^{\Delta}(t)$ are respectively the left nabla and right delta derivatives of f in t.

It is clear that if t is left scattered or right scattered,

$f_{-}^{\nabla}(t) = f^{\nabla}(t)$ and $f_{-}^{\nabla}(t) = f_{+}^{\Delta}(t)$,

Then we have,

$f_{-}^{\nabla}(t) = f^{\nabla}(t) = f_{+}^{\Delta}(t) = f^{\Delta}(t)$

Elsewhere, the function is neither right delta nor nabla differentiable at t.

Remark 4.28

Let $f: \mathbb{T} \to \mathbb{R}$ be quasiconvex and $t \in \mathbb{T}_{k}^{k}$ such that there exist $f_{-}^{\nabla}(t)$ and $f_{+}^{\Delta}(t)$

And suppose $S_{\alpha}(f)$ is closed and $\overline{S_{\alpha}}(f)$ is open. Then the function is semicontinuous.

Now, if t is scattered, $S_{\alpha}(f)$ is closed and $\overline{S_{\alpha}}(f)$ is open, then the existence of $f_{-}^{\nabla}(t)$ and $f_{+}^{\Delta}(t)$ is the existence of $f^{\nabla}(t)$ and $f^{\Delta}(t)$ and which implies the semicontinuity of f in $\mathbb{T}$.

If t is dense then we have $f_{-}^{1}(t)$ and $f_{+}^{1}(t)$ as finite numbers.

Using the above, we present a theorem which is the quasiconvex variant of Theorem 4.1 of the Dinu (2008).

Theorem 4.29

Let $f : [q, r]_{\mathbb{T}} \to \mathbb{R}$ be quasiconvex function. Then for all $a, b \in [q, r]_{\mathbb{T}}$ with $a < b$,we have $f_{-}^{\nabla}(a) \leq f_{+}^{\Delta}(a) \leq f_{-}^{\nabla}(b) \leq f_{+}^{\Delta}(b)$ and hence both f_{-}^{∇} and f_{+}^{Δ} exist and they are increasing on $[q, r]_{\mathbb{T}}$.

Proof

Let $x < y < a < z \in [q, r]_{\mathbb{T}}$.Then from the definition of convexity and Remark 4.28, we get

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(y) - f(a)}{y - a} < \frac{f(z) - f(a)}{z - a}$$

Suppose a is right scattered and left dense and y approaches a, we have

$$\lim_{y \to a} \frac{f(y) - f(a)}{y - a} = f^{\nabla}(a) \leq \frac{f(z) - f(a)}{z - a}$$

We notice that

$$\mathcal{F} : [q, r]_{\mathbb{T}} \to \mathbb{R}, \mathcal{F}(a) = \frac{f(y) - f(a)}{y - a} \tag{31}$$

Is nondecreasing and bounded above as a function of a.

If we substitute $z = \sigma(a)$ into (31), we obtain

$$f^{\nabla}(a) \leq \frac{f(\sigma(a)) - f(a)}{\sigma(a) - a} = f^{\Delta}(a)$$

$$f^{\nabla}(a) \leq f^{\Delta}(a)$$

A similar argument will give the same conclusion for a being left scattered and right dense. Assume that a is left scattered and right dense and z approaches a, we have

$$\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f^{\Delta}(a) \geq \frac{f(y)-f(a)}{y-a} \tag{32}$$

If we put $y = \rho(a)$ into (32), we arive at

$$f^{\Delta}(a) \geq \frac{f(\rho(a))-f(a)}{\rho(a)-a} = f^{\nabla}(a) \tag{33}$$

Re-writing (33), we get (31).

If a is an isolated point, that is, $\rho(a) < a < \sigma(a)$, then

$$\frac{f(\rho(a))-f(a)}{\rho(a)-a} \leq \frac{f(\sigma(a))-f(a)}{\sigma(a)-a} \equiv f^{\nabla}(a) \leq f^{\Delta}(a) \tag{34}$$

If a is dense, that is, $\rho(a) = a = \sigma(a)$,

then making y and z to tend to a and using the nondecreasing function $\mathcal{F}$, we get,

$$\lim_{y\to a, y<a} \frac{f(y)-f(a)}{y-a} \leq \lim_{z\to a, z>a} \frac{f(z)-f(a)}{z-a}$$

Thus, for every $a \in [q,r]_{\mathbb{T}}$, we have

$$f_{-}^{\nabla}(a) \leq f_{+}^{\Delta}(a) \tag{35}$$

Conversly, for $a < z < w < b$, we have

$$\frac{f(z)-f(a)}{z-a} \leq \frac{f(w)-f(a)}{w-a} \leq \frac{f(w)-f(b)}{w-b} \tag{36}$$

Concise form of (35) gives

$$f_{+}^{\Delta}(a) \leq f_{-}^{\nabla}(b) \tag{37}$$

Combining (36) and (37) concludes the proof.

Remark 4.30

From Remark 4.28 and Theorem 4.29, the semicontinuity of a quasiconvex function in $[q,r]_{\mathbb{T}}$ is well established.

Next, the subdifferential of a quasiconvex function is presented. This is a time scale variant of the quasiconvex subdifferential introduced by Daniilidis et al (2002).

Definition 4.31 The quasiconvex subdifferential $\partial^q f : \mathbb{T} \rightarrow \mathbb{R}$ of a lower semicontinuous function f is defined for all $t \in I_{\mathbb{T}}$ as follows

$$\partial^q f(t) = \begin{cases} \partial f(t) \cap N_{S_\alpha(f)}, & \text{if } N_{\overline{S_\alpha}(f)} \neq \{0\} \\ \emptyset, & \text{if } N_{\overline{S_\alpha}(f)} = \{0\} \end{cases}$$

where $\partial f(t) = \{t^* \in \mathbb{R} : f(s) \geq f(t) + < t^*, s - t >, s \in \mathbb{T}\}$

whenever f is convex;

$N_{S_\alpha(f)} = \{t^* \in \mathbb{R} : \forall s \in \mathbb{I}_{\mathbb{T}} , < t^*, s - t > \leq 0\}$

is the normal cone to sublevel set.

$S_\alpha(f)$ and $N_{\overline{S_\alpha}(f)}$ is the normal cone to the strict sublevel set $\overline{S_\alpha}(f)$ and "$< >$" is an inner product.

The set of all such t^* is called the subdifferential of f at t denoted by $\partial^q f(t)$. This subdifferential provides the gradients of the lines that touch the graph of the function. The subdifferential is usually a nom empty convex compact set or convex closed set but however it can be an empty set (Daniilidis et al, 2002).

Mostly, the subdifferential generalizes the derivative of the functions at points that are not differentiable. However, the subdifferential of a function be found espercially when it is continuous.

The function $f: \mathbb{I}_{\mathbb{T}} \rightarrow \mathbb{R}$ admits a hyperplane at $t \in \mathbb{I}_{\mathbb{T}}$ if there exists a $t^* \in \mathbb{R}$ such that

$$f(s) \geq f(t) + t^*(s - t), \forall s \in \mathbb{I}_{\mathbb{T}} \tag{38}$$

Thus, equation (38) becomes

$$\frac{f(s) - f(t)}{s - t} \geq t^*$$

Proposition 4.32 Let the restriction of f on line segments be continuous (that is f is radially continuous). Then

(i). $\forall\, t \in \mathbb{I}_{\mathbb{T}}$, we have

$$\partial^q f(t) = \begin{cases} \partial f(t) \cap N_{S_\alpha(f)}, & \text{if } N_{\overline{S_\alpha}(f)} \neq \{0\} \\ \emptyset, & \text{if } N_{\overline{S_\alpha}(f)} = \{0\} \end{cases}$$

(ii). $\partial^q f(t) / \{0\} \subseteq \partial f(t)$

Proof

(i). If $0 \in \partial f(t), \partial f(t) = \mathbb{R}$. Hence, if $\partial f(t) \neq \emptyset$, then

$N_{\overline{S_\alpha}(f)} \neq \{0\}$. Then, we have to show that if $\partial f(t) \neq \emptyset$, then $N_{\overline{S_\alpha}(f)} = \{0\}$.

Note that $t^* \in N_{S_\alpha(f)} \Rightarrow < t^*, s - t > \Rightarrow f(s) > f(t), \forall\, s \in \mathbb{T}$

Therefore, we always have have $0 \in N_{S_\alpha(f)}$. Hence, its conclusive that $\partial^q f(t) = \emptyset$

Whenever $\partial f(t) = \emptyset$. Furthermore, we show that $N_{S_\alpha(f)} / \{0\} \subseteq \partial f(t)$. Suppose

$t^* \in N_{S_\alpha(f)} / \{0\}$ and assume that $< t^*, s - t > \geq 0$.

We choose $h \in \mathbb{T}$ such that $< t^*, h > > 0$. For any $\lambda > 0$, we get

$< t^*, s + \lambda h - t > > 0$ and hence $f(s + \lambda h) > f(t)$.

Letting $\lambda \to 0$ and since f is radially continuous $f(s) \geq f(t)$, that is

$t^* \in \partial f(t)$. Thus if $\partial f(t) \neq \emptyset$, then

$\partial^q f(t) = \partial f(t) \cap N_{S_\alpha(f)}$.

(ii). The second statement follows from the inclusions:

$$\partial^q f(t) / \{0\} \subseteq N_{S_\alpha(f)} / \{0\} \subseteq N_{\overline{S_\alpha}(f)} / \{0\} \subseteq \partial^q f(t)$$

Definition 4.33 Let $\partial^q : \mathbb{T} \to \mathbb{R}$ be a multivalued operator. Then ∂^q is cyclically quasimonotone, if for any $n \geq 1$ and $t_1, t_2, \dots, t_n \in \partial^q$, there exists $i \in \{1,2, \dots, n\}$ such that

$< t^*{}_i, t_{i+1} - t_i > \geq 0, \forall\, t^*{}_i \in \partial^q(t^*{}_i)$, (where $t_{n+1} = t_1$). If n is restricted to $n = 2$, then ∂^q is quasimonotone.

Proposition 4.34 For every lower semicontinuous quasiconvex function, the quasiconvex differential $(\partial^q f)$ is quasimonotone.

Proof

Let $t_i \in \mathbb{T}, i = 1,2$ and $t^*{}_i \in N_{S_\alpha(f)}$ such that

$< t^*{}_i, t_{i+1} - t_i > > 0$ for all i (where $t_2 = t_1$),

then $f(t_{i+1}) > f(t_i)$ for all i. By transivity, we get

$f(t_2) < f(t_1)$, hence a contradiction.

Thus, $< t^*{}_i, t_2 - t_1 > > 0$ for $i = 1,2$ and hence from Definition 4.33 f is quasimonotone.

Theorem 4.35 Let $f : [q, r]_{\mathbb{T}} \to \mathbb{R}$ be a quasiconvex function on a time scale. Then $\partial^q f(x) = \emptyset, \forall\, x \in (a, b)_{\mathbb{T}}$. For any function $g : [q, r]_{\mathbb{T}} \to \mathbb{R}$ such that $g(x) \in \partial^q f(t)$, verifies the inequality

$f^\nabla(x) \leq g(x) \leq f^\Delta(x)$

For all $x \in (a, b)_{\mathbb{T}}$ and thus f is nondecreasing.

Proof

Let $x < u \leq y \in [q, r]_{\mathbb{T}}$. Since every quasiconvex function is convex, we have

$$\frac{f(u) - f(x)}{u - x} \leq \frac{f(y) - f(x)}{y - x}$$

If u approaches x, then $\frac{f(u)-f(x)}{u-x}$ tends to $f_+^1(x)$ and if x is right dense and

right scattered, then $\frac{f(u)-f(x)}{u-x}$ tends to $f^{\Delta}(x)$. This gives

$f(y) \geq f(x) + f^{\Delta}(x)(y-x)$

Arguing the same way for $y \leq u < t \in [q,r]_{\mathbb{T}}$, we have

$f(y) \geq f(x) + f^{\nabla}(x)(y-x)$

Using Theorem 4.31, we have

$$f^{\nabla}(x) \leq f^{\Delta}(x) \tag{39}$$

and hence (39) holds for all $x \in (a,b)_{\mathbb{T}}$ and $y \in [q,r]_{\mathbb{T}}$. $f^{\nabla}(x), f^{\Delta}(x) \in N_{S_{\alpha}(f)}$

and therefore the left nabla and right delta derivatives belong to

and therefore the left nabla and right delta derivatives $\partial^q f(x)$.Thus, for every

$g(x) \in [f^{\nabla}(x), f^{\Delta}(x)]$, we have

$f^{\nabla}(x) \leq g(x) \leq f^{\Delta}(x)$

Remark 4.36 If $f : [q,r]_{\mathbb{T}} \to \mathbb{R}$ is a quasiconvex function and $g : \mathbb{T} \to \mathbb{R}$ is a function

such that $g(z) \in \partial^q f(z)$, for all $z \in \mathbb{T}_k^k$, then

$f(y) \geq \sup\{f(x) + x^*(y-x)\}$

for all $x^* \in \mathbb{R}$ and $y \in (q,r)_{\mathbb{T}}$. Moreover, if f is lower semicontinuous, then the above

relation holds for all $y \in [q,r]_{\mathbb{T}}$.

Theorem 4.37 Let $f : [q,r]_{\mathbb{T}} \to \mathbb{R}$ be a function such that $\partial^q f(x) \neq \emptyset$ for all $(q,r)_{\mathbb{T}}$.

Then f is quasiconvex.

Proof

Let $x, y \in (q,r)_{\mathbb{T}}$ and $\theta \in [0,1]$such that $\theta x + (1-\theta)y \in (q,r)_{\mathbb{T}}$.

For every $\omega \in \partial^q(\theta x + (1-\theta)y)$, we get

$$f(y) \geq f(\theta x + (1-\theta)y) - \theta(x-y)\omega \tag{40}$$

$$f(x) \geq f(\theta x + (1-\theta)y) + (1-\theta)(x-y)\omega \tag{41}$$

By multiplying (40) by $(1-\theta)$ and (41) by θ, we have

$$(1-\theta)f(y) \geq (1-\theta)f(\theta x + (1-\theta)y) - \theta(1-\theta)(x-y)\omega \tag{42}$$

$$\theta f(x) \geq \theta f(\theta x + (1-\theta)y) + \theta(1-\theta)(x-y)\omega \tag{43}$$

Summing (42) and (43), we have

$$\theta f(x) + (1-\theta)f(y) \geq (1-\theta)f(\theta x + (1-\theta)y) + \theta f(\theta x + (1-\theta)y) \tag{44}$$

Simplifying (44), we get

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y) \leq \max\{f(x), f(y)\}$$

Thus,

$$f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$$

Hence the quasiconvexity of f

4.4 Applications of Quasiconvex functions on Time Scales

We present some Jensen type inequalities for quasiconvex functions and give some applications in the area of probability theory and mathematical finance in this section.

Proposition 4.38 For an arbitrary mapping $f: \mathbb{T} \to (0,1)$ and $s_i \in \mathbb{T}$ being convex for all $i = 1,2, \ldots, n$, we can define the map $g_{s_i} : [0,1] \to \mathbb{R}$ by $g_{s_i}(t_i) \leq f(\sum_{i=1}^{n} s_i t_i)$,

For $s_i \in [0,1]$.

The following statements are equivalent:

(i). f is quasiconvex on $\mathbb{T}$.

(ii). For every $s_i \in \mathbb{T}$, the mapping g_{s_i} is qiasiconvex on [0, 1].

Proof

Assume that f is quasiconvex on $\mathbb{T}$.

Let $t_i \in [0,1]$ and $\alpha_i \geq 0$ with $\sum \alpha_i = 1$. Then

$$g_{s_i}(\sum_{i=1}^{n} \alpha_i t_i) = f(\sum_{i=1}^{n} \alpha_i s_i t_i)$$

We proof by induction by letting t_1, t_2 be two fixed points in $\mathbb{T}$ and $i = 1,2$ with

$\alpha_1 + \alpha_2 = 1$.

$$g_{s_i}(\sum_{i=1}^{2} \alpha_i t_i) = f(\sum_{i=1}^{2} \alpha_i s_i t_1 + [1 - \sum_{i=1}^{2} \alpha_i s_i] t_2)$$

$$g_{s_i}(\sum_{i=1}^{2} \alpha_i t_i) = f(\alpha_1 s_1 t_1 + \alpha_2 s_2 t_2 + [1 - (\alpha_1 s_1 + \alpha_2 s_2)] t_2)$$

$$g_{s_i}(\sum_{i=1}^{2} \alpha_i t_i) = f(\alpha_1 s_1 t_1 + \alpha_2 s_2 t_2 + t_2 - \alpha_1 s_1 t_2 - \alpha_2 s_2 t_2) \quad (45)$$

Rewriting (45), we have

$$g_{s_i}(\sum_{i=1}^{2} \alpha_i t_i) = f(\sum_{i=1}^{2} \alpha_i [s_i t_1 + (1 - s_i) t_2])$$

$$g_{s_i}(\sum_{i=1}^{2} \alpha_i t_i) \leq \max_{i=1,2}[f(s_i t_1 + (1 - s_i) t_2)]$$

$$g_{s_i}(\sum_{i=1}^{2} \alpha_i t_i) \leq \max_{i=1,2}\{g_{s_i}(t_i)\}$$

Thus,

$$g_{s_i}(\sum_{i=1}^{n} \alpha_i t_i) \leq \max_{i=1,2,\dots,n}\{g_{s_i}(t_i)\}$$

which shows that the mapping g_{s_i} is quasiconvex on $[0,1]$.

Conversely, suppose that (ii) holds. Then for any isolated points $s_1, s_2 \in \mathbb{T}$ and $t \in [0,1]$

we have,

$$f(ts_1 + (1 - t)s_2) = g_{s_1,s_2}(t)$$

$$f(ts_1 + (1 - t)s_2) = g_{s_1,s_2}((1 - t) \cdot 0 + t \cdot 1)$$

$$f(ts_1 + (1 - t)s_2) = \max_{i=1.2}\{g_{s_1,s_2}(0), g_{s_1,s_2}(1)\}$$

$$f(ts_1 + (1 - t)s_2) \leq \max\{f(s_1), f(s_2)\}$$

Proposition 4.39 Suppose that φ_k is quasiconvex on $[0,1]$ for $k = 1,2 \dots, n$. Then

$\varphi = \max_{1\le k\le n} \varphi_k$ is quasiconvex on $[0,1]$.

Proof

Let $s_1, s_2 \in [0,1]$ be isolated points. Then

$\varphi(\alpha_1 s_1 + \alpha_2 s_2) = \max_{1\le k\le n} \varphi_k (\alpha_1 s_1 + \alpha_2 s_2)$

$\varphi(\alpha_1 s_1 + \alpha_2 s_2) \le \max_{1\le k\le n} \max_{i=1,2} \varphi_k(s_i)$

$\varphi(\alpha_1 s_1 + \alpha_2 s_2) \le \max\limits_{i=1,2} \max\limits_{1\le k\le n} \varphi_k(s_i)$

$\varphi(\alpha_1 s_1 + \alpha_2 s_2) \le \max\limits_{i=1,2} \varphi(s_i)$

which establishes the quasiconvexity of φ.

4.4.1 Discrete Probabilistic Interpretation of Jensen's Inequality for Quasiconvex functions On Time Scales

Suppose that $S = \mathbb{Z}$ is a discrete random variable such that $S = \{s_1, s_2, \dots, s_n\} \subseteq \mathbb{T}$ with probabilities $P(S = s_i) = \theta_i$ where $\theta_i \ge 0$ with $\sum_{i=1}^{n} \theta_i = 1$. Let $f(s)$ be an arbitrary discrete probability mass function. Then the following properties are satisfied:

Property 1: $f(s_i) \ge 0$ for $i = 1,2, \dots, n$.

Property 2: $\sum_{i=1}^{n} f(s_i) = 1$, where the summation is over all the possible values of the random variable S.

Property 3: The expectation of S is $E(S) = \sum_{i=1}^{n} \theta_i s_i$.

Example 4.40

From Table 4.1, find the expectation $E(S)$ and show that for an arbitrary function $f: S \subseteq \mathbb{T} \to [0,1]$ we can define a mapping (expectation function)

$g_{s_i}: [0,1] \to \mathbb{R}$ by

$g_{s_i}(\theta_i) \leq f(E(S))$.

Table 4.1 Probability Distribution of S

S	1	2	3
$f(s) =$	$\frac{5}{21}$	$\frac{1}{3}$	$\frac{3}{7}$

Solution

The probability mass function is

$$f(s) = \begin{cases} \frac{1}{21}(2s+3), & s = 1,2,3 \\ 0, & \text{else where} \end{cases}$$

$$E(S) = \sum_{i=1}^{3} s_i f(s_i) = 1 \cdot \frac{5}{21} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{3}{7} = \frac{46}{21} = 2.1905$$

Let $g_{s_i}(\theta_i) \leq f(E(S))$.

Thus,

$$g_{s_i}(\theta_i) \leq f\big(E(S)\big) = f\left(\frac{46}{21}\right) = \frac{1}{21}\left(2\left(\frac{46}{21}\right) + 3\right) = \frac{155}{441} = 0.35$$

$$g_{s_i}(\theta_i) \leq f(\textstyle\sum_{i=1}^{n} \theta_i s_i) = f(\theta_1 s_1 + \theta_2 s_2 + \theta_3 s_3)$$

$\theta_i \in [0,1]$.

$f(\theta_1 s_1 + \theta_2 s_2 + \theta_3 s_3) \leq \max_{i=1,2,3}\{f_{s_i}\}$.

Thus, we have

$g_{s_i}(\theta_i) \leq \max_{i=1,2,3}\{f_{s_i}\}$.

$$g_{s_1}(\theta_1) = f_{s_1} = \frac{5}{21} = 0.24;\ g_{s_2}(\theta_2) = f_{s_2} = \frac{1}{3} = 0.66;\ g_{s_3}(\theta_3) = f_{s_3} = \frac{3}{7} = 0.43$$

$g_{s_i}(\theta_i) = 1.3$.

$1.3 < 2.1905$ and therefore confirms proposition 4.38.

Proposition 4.41

Suppose that $\varphi : \mathbb{I}_{\mathbb{T}} \to \mathbb{R}$ is quasiconvex and S is a discrete random variable taking values in $\mathbb{I}_{\mathbb{T}}$ with $S = \{s_1, s_2, \dots, s_n\} \subset \mathbb{T}$ and probabilities $f(S = s_i) = \theta_i$, then $\varphi(E(S)) \leq E(\varphi(S)$.

Proof

Suppose that $\varphi : \mathbb{I}_{\mathbb{T}} \to \mathbb{R}$ is quasiconvex and $s_1, s_2, \dots, s_n \in S$ are arbitrary random variables which are right scattered with

$\theta_1, \theta_2, \dots, \theta_n \in (0,1)$ arbitrary weights. The arithmetic mean

$E(S) = \bar{s} = \sum_{i=1}^{n} \theta_i\, s_i$ is a point in $\mathbb{I}_{\mathbb{T}}$.

Therefore, the support function below satisfies all $s \in \mathbb{I}_{\mathbb{T}}$, such that

$$\varphi(\bar{s}) + \lambda(\bar{s})(s - \bar{s}) \leq \varphi(s) \tag{46}$$

where $\lambda(\bar{s})$ is the gradient at $\bar{s}$.

Putting $s = s_i$, multiplying (46) by θ_i and summing gives

$$\sum_{i=1}^{n}[\theta_i\, \varphi(\bar{s}) + \lambda(\bar{s})(s_i - \bar{s})\theta_i] \leq \sum_{i=1}^{n} \theta_i \varphi(s_i) = E(\varphi(s))$$

$$\sum_{i=1}^{n}[\theta_i\, \varphi(\bar{s}) + \sum_{i=1}^{n} \lambda(\bar{s})(s_i - \bar{s})\theta_i] \leq E(\varphi(s))$$

$$\sum_{i=1}^{n}[\theta_i\, \varphi(E(S) + \sum_{i=1}^{n} \lambda(E(S))(s_i - E(S))\theta_i] \leq E(\varphi(s))$$

$$\varphi\big(E(S)\big) + \sum_{i=1}^{n} \lambda(E(S))\, s_i\theta_i - \sum_{i=1}^{n} \lambda(E(S))E(S))\theta_i \ \leq E(\varphi(s))$$

$$\varphi\big(E(S)\big) + \lambda(E(S)) \sum_{i=1}^{n} s_i\theta_i - \lambda(E(S))E(S)) \sum_{i=1}^{n} \theta_i \ \leq E(\varphi(s))$$

$$\varphi\big(E(S)\big) + \lambda\big(E(S)\big)E(S) - \lambda(E(S))E(S)) \leq E(\varphi(s))$$

$$\varphi\big(E(S)\big) \leq E(\varphi(s))$$

This is the well-known arithmetic-mean inequality for $\mathbb{T} = \mathbb{Z}$.

4.4.2 Continuous Probabilistic Interpretation of Jensen's Inequality for Quasiconvex functions on Time Scales

Proposition 4.42

Let $f(t)$ be a probability density function for random variable $S \in \mathbb{T}$ such that $S \in [a, b]$.

suppose $f(t)$ is integrable, $f(t) \geq 0$ and $\int_a^b f(t)dt = 1$. The expectation of S is

$\bar{t} = E(S) = \int_a^b tf(t)dt$. Therefore,

$\varphi\big(E(S)\big) \leq E\big(\varphi(S)\big)$.

Proof

Let t be left and right dense in an interval $[a, b]$. Since the probability density function is quasiconvex, there exist a subdifferential

$\varphi: S \to \mathbb{R}$ which is convex for all $t \in S$. Then

$$\varphi(\bar{t}) + \lambda(\bar{t})(t - \bar{t}) \leq \varphi(t) \quad (47)$$

Integrating (47), we have

$$\int_a^b [\varphi(\bar{t}) + \lambda(\bar{t})(t - \bar{t})]\, f(t)\Delta t \leq \int_a^b \varphi(t)\, f(t)\Delta t$$

$$\int_a^b \varphi(\bar{t})\, f(t)\Delta t + \int_a^b \lambda(\bar{t})(t - \bar{t})f(t)\Delta t \leq \int_a^b \varphi(t)\, f(t)\Delta t$$

$$\varphi(\bar{t}) \int_a^b f(t)\Delta t + \int_a^b \lambda(\bar{t})t\, f(t)\Delta t - \int_a^b \lambda(\bar{t})\bar{t}\, f(t)\Delta t \leq \int_a^b \varphi(t)\, f(t)\Delta t$$

$$\varphi(\bar{t}) + \lambda(\bar{t})\bar{t} - \lambda(\bar{t})\bar{t} \leq E(\varphi(t)) \quad (48)$$

Simplifying (48) gives $\varphi(E(S)) \leq E(\varphi(S))$ confirming the proposition.

This is the continuous time scale version which is the same as the classical Jensen inequality when $\mathbb{T} = \mathbb{R}$.

Proposition 4.43 Let $[a, b] \in \mathbb{T}$ and $s, t \in (c, d) \in \mathbb{R}$. Suppose that $f: [a, b] \to (c, d)$ is right dense continuous; $\mathcal{G}: (c, d) \to \mathbb{R}$ is quasiconvex. Then

$$\mathcal{G}\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \leq \frac{\int_a^b \mathcal{G}(f(t))\Delta t}{b-a}.$$

Proof

Let $\bar{s} \in (c, d)$. Then there exists a $\beta \in \mathbb{R}$ such that

$$\mathcal{G}(s) - \mathcal{G}(\bar{s}) \geq \beta(s - \bar{s}) \quad (49)$$

for all $s \in (c, d)$.Since f is rd-continuous,

$$\bar{s} = \frac{\int_a^b f(\tau)\Delta\tau}{b-a} \quad (50)$$

$\mathcal{G}(f(t))$ is also rd-continuous and hence we can apply (51) and set $s = f(t)$ and integrate from a to b to get

$$\int_a^b \mathcal{G}(f(t))\Delta t - (b-a)\mathcal{G}\left(\frac{\int_a^b f(\tau)\Delta\tau}{b-a}\right) = \int_a^b \mathcal{G}(f(t))\Delta t - (b-a)\mathcal{G}(\bar{s}) \quad (51)$$

From (49) $(b-a)\mathcal{G}(\bar{s}) = \int_a^b \mathcal{G}(\bar{s})\Delta t$ and therefore (51) yields

$$\int_a^b \mathcal{G}(f(t))\Delta t - (b-a)\mathcal{G}\left(\frac{\int_a^b f(\tau)\Delta\tau}{b-a}\right) = \int_a^b [\mathcal{G}(f(t) - \mathcal{G}(\bar{s})]\Delta t \quad (52)$$

Substituting (49) into (52) we have

$$\int_a^b \mathcal{G}(f(t))\Delta t - (b-a)\mathcal{G}\left(\frac{\int_a^b f(\tau)\Delta\tau}{b-a}\right) \geq \int_a^b \beta(s - \bar{s})\Delta t \quad (53)$$

But $s = f(t)$ and therefore (127) gives

$$\int_a^b \mathcal{G}(f(t))\Delta t - (b-a)\mathcal{G}\left(\frac{\int_a^b f(\tau)\Delta\tau}{b-a}\right) \geq \beta \int_a^b (f(t) - \bar{s})\Delta t \quad (54)$$

$$\int_a^b \mathcal{G}(f(t))\Delta t - (b-a)\mathcal{G}\left(\frac{\int_a^b f(\tau)\Delta\tau}{b-a}\right) \geq \beta\left(\int_a^b f(t)\Delta t - \int_a^b \bar{s}\,\Delta t\right)$$

$$\int_a^b \mathcal{G}(f(t))\Delta t - (b-a)\mathcal{G}\left(\frac{\int_a^b f(\tau)\Delta\tau}{b-a}\right) \geq \beta[(b-a)\bar{s} - \bar{s}(b-a)]$$

$$\int_a^b \mathcal{G}(f(t))\Delta t - (b-a)\mathcal{G}\left(\frac{\int_a^b f(\tau)\Delta\tau}{b-a}\right) \geq \beta(0)$$

$$\int_a^b \mathcal{G}(f(t))\Delta t - (b-a)\mathcal{G}\left(\frac{\int_a^b f(\tau)\Delta\tau}{b-a}\right) \geq 0$$

This yields the Jensen's inequality.

Example 4.44

The probability density function of a random variable S is given by

$$f(s) = \begin{cases} 0 & s < 0 \\ \frac{s}{2} & 0 \leq s \leq 2 \\ 0 & s > 0 \end{cases}$$

Find the cumulative density function $\mathcal{G}(s)$ and verify that Proposition 4.43 holds.

Solution

If $s < 0$, then

$$\mathcal{G}(s) = \int_{-\infty}^{t} f(t)dt = 0.$$

If $0 \leq s \leq 2$, then

$$\mathcal{G}(s) = \int_{-\infty}^{o} f(t)dt + \int_0^s f(t)dt = 0 + \int_0^s \frac{t}{2}dt = \left[\frac{t^2}{4}\right]_0^s = \frac{s^2}{4}$$

If $s > 0$, then

$$\mathcal{G}(s) = \int_{-\infty}^{o} f(t)dt + \int_0^2 f(t)dt + \int_2^s f(t)dt = \int_0^2 \frac{t}{2}dt = \left[\frac{t^2}{4}\right]_0^2 = \frac{4}{4} = 1$$

Thus,

$$\mathcal{G}(s) = \begin{cases} 0 & s < 0 \\ \frac{s^2}{4} & 0 \leq s \leq 2 \\ 0 & s > 0 \end{cases}$$

From Proposition 4.43, we have

$$\mathcal{G}\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \leq \frac{\int_a^b \mathcal{G}(f(t))\Delta t}{b-a}$$

$[0,2] = [a, b]$ and $(0,1) = (c, d)$.

Solving the left-hand side of Proposition 4.43, we have

$$\int_0^2 f(t)\Delta t = \int_0^2 \frac{t}{2}\Delta t = \left[\frac{t^2}{4}\right]_0^2 = \frac{4}{4} = 1$$

$$\mathcal{G}\left(\frac{1}{2-0}\right) = \mathcal{G}\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^2}{4} = \frac{1}{16}$$

Solving the left-hand side of Proposition 4.43, we get

$$\frac{\int_0^2 \mathcal{G}\left(\frac{t}{2}\right)\Delta t}{2-0} = \frac{\int_0^2 \left(\frac{t}{4}\right)^2}{2-0} = \frac{\int_0^2 \left(\frac{t^2}{16}\right)\Delta t}{2} = \frac{\left[\frac{1}{3}.\frac{t^3}{16}\right]_0^2}{2} = \frac{\frac{8}{48}}{2} = \frac{1}{12}$$

Therefore, $\frac{1}{16} < \frac{1}{12}$, which verifies Proposition 4.43.

4.4.3 Jensen's Inequality for Monetary Utility Functions on Time Scales

Monetary utility functions are non- linear functions that are bounded and asymmetric about the origin. They have attracted much attention in mathematical finance in recent times because of their usefulness and profound applications in the decision making process. In situations where outcomes of choices influence utility through gains or losses of money which is usual in the business environment, the optimal choice for a given decision depends on the possible outcomes of all other decisions in the same period of time.

Jensen's inequality holds for classical expectation, which in terms of operator, can be seen as a particular type of monetary utility function (Liu and Jiang, 2012). The interest in this section is to examine the application of quasiconcave (the negative of quasiconvex) functions on time scales. The monetary utility function is quasiconcave in nature.

Notations and Assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with Ω describing the set of all possible outcomes; $\mathcal{F}$ the collection of complex events used to characterize groups of outcomes and $\mathbb{P}$ the probability measure function.

Assume that $\mathbb{L}^{\infty} = \{ \{x_n\}_{n=1}^{\infty} : |x_n| \leq M, \forall\, n \in \mathbb{N}, M \text{ is a constant}\}$ is a space of bounded random variables.

Definition 4.45 A function $\mathcal{U}: \mathbb{L}^{\infty} \to \mathbb{R}$ is called a monetary utility function if it is nondecreasing with respect to the order of $\mathbb{L}^{\infty}$ and satisfies

(i) Normalization condition: $\mathcal{U}(S) \geq 0$ if $S \geq 0$.

(ii) Quasiconcavity: $\mathcal{U}(\lambda S + (1-\lambda)T) \geq \min\{\mathcal{U}(S), \mathcal{U}(T)\}$, for all $S, T \in \mathbb{L}^{\infty}$ and $a \leq \lambda \leq b$ $(0 \leq \lambda \leq 1)$.

(iii) Monotonicity: $\mathcal{U}(S) \geq \mathcal{U}(T)$ for all $S, T \in \mathbb{L}^{\infty}$ such that $S \geq T$.

(iv) Monetary or cash invariance property: $\mathcal{U}(S + m) = \mathcal{U}(S) + m$, for all $S \in \mathbb{L}^{\infty}$ and $m \in \mathbb{L}^{\infty}$.

(v) Fatou property: If $\{\sup \|S_n\|\}_{n=1}^{\infty} < \infty$, if $S_n \to S$ in probability, then $\mathcal{U}(S) \geq \limsup \cup (S_n)$.

Remark 4.46 The monotonicity and monetary property suggest that $\mathcal{U}$ is finite and Lipschitz-continuous on $\mathbb{L}^{\infty}$. Thus, the normalization $\mathcal{U}(0) = 0$ does not restrict the generality as it can be obtained by adding a constraint (Jouini et al, 2008).

Proposition 4.49 is of vital importance in order to establish our main results. This proposition is the same as proposition 2.1 in Liu and Jiang (2012) though we imposed quasiconcavity on the monetary utility function.

Proposition 4.47 Let $\mathcal{U}: \mathbb{L}^{\infty} \to \mathbb{R}$ be a monetary utility function which is quasiconcave.

Then for any $\lambda \in \mathbb{R}$, $S \in \mathbb{L}^{\infty}$, the inequalities

(i). $\mathcal{U}(\lambda S) \geq \lambda \mathcal{U}(S)$, if $a \leq \lambda \leq b$ (55)

(ii). $\mathcal{U}(\lambda S) \leq \lambda \mathcal{U}(S)$, if $\lambda \leq a$ or $\lambda \geq b$ (56)

hold.

Proof

(i). For $a \leq \lambda \leq b$ and quasiconcavity of $\mathcal{U}$, we have

$$\mathcal{U}(\lambda S + (1-\lambda)T) \geq \lambda \mathcal{U}(S) + (1-\lambda)\mathcal{U}(T) \geq \min\{\mathcal{U}(S), \mathcal{U}(T)\},$$

for all $S, T \in \mathbb{L}^{\infty}$.

For the property of monotonicity of $\mathcal{U}$ and $S \geq T$, we have

$$\mathcal{U}(\lambda S + (1-\lambda)T) \geq \lambda \mathcal{U}(S) + (1-\lambda)\mathcal{U}(T) \geq \mathcal{U}(T).$$

$$\mathcal{U}(\lambda S + (1-\lambda)T) \geq \lambda \mathcal{U}(S) + \mathcal{U}(T) - \lambda \mathcal{U}(T) - \mathcal{U}(T) \geq 0$$

$$\mathcal{U}(\lambda S + (1-\lambda)T) \geq \lambda \mathcal{U}(S) - \lambda \mathcal{U}(T)$$

Take $T = 0, \mathcal{U}(0) = 0$ (Normalization condition)

Thus,

$$\mathcal{U}(\lambda S) \geq \lambda \mathcal{U}(S).$$

(ii). Consider $\lambda \geq b$, then $a \leq \frac{1}{\lambda} \leq b$. By (55),

$$\mathcal{U}\left(\left(\frac{1}{\lambda}\right)(\lambda S)\right) \geq \frac{1}{\lambda}\mathcal{U}(\lambda S)$$

$$\lambda \mathcal{U}\left(\left(\frac{1}{\lambda}\right)(\lambda S)\right) \geq \mathcal{U}(\lambda S)$$

$$\lambda \mathcal{U}(S) \geq \mathcal{U}(\lambda S)$$

For $-b \leq \lambda \leq a$, then $a \leq -\lambda \leq b$. By (55)

$$\mathcal{U}((-\lambda)S) \geq (-\lambda)\mathcal{U}(S) \tag{57}$$

But,

$$0 = \mathcal{U}(0) = \mathcal{U}\left(\tfrac{1}{2}\,\lambda S + \tfrac{1}{2}(-\lambda S)\right) \geq \tfrac{1}{2}\mathcal{U}(\lambda S) + \tfrac{1}{2}\mathcal{U}((-\lambda)S)$$

$$0 \geq \tfrac{1}{2}\mathcal{U}(\lambda S) + \tfrac{1}{2}\mathcal{U}((-\lambda)S)$$

$$-\mathcal{U}(\lambda S) \geq \mathcal{U}((-\lambda)S)$$

$$\mathcal{U}((-\lambda)S) \leq -\mathcal{U}(\lambda S) \tag{58}$$

Combining (57) and (58), we have

$$-\lambda\mathcal{U}(S) \leq \mathcal{U}((-\lambda)S) \leq -\mathcal{U}(\lambda S)$$

Thus,

$$-\lambda\mathcal{U}(S) \leq -\mathcal{U}(\lambda S) \quad \text{or}$$

$$\lambda\mathcal{U}(S) \geq \mathcal{U}(\lambda S)$$

This completes the proof of proposition 4.47

Next, we consider two theorems that characterize Jensen's inequality for monetary utility functions. Here, we disregard the conditions in Liu and Jiang (2012) and establish similar results.

Theorem 4.48 Let $\mathcal{U}: \mathbb{L}^\infty \to \mathbb{R}$ be any monetary utility function for all $S \in \mathbb{L}^\infty$. Then for any convex function φ on $\mathbb{R}$, we have

$$\varphi(\mathcal{U}(S)) \leq \mathcal{U}(\varphi(S)).$$

Proof

Based on the subdifferential inequality in (Dinu, 2008), we have

$$\varphi(s) \geq \varphi(\mathcal{U}(S)) + \lambda(s - \mathcal{U}(S)) \quad \text{for all } s \in \mathbb{L}^\infty \text{ and } \lambda \in \mathbb{R}.$$

For generality of s, we have

$$\varphi(S) \geq \varphi(\mathcal{U}(S)) + \lambda(S - \mathcal{U}(S)).$$

Considering the monotonicity and cash invariance of $\mathcal{U}$ together with (55) under proposition 4.47, then we obtain

$$\mathcal{U}(\varphi(S)) \geq \mathcal{U}\left(\varphi(\mathcal{U}(S)) + \lambda(S - \mathcal{U}(S))\right).$$

$$\mathcal{U}(\varphi(S)) \geq \varphi(\mathcal{U}(S)) + \lambda\mathcal{U}(S) - \lambda\mathcal{U}(S).$$

$$\mathcal{U}(\varphi(S)) \geq \varphi(\mathcal{U}(S)).$$

Hence,

$$\varphi(\mathcal{U}(S)) \leq \mathcal{U}(\varphi(S)).$$

Theorem 4.49 Let ψ be any concave function on $\mathbb{R}$. Then for any $S \in \mathbb{L}^\infty$ and any monetary utility function $\mathcal{U}: \mathbb{L}^\infty \rightarrow \mathbb{R}$, the inequality

$$\psi(\mathcal{U}(S)) \geq \mathcal{U}(\psi(S)).$$

Proof

Again from Dinu (2008),

$$\psi(S) \leq \psi(\mathcal{U}(S)) + \lambda(S - \mathcal{U}(S)), \lambda \in \mathbb{R}.$$

Following the same steps in Theorem 4.49, we have

$$\psi(\mathcal{U}(S)) \geq \mathcal{U}(\psi(S)).$$

We see that it is possible to get Jensen inequality for quasiconcave monetary utility functions with respect to convex and concave functions. It is well established that Jensen's inequality is not true for all monetary utility functions even when the associated convex or concave function is linear (Liu and Jiang, 2012).

Next, we consider when φ and ψ are quasiconvex and quasiconcave respectively.

Theorem 4.50 Let φ be a quasiconvex function on $\mathbb{R}$. Suppose that $\mathcal{U}: \mathbb{L}^\infty \rightarrow \mathbb{R}$ is a monetary utility function, then

$\varphi(\mathcal{U}(S)) \leq \mathcal{U}(\varphi(S))$ for all $S \in \mathbb{L}^{\infty}$.

Proof

Based on the subdifferential for quasiconvex function in Daniilidis et al (2002), we have

$\lambda(s - \mathcal{U}(S)) \geq 0 \Longrightarrow \varphi(S) \geq \varphi(\mathcal{U}(S)).$

$\lambda s - \lambda\mathcal{U}(S) \geq 0 \Longrightarrow \varphi(S) \geq \varphi(\mathcal{U}(S)).$

From the cash invariance property of $\mathcal{U}$, we have

$\mathcal{U}(\lambda s - \lambda\mathcal{U}(S)) \geq 0 \Longrightarrow \mathcal{U}(\varphi(S)) \geq \mathcal{U}(\varphi(\mathcal{U}(S))).$

$\mathcal{U}(\lambda s) \geq \lambda\mathcal{U}(S)) \Longrightarrow \mathcal{U}(\varphi(S)) \geq \varphi(\mathcal{U}(S)).$

$\mathcal{U}(\lambda s) \geq \lambda\mathcal{U}(S)) \Longrightarrow \varphi(\mathcal{U}(S)) \leq \mathcal{U}(\varphi(S)).$

It is possible to establish the Jensen inequality for monetary utility functions with respect to quasiconcave functions (Theorem 4.51) and prove based on the subdifferential inequality in (Daniilidis et al, 2002). Since the proof follows the procedure as Theorem 4.50, we state only the theorem.

Theorem 4.51 Let ψ be a quasiconcave function on $\mathbb{R}$ and $\mathcal{U}: \mathbb{L}^{\infty} \to \mathbb{R}$ be any monetary utility function. For all $S \in \mathbb{L}^{\infty}$, then

$\psi(\mathcal{U}(S)) \geq \mathcal{U}(\psi(S)).$

The example below shows that Jensen's inequality is true for all quasiconcave monetary utility functions with respect to certain quasiconvex and quasiconcave functions.

Example 4.52 Let $\varphi(S) = \frac{\lambda S+c}{\gamma S+d}, (\gamma S + d \neq 0, \lambda, \gamma \in \mathbb{R})$ and $\psi(S) = \frac{\alpha S+e}{\beta S+f}, (\beta S + f \neq 0, \alpha, \beta \in \mathbb{R})$ be quasiconvex and quasiconcave functions respectively on $\mathbb{R}$. We see that Jensen's inequality is true for all quasiconcave monetary utility functions in respect of quasiconcave and quasiconvex functions that are linear-fractionals.

$$\varphi(s)=\frac{\lambda s+c}{\gamma s+d}=\frac{\frac{\lambda}{\gamma}(\gamma s+d)+c-\frac{\lambda d}{\gamma}}{\gamma s+d}$$

$$\varphi\big(\mathcal{U}(S)\big)=\frac{\frac{\lambda}{\gamma}(\gamma\mathcal{U}(s)+d)+c-\frac{\lambda d}{\gamma}}{\gamma\mathcal{U}(S)+d}\leq\frac{\mathcal{U}\left(\frac{\lambda}{\gamma}(\gamma S+d)+c-\frac{\lambda d}{\gamma}\right)}{\mathcal{U}(\gamma S+d)}$$

$$\varphi\big(\mathcal{U}(S)\big)\leq\mathcal{U}\left(\frac{\frac{\lambda}{\gamma}(\gamma S+d)+c-\frac{\lambda d}{\gamma}}{\gamma S+d}\right)$$

Thus,

$$\varphi\big(\mathcal{U}(S)\big)\leq\mathcal{U}\big(\varphi(S)\big).$$

$$\psi(s)=\frac{\alpha s+e}{\beta s+f}=\frac{\frac{\alpha}{\beta}(\beta s+f)+e-\frac{\alpha f}{\beta}}{\beta s+f}$$

$$\psi(s)=\frac{\frac{\alpha}{\beta}(\beta\mathcal{U}(S)+f)+e-\frac{\alpha f}{\beta}}{\beta\mathcal{U}(S)+f}\geq\frac{\mathcal{U}\left(\frac{\alpha}{\beta}(\beta S+f)+e-\frac{\alpha f}{\beta}\right)}{\mathcal{U}(\beta S+f)}$$

$$\psi(s)\geq\mathcal{U}\left(\frac{\frac{\alpha}{\beta}(\beta S+f)+e-\frac{\alpha f}{\beta}}{\beta S+f}\right)$$

Therefore,

$$\psi(\mathcal{U}(S))\geq\mathcal{U}\big(\psi(S)\big).$$

Thus Jensen's inequality holds.

We examine the entropic utility function in Liu and Jiang (2012) which is similar to Acciaio (2007) and defined as

$$\mathcal{U}(S)=-\mathrm{In}\mathbb{E}[\exp(-S)]$$

via quasiconvex and quasiconcave functions. The example below serves as an illustration.

Example 4.53 Let $\varphi(s)=\frac{s+1}{s+2}$ and $\psi(s)=-\frac{s+1}{s+2}$ be quasiconvex and quasiconcave functions respectively. Choose $S\in\mathbb{L}^{\infty}$ such that $\mathbb{P}(S=0)=\mathbb{P}(S=1)=0.5$.

Thus, for $S=0$,

$\varphi(0)=\frac{1}{2}$ and $\mathcal{U}(0)=-\text{In}\mathbb{E}[\exp(-0)]=0.$

$\varphi(\mathcal{U}(0))=\varphi(0)=\frac{1}{2}.$

$\mathcal{U}(\varphi(0))=\mathcal{U}\left(\frac{1}{2}\right)=-\text{In}\mathbb{E}\left[\exp\left(-\frac{1}{2}\right)\right]=\frac{1}{2}.$

Taking $S=1$, we have

$\varphi(1)=\frac{2}{3}$ and $\mathcal{U}(1)=-\text{In}\mathbb{E}[\exp(-1)]=1.$

Therefore,

$\varphi(\mathcal{U}(1))=\varphi(1)=\frac{2}{3}.$

$\mathcal{U}(\varphi(1))=\mathcal{U}\left(\frac{2}{3}\right)=-\text{In}\mathbb{E}\left[\exp\left(-\frac{2}{3}\right)\right]=\frac{2}{3}.$

Similarly, for $S=0$,

$\psi(0)=-\frac{1}{2}$ and $\mathcal{U}(0)=-\text{In}\mathbb{E}[\exp(-0)]=0$

$\psi(\mathcal{U}(0))=\psi(0)=-\frac{1}{2}.$

$\mathcal{U}(\psi(0))=\mathcal{U}\left(-\frac{1}{2}\right)=-\text{In}\mathbb{E}\left[\exp\left(\frac{1}{2}\right)\right]=-\frac{1}{2}.$

For $S=1$, we have

$\psi(1)=-\frac{2}{3}$ and $\mathcal{U}(1)=-\text{In}\mathbb{E}[\exp(-1)]=1.$

$$\psi(\mathcal{U}(1))=\psi(1)=-\frac{2}{3}$$

$\mathcal{U}(\psi(0))=\mathcal{U}\left(-\frac{2}{3}\right)=-\text{In}\mathbb{E}\left[\exp\left(\frac{2}{3}\right)\right]=-\frac{2}{3}.$

An application of Jensen's inequality for monetary utility functions is given which is similar to the results in Liu and Jiang (2012).

Example 4.54

Using the entropic utility of the future outcome S, it is possible to estimate the entropic utility of S^+ or S^-. Jensen's inequality is a useful tool. Suppose that $\varphi(S) = S^+$ and $\psi(S^-) = S^-$ are quasiconvex and quasiconcave functions satisfying theorems 3.3 and 3.4, we have

$-\text{In}\ \mathbb{E}[\exp(-S^+)] \geq -\text{In}\ \mathbb{E}[\exp(-(-S^+))]$.

$-\text{In}\ \mathbb{E}[\exp(-S^-)] \leq -\text{In}\ \mathbb{E}[\exp(-(-S^-))]$.

Jensen's inequality for quasiconcave type monetary utility functions is examined. Examples 4.53 and 4.54 show that the inequality of Jensen holds for for quasiconcave and quasiconvex functions of linear fractional form and that the Jensen inequality is a useful tool for estimating the entropic utility of a future outcome.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.0 Introduction

The main results of the study are summarized and some conclusions and recommendations are drawn in relation to quasiconvex functions on time scales.

5.1 Summary of findings

- Time scales analogues of quasiconvex functions have been developed. Thus, the study unified and extended corresponding continuous and discrete versions in the literature.
- The time scales version for the subdifferential of a quasiconvex is introduced in a similar way that Dinu's subdifferential is related to the convex ones.
- Some Jensen inequalities for quasiconvex functions on time scales have been presented and applied in the areas of probability theory and mathematical finance. The study investigated quasiconcave-type monetary utility function and established that the Jensen inequality holds for such a monetary utility function regarding some convex, concave, quasiconvex and quasiconcave functions.

5.2 Conclusion

In the study, the structure underlying quasiconvex functions can be presented in the context of time scales. Some properties such as set relations, semicontinuity, differentiability and inequalities of quasiconvex functions in the domain of time scales were established. Also the study defined the subdifferential of a quasiconvex function on time scales with the condition of convexity of the function .This study further proved that the Jensen inequality holds for a quasiconcave monetary utility function in relation to quasiconvex and quasiconcave functions of linear fractional form.

5.3 Recommendations

It is recommended that further studies be carried out on quasiconvex functions on time scales in other areas such as boundedness, extreme values, and transformations to ascertain the validity of the properties.

Furthermore, it is recommended that further investigation be done on the applications of quasiconvex functions on time scales in optimization, economics, mathematical modeling and among others.

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