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#### UNIVERSITY FOR DEVELOPMENT STUDIES, TAMALE

# MATHEMATICAL INVESTIGATION INTO CHEMICALLY REACTING MAGNETOHYDRODYNAMIC FLOW WITH RADIATION AND CONVECTIVE BOUNDARY CONDITIONS

BY

# ARTHUR EMMANUEL MAURICE (B.A. Mathematics and Statistics) (UDS/MM/0013/12)

THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY FOR DEVELOPMENT STUDIES, IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF A MASTER OF SCIENCE

**DEGREE IN MATHEMATICS** 

**JANUARY**, 2015

## DECLARATION

#### Student

I hereby declare that this thesis is the result of my own original work and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature: 12 ag Date: 27/01/2-15 Name: ARTHUR EMMANUEL MAURICE

#### Supervisor

I hereby declare that this preparation and presentation of the thesis was supervised in accordance with the guidelines on supervision of thesis laid down by the University for Development Studies.

Date: 25/01/2015 Principal Supervisor's Signature: Name: Ing. Mr. Lelui Yakuba Irrahim

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STUDIES

### ABSTRACT

This thesis is a mathematical investigation into the effects of different parameters on the detailed flow of electrically conducting fluid with heat and mass transfer in the presence of thermal radiation with convective boundary conditions. A literature review is incorporated and the approximate consistency between different investigations shown. The problems related to the flow, heat and mass transfer over a flat plate in a stream of cold fluid in the presence of radiation and magnetic field have been modeled. The governing boundary layer equations have been developed and transformed into a self-similar form. The similarity equations were solved numerically using the Newton Raphson shooting iteration method together with Runge-Kutta Fourth-order integration scheme. The Effects of embedded parameters such as Prandtl number, local Biot number, magnetic parameter, radiation parameter, Brinkmann number, Schmidt number and the reaction rate parameter, on the fluid velocity, temperature profile, concentration profile, local skin friction, local Nusselt number and local Sherwood number in the flow regime are depicted both tabular and graphical form and discussed quantitatively. It is concluded that for this particular flow, the magnetic field strength is the only embedded parameter that helps control the flow kinematics and enhances both the heat and mass transfer process. In the same way, embedded parameters associated with thermal radiation, convective heating and viscous dissipation controlled to enhance the heat transfer process when controlled. Meanwhile, the Schmidt number and the reaction rate parameters contribute well to enhancing mass transfer if carefully controlled.

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## ACKNOWLEDGEMENTS

My first and foremost gratitude goes to the Almighty God for supplying my needs throughout my two –year course of study. He is the most powerful, compassionate, kind and merciful. Secondly, my heartfelt thanks go to my supervisor Dr. Seini Yakubu Ibrahim for his tireless effort, tolerance, patience and dedicated supervision from the beginning of this thesis to its logical conclusion. I have a definite conclusion that he is the master of his field. I always found him rich in new ideas. His genuine concern and friendship are irreplaceable. Finally, my profound appreciation goes to all my course mates and friends who have contributed in diverse ways to my success. God bless you all.



# **DEDICATION**

This work is dedicated to my beloved parents; Mr. Augustine Arthur and Mrs. Agnes Arthur (deceased) and my siblings; Mark, Sota, Ben (deceased), Sabastine, Esther, Ruth and Agnes.

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## NOMENCLATURE

- (x, y) Cartesian Coordinates
- $C_s$  Plate surface concentration
- $C_{\infty}$  Free stream concentration
- C Fluid chemical species concentration vector
- D Diffusion coefficient
- f Dimensionless stream function
- Nu Nusselt number
- $T_{\infty}$  Free stream temperature
- Sh Sherwood number
- *T* Fluid temperature vector
- Pr Prandtl number
- $T_s$  Plate surface temperature
- κ Thermal conductivity coefficient
- Sc Schmidt number
- V Fluid velocity vector
- (u, v) Velocity components in Cartesian Coordinates
- K' Mean absorption coefficient
- Ra Thermal radiation parameter
- $B_0$  Magnetic field of constant strength
- y Reaction rate
- $q_r$  Radiative heat flux



- *Hax* Local magnetic field parameter
- Br Brinkmann number
- $Bi_x$  Local Biot number
- $\beta_x$  Local reaction rate parameter
- $\Phi$  Viscous dissipation function
- $q_m$  Mass diffusion flux
- $h_s$  Heat transfer coefficient at plate surface

## **Greek Letters**

- $\mu$  Coefficient of viscosity
- $\rho$  Density of the fluid
- $\phi$  Dimensionless concentration
- $\theta$  Dimensionless temperature
- $\eta$  Dimensionless variable
- v Kinematic viscosity
- $\psi$  Stream function
- $\sigma^*$  Stefan-Boltzmann constant
- $\sigma$  Fluid electrical conductivity



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## **CHAPTER ONE**

## **INTRODUCTION AND BACKGROUND OF THE STUDIES**

## 1.1 Background of the Studies

The science of fluid dynamics encompasses the motion of gasses and liquids. It is a study of the forces that are responsible for the motion of fluid and the interaction between the fluid and solid boundaries. Fluid dynamics is central to almost all sciences and engineering and touches almost every aspect of human daily life. Fluid dynamics imparts one way or the other on defense, transportation, manufacturing, environment, medicine, and energy among others. From predicting the aerodynamic behavior of moving vehicles to the movement of biological fluids in human body, weather predictions, cooling of electronic components, performance of micro-fluidic devices, demand a detailed understanding of the subject of fluid dynamics and a substantial research. Due to the complexity of the subject and a breadth of its applications, fluid dynamics is proven to be a highly exciting and challenging subject of modern sciences. The quest for deeper understanding of the subject has not only inspired the development of the subject itself, but has also suggested the progress in the supporting areas, like, applied mathematics, numerical computing and experimental techniques.

The Navier Stokes equations generalized the governing equations for fluid dynamics. It is sometimes coupled with the transport conservation equations for energy and concentration. The Navier-Stoke equations are highly nonlinear and exact solutions are only available for some very special cases. In most cases, approximate solutions

are obtained using analytical and numerical computations. These methods enable the study of the interaction between moving conducting fluids with electrical and magnetic fields which is associated with electro-fluid-mechanical energy conversion as observed in liquids, gases, two-phase mixtures, or plasma. Numerous scientific and technical applications exist such as heating and flow control in metal processing, power generation from two-phase mixtures or seeded high-temperature gases, magnetic fields in planetary bodies. Different terms have been used in reference to the broad field of electromagnetic effects in conducting fluids. Commonly used terms include; Magneto-Fluid- Mechanics, Magneto-Gas-Dynamics, and Magneto-Hydrodynamics (MHD). MHD devices have been in use since the early part of the 20th century. For instance, an MHD pump prototype was built as early as 1907 (Northrup, 1907). In more recent times, MHD devices have been used for stirring, levitating, and otherwise controlling flows of liquid metals for metallurgical processing and other applications (Kolesnichenko, 1990).

Gas-phase MHD is probably best known in MHD power generation. Since the 1950s (Sporn and Kantrowitz, 1959; Steg and Sutton, 1960), major efforts have been made to improve electric conversion efficiency, increase reliability by eliminating moving parts, and reduce emissions from coal and gas plants. For example, research has shown the possibility of seawater propulsion using MHD (Graneau, 1989) and control of turbulent boundary layers to reduce drag (Tsinober, 1990). Extensive worldwide research on magnetic confinement of plasmas has led to attainment of conditions approaching those needed to sustain fusion reactions (Baker *et al.*, 1998). In MHD flow, interaction with a magnetic field can alter the velocity and pressure

characteristics of the flow and can also significantly delay the onset of turbulent fluctuations. These effects together or individually can dramatically alter the heat and mass transfer characteristics and hence the fluid drags on the surface. Applications of such phenomena include cooling systems for magnetic fusion reactors and reduced-drag ship hulls and airplane fuselages. The MHD force can be applied in such a way that useful work can be done.

Many researchers have analyzed hydromagnetic flows under various conditions. Anderson (1992) examined the influence of uniform magnetic field on the motion of an electrically conducting fluid past a flat and impermeable elastic sheet and obtained closed form solutions of the momentum boundary layer equation. The unsteady MHD convective heat transfer past a semi-infinite vertical porous moving plate with variable suction has been reported by Kim (2000). Similarly, Chamkha and Khaled (2001) investigated similarity solutions for hydromagnetic simultaneous heat and mass transfer by natural convection from an inclined plate with internal heat generation or absorption. Seddeek (2001) then analyzed the thermal radiation and buoyancy effects on MHD free convection heat generation flow over an accelerating permeable surface with temperature dependent viscosity. In all these works, it was observed that the heat transfer on hydromagnetic flow depended on the orientation of the surface on which the fluid was flowing.

Ouaf (2005) numerically analyzed the exact solution of thermal radiation on MHD flow over a stretching porous sheet whilst Cortell (2006) studied the influence of transverse magnetic field on the flow and heat transfer in an electrically conducting second grade fluid over a stretching sheet subject to suction. Subhas *et al.*, (2007) investigated the viscoelastic MHD flow and heat transfer over a stretching sheet with

viscous and ohmic dissipation whilst Barletta and Celli (2007) investigated the effects of joule heating and viscous dissipation of mixed convection MHD flow in vertical channels. The influence of radiation on MHD free convection from a vertical flat plate embedded in porous media with thermophoretic deposition of particles was analyzed by Rashad (2007). Ahmed et al., (2007) theoretically investigated the steady MHD free convective and mass transfer boundary layer of an electrically conducting fluid through a porous medium. The study was an extension of earlier work by Raptis and Kafousias (1982). It was realized that varying conditions of the surface of flow-like stretching as encountered in polymer extrusion or porosity as in manufacturing of textiles affect the rate at which heat and mass are transferred. Some researchers have been led by this to include various physical aspects of the problem of combined heat and mass transfer (Salem (2006); Ishak et al., (2008); and Makinde and Ogulu (2008)). Furthermore, Tezer-Sezgin and Bozkaya (2008) studied the boundary element of a Magnetohydrodynamic flow in an infinite region while Abass et al., (2008) presented solutions for hydromagnetic flow in a viscoelastic fluid due to an oscillating stretching surface. Amkadni et al., (2008) provided exact solutions for laminar MHD flow over a stretching flat plate.

The last three decades have seen an increase in the interest of MHD free or mixed convection from a plate embedded in porous media. This is largely due to its application in industry involving heat exchanger design, petroleum production, filtration, chemical catalytic reactors, and MHD generators. Other applications of free convection in engineering include heat rejection into the environment such as in water bodies like lakes, rivers and the sea. Thermal energy storage systems such as solar ponds and condenser of power plants are common examples. Makinde and Sibanda (2008) investigated the boundary layer problem of MHD mixed convective flow with heat and mass transfer past a vertical plate in a porous medium with constant wall suction while Abdelkhalek (2008) presented numerical results on heat and mass transfer in MHD free convection from a moving permeable vertical surface by a perturbation technique. Ibrahim and Makinde (2010; 2011) reported interesting results on MHD boundary layer flow involving chemical reaction and radiation effects on stretching and porous surfaces. Heat and mass transfer by MHD mixed convection stagnation point flow toward a vertical plate embedded in a highly porous medium with radiation and internal heat generation was recently reported by Makinde (2012). Singh and Makinde (2012) then presented a computational dynamics of MHD free convection flow along an inclined plate with Newtonian heating in the presence of volumetric heat generation whilst Makinde (2012) studied entropy analysis for MHD boundary layer flow and heat transfer over a flat plate with convective surface boundary conditions. Alam et al., (2013) analyzed the heat and mass transfer in MHD free convection flow over an inclined plate with hall current whilst Alia et al., (2013) presented numerical results of radiation effects on MHD free convection flow along vertical flat plate in the presence of Joule heating and heat generation. Alireza et al., (2013) analytically studied the MHD stagnation point flow and heat transfer over a permeable stretching sheet with chemical reaction. Ibrahim (2013) analyzed the heat and mass transfer effects on steady MHD flow over an exponentially stretching surface with viscous dissipation, heat generation and radiation. Seini and Makinde (2013) then studied MHD boundary layer flow due to exponential stretching surface with radiation and chemical reaction whilst Raja et al., (2013) presented results of Magneto-convection over a semi- infinite porous plate with heat generation. Moreover, Uddin et al., (2013) studied MHD forced convective laminar boundary

layer flow from a convectively heated moving vertical plate with radiation and transpiration effect. Arthur and Seini (2014) also investigated the MHD thermal stagnation point flow towards a stretching porous surface. In this study, the combined effect of Radiation and viscous dissipation in a hydromagnetic fluid flow with convective boundary conditions are investigated.

## **1.2 Problem Statement**

The need to control the cooling process in manufacturing processes led to a major paradigm shift in many industrial and engineering activities such in chemical engineering processes and electrical system design (Arthur and Seini, 2014). It has been observed in literature that the fluid mechanical properties of the penultimate product depend mainly on the cooling liquid used and the rate of stretching. Conventional fluids such as water and air are the most widely used media in cooling systems. The rate of heat and mass exchange achievable using these conventional fluids is unsuitable for certain sheet materials. It has been proposed to alter flow kinematics so that it would lead to a slower rate of solidification, as compared with water. Among the many techniques of controlling flow kinematics, the idea of using magnetic fields appears to be the most attractive. Using magnetic field is preferred because of the ease of implementation and its non-intrusive nature. Many researchers have been attracted to the computational dynamics of MHD fluid flow with heat and mass transfer over surfaces of various orientations. It is worth noting that many investigations into MHD flow have been reported in the literature. However, the combined effect of radiation and viscous dissipation, which adds up to heat transfer, with convective boundary condition as encountered in heat exchanger design technology is very limited in the literature. This study aims to investigate the

combined effects of radiation and viscous dissipation in a hydromagnetic flow with convective boundary conditions in heat and mass transfer.

## **1.3 Research Objectives**

This research aims at achieving two categories of objectives: the general and the specific objectives.

### 1.3.1 General Objective

To investigate the combined effects of radiation and viscous dissipation on hydromagnetic flow with convective boundary conditions

### 1.3.2 Specific Objectives

The specific objectives of the study are to:

- i. Develop a mathematical model governing the combined effects of radiation and viscous dissipation.
- ii. Employ the techniques of similarity analysis to solve the partial differential equations modelling the problem.
- iii. Analyse the effects of various controlling parameters on the problem.

## 1.4 Relevance of the Research

The study is relevant to the design of experiments both physically and numerically. It could be used in reporting of experimental results in the field of MHD ship propulsion, flow and flight control in aerospace engineering and solar energy fusion. This will help solve problems arising from melts by electromagnetic forces caused in conducting media by external electromagnetic fields. It will also help in enhancing

technologies in various types of electromagnetic pumps and valves for transportation batching, and stirring metals, which affect the convective heat and mass transfer in crystallization process in metallurgy and in liquid metal cooling systems for nuclear technology where radiative and viscous heating is inherent.

The study seeks to discover parameters of importance for building of power engineering devices, metallurgical and electrical technologies in which MHD effects play essential roles and determine the efficiency of operation when strong electromagnetic fields are used.

The field of MHD technology comprising of various types of MHD relays, communication switches, and units of automation with liquid working body would benefit from parameters and the rigorous analysis from this research. The work will also serve as reference material for future works in the areas of nuclear studies, manufacturing, mining, petro-chemical, aerospace and automotive industries.

### **1.5 Properties of Fluids**

#### **1.5.1** Nonlinearity

Any physical systems can be modeled by a set of nonlinear equations. These nonlinearities in dynamical systems make solutions of most problems difficult to obtain. It is often necessary to approximate these non-linear equations to systems of linear equations by making some appropriate assumptions to obtain direct solutions. In fluid flow problems, nonlinearities are mostly due to the convective acceleration associated with the change of velocity over position. Thus, any convective flow, whether turbulent or not will involve nonlinearities.

## 1.5.2 Density of a substance

Density  $(\rho)$  is defined as mass (m) per unit volume (V). That is,

$$\rho = \frac{m}{V} \tag{1.1}$$

unit mass. The density of a substance, in general depends on temperature and pressure. The density of most gases is proportional to pressure and inversely proportional to temperature. Liquids and solids, on the other hand, are essentially incompressible substances, and the variation of their density with pressure is usually negligible. The density of liquids and solids depends more strongly on temperature than it does on pressure.

#### 1.5.3 Viscosity

Fluid tends to flow when acted upon by forces because they cannot resist the induced shearing effects. Prandtl (1904) observed that when a fluid flows over a surface, its particles directly in touch with the surface move with it, but with stationary surfaces, particles of fluid have zero velocities. The velocities of particles increases as the distance from the surface increase and reaches the maximum, called the free stream value, at a region far away from the object. Thus, shearing stresses opposing the relative motion of fluid develop with magnitude depending on the velocity gradient from layer to layer. For fluids obeying Newton's law of viscosity, the x- coordinate is taken to act along the direction of flow with velocity, u. The shear stress ( $\tau$ ) in a fluid at a distance, y from the surface is given by:

$$\tau = \mu \frac{du}{dy} \tag{1.2}$$

Where  $\mu$  is the co-efficient of dynamic viscosity defined as the shear force per unit area required to drag one layer of fluid with unit velocity past another layer a unit distance away from it in the fluid. The SI units for dynamic viscosity is the Newton-Seconds per Meter Squared (Nsm<sup>-2</sup>)

## **1.5.4 Kinematic Viscosity**

The kinematic viscosity is defined as the ratio of the dynamic viscosity ( $\mu$ ) to the mass density. It is measured in Squared Meters per Second (m<sup>2</sup>s<sup>-1</sup>) and mathematically expressed as

$$v = \frac{\mu}{\rho} \tag{1.3}$$

## **1.5.5 Internal Energy**

The energy stored in a system due to the molecular interaction and bonding is referred to as the internal energy. A given mass of a viscous fluid may be viewed as a thermodynamic system that stores various forms of energies. The internal energy of a fluid includes the energy due to translation, rotation, vibration and the energy of molecular dissociation as well as energy of electronic excitation of the molecules. Internal energy has units of Jmol<sup>-1</sup>.

#### **1.5.6 Thermal Conductivity**

The thermal conductivity,  $\kappa$  is a measure of a material's ability to conduct heat. It relates the vector rate of heat transfer per unit area, q (also called the heat flux) to the vector gradient of temperature,  $\nabla T$ . For solids and liquids the Fourier's law of heat conduction is given as:

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$$q = -\kappa \nabla T \,, \tag{1.4}$$

where the negative sign indicates that heat flux is positive in the direction of decreasing temperature.

## 1.6 Modes of Heat Transfer

Basically, there are three modes of heat transfers. These are conduction, radiation, and convection. Conduction is an exchange of energy by direct interaction between molecules of a substance having temperature differences. It occurs in gases, liquids, or solids and has a strong basis in the molecular kinetic theory of Physics. Radiation is a transfer of thermal energy in the form of electromagnetic waves. Like electromagnetic radiation (light, X-rays, microwaves), thermal radiation travels at the speed of light, passing most easily through a vacuum or a nearly transparent gasses. Liquids containing gases, such as carbon dioxide and water vapor, and glasses transmit only a portion of incident radiation, while most of solids are essentially opaque to radiation. Convection remains the basic mode of heat transfer among the fluids.

#### 1.6.1 Convection

Convection in general terms refers to the movement of molecules within fluid. Convection is one of the major modes of heat and mass transfer in fluids. Convective heat and mass transfer take place through both diffusion (the random Brownian motion of individual particles in the fluid) and by advection, in which matter or heat is transported by the larger-scale motion of currents in the fluid. In the context of heat and mass transfer, the term "convection" is used to refer to the sum of advective and diffusion.

### **1.6.1.1 Convective Heat Transfer Coefficient**

Convection is the transfer of energy by conduction and radiation in moving fluid media. The motion of the fluid is an essential part of convective heat transfer. A key step in calculating the rate of heat transfer by convection is the calculation of the heattransfer coefficient.

In many cases of industrial importance, heat is transferred from one fluid, through a solid wall, to another fluid. The transfer occurs in a heat exchanger.

#### **1.6.1.2 Heat Transfer Coefficient**

The local rate of convective heat transfer between a surface and a fluid is given by Newton's law of cooling

$$q = h_s (T_s - T), \tag{1.5}$$

where  $h_s$  is the local heat transfer coefficient at the surface,  $T_s$  is the temperature of the surface, T is the bulk fluid temperature and q is the energy flux defined by Fourier's law given in equation (1.4).

#### **1.7 Computational Approach**

Mathematically speaking, the differential equations describing the hydromagnetic boundary layer flow interaction with heat transfer over a flat surface constitute a nonlinear problem in an unbounded computational domain. The theory of nonlinear differential equations is quite elaborate and their solutions remain an extremely important problem of practical relevance in industrial and engineering systems. Approximate solutions for the nonlinear systems of differential equations modeling the MHD flow over a flat surface will be constructed using the fourth order Runge-Kutta integration scheme coupled with numerical shooting technique, Roche (1998). Both numerical and graphical results will be presented and discussed quantitatively with respect to various parameters embedded in the problem.

#### 1.7.1 Runge-Kutta (RK)

Runge-Kutta (RK) methods are based on Taylor expansion formulae, but yield in general better algorithms for solutions of an ODE (Hjorth-Jensen (2013). The basic philosophy is that it provides an intermediate step in the computation of  $y_{i+1}$ .

To see this, consider first the following definitions

$$\frac{dy}{dt} = f(t, y), \tag{1.6}$$

$$y(t) = \int f(t, y)dt, \qquad (1.7)$$

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t, y) dt.$$
 (1.8)

To demonstrate the philosophy behind RK methods, let us consider the second-order RK method, RK2. The first approximation consists in Taylor expanding f(t, y) around the centre of the integration interval  $t_i$  to  $t_{i+1}$ , i.e., at  $t_i + h/2$ , h being the step. Using the midpoint formula for an integral, defining  $y(t_i + h/2) = y_{i+1/2}$  and  $t_i + h/2 = t_{i+1/2}$ , we obtain

$$\int_{t_i}^{t_{i+1}} f(t, y) dt \approx h f(t_{i+1/2}, y_{i+1/2}) + O(h^3)$$
(1.9)

This means in turn that we have

$$y_{i+1} = y_i + hf(t_{i+1/2}, y_{i+1/2}) + O(h^3)$$
(1.10)

However, we do not know the value of  $y_{i+1/2}$ . Here comes thus the next approximation, namely, we use Euler's method to approximate  $y_{i+1/2}$ . We have then

$$y_{(i+1/2)} = y_i + \frac{h}{2} \frac{dy}{dh} = y(t_i) + \frac{h}{2} f(t_i, y_i).$$
(1.11)

This means that we can define the following algorithm for the second-order Runge-Kutta method, RK2.

$$k_1 = hf(t_i, y_i),$$
 (1.12)

$$k_2 = hf(t_{i+1/2}, y_i + k_1/2), \qquad (1.13)$$

$$y_{i+1} \approx y_i + k_2 + O(h^3)$$
 (1.14)

The difference between the previous one-step methods is that we now need an intermediate step in our evaluation, namely  $t_i + h/2 = t_{(i+1/2)}$  where we evaluate the derivative f. This involves more operations, but the gain is a better stability in the solution. The fourth-order Runge-Kutta, RK4, which we will employ in the solution of various differential equations below, is easily derived. The steps are as follows. We start again with (1.7), but instead of approximating the integral with the midpoint rule, we use now Simpson's rule at  $t_i + h/2$ , with h being the step. Using Simpson's formula for an integral, defining  $y(t_i + h/2) = y_{i+1/2}$  and  $t_i + h/2 = t_{i+1/2}$ , we obtain

$$\int_{t_i}^{t_{i+1}} f(t, y) dt \approx \frac{h}{6} \Big[ f(t_i, y_i) + 4f(t_{i+1/2}, y_{i+1/2}) + f(t_{i+1}, y_{i+1}) \Big] + O(h^5) \quad (1.15)$$

This means in turn that we have

$$y_{i+1} = y_i + \frac{h}{6} \Big[ f(t_i, y_i) + 4f(t_{i+1/2}, y_{i+1/2}) + f(t_{i+1}, y_{i+1}) \Big] + O(h^5)$$
(1.16)

However, we do not know the values of  $y_{i+1/2}$  and  $y_{i+1}$ . The fourth-order Runge-Kutta method splits the midpoint evaluations in two steps, that is we have

$$y_{i+1} \approx y_i + \frac{h}{6} \Big[ f(t_i, y_i) + 2f(t_{i+1/2}, y_{i+1/2}) + 2f(t_{i+1/2}, y_{i+1/2}) + f(t_{i+1}, y_{i+1}) \Big], \quad (1.17)$$

since we want to approximate the slope at  $y_{i+1/2}$  in two steps. The first two function evaluations are as for the second order Runge-Kutta method. Thus, the algorithm consists in first calculating  $k_1$  with  $t_i$ ,  $y_1$  and f as inputs. Thereafter, we increase the step size by h/2 and calculate  $k_2$ , then  $k_3$  and finally  $k_4$ . With this caveat, we can then obtain the new value for the variable y. It results in four function evaluations, but the accuracy is increased by two orders compared with the second-order Runge-Kutta method. The fourth order Runge-Kutta method has a global truncation error which goes like  $O(h^4)$ .

#### 1.7.2 Shooting method

In many physics applications we encounter differential equations like

$$\frac{d^2 y}{dx^2} + k^2(x)y = F(x), \quad a \le x \le b,$$
(1.18)



with boundary conditions

$$y(a) = \alpha, y(b) = \beta. \tag{1.19}$$

We can interpret F(x) as an inhomogeneous driving force while k(x) is a real function. If it is positive the solutions y(x) will be oscillatory functions, and if negative they are exponentionally growing or decaying functions (Hjorth-Jensen (2013). To solve this equation we could start with for example the Runge-Kutta method or various improvements to Euler's method, as discussed in the previous chapter. Then we would need to transform this equation to a set of coupled first-order equations. We could however start with the discretized version for the second derivative. We discretize our equation and introduce a step length h = (b-a)/N, with N being the number of equally spaced mesh points. Our discretized second derivative reads at a step  $x_i = a + ih$  with i = 0, 1, ...

$$y_i'' = y_i + \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} + O(h^2), \qquad (1.20)$$

leading to a discretized differential equation:

$$F_{i} = k_{i}^{2} y_{i} + \frac{y_{i+1} + y_{i-1} - 2y_{i}}{h^{2}} + O(h^{2})$$
(1.21)

Recall that the fourth-order Runge-Kutta method has a local error of  $O(h^4)$ . Since we want to integrate our equation from  $x_0 = a$  to  $x_N = b$ , we rewrite it as

$$y_{i+1} \approx -y_{i-1} + y_i \left( 2 - h^2 k_i^2 + h^2 F_i \right)$$
(1.22)



Starting at i = 1 we have after one step:

$$y_2 \approx -y_0 + y_1 \left( 2 - h^2 k_1^2 + h^2 F_1 \right)$$
 (1.23)

Irrespective of method to approximate the second derivative, this equation uncovers our first problem. While  $y_0 = y$  (a) = 0, our function value  $y_1$  is unknown, unless we have an analytic expression for y(x) at x = 0. Knowing  $y_1$  is equivalent to knowing y'at x = 0 since the first derivative is given by

$$y'_i \approx \frac{y_{i+1} - y_i}{h}.$$
 (1.24)

## **1.8 Organisation of Report**

This dissertation is organised into five chapters. Chapter one focuses on the general introduction of the study which includes the background of the study. The problem is stated and the research methodology outlined. In chapter two, the relevant mathematical formulae have been derived. Chapter three applies these formulae to solving a practical problem of relevance in industry whilst chapter four presents the results and discussions. Chapter five includes summary of findings, conclusion and recommendation for industry and future researchers.

## **CHAPTER TWO**

## **MODELS FORMULATION**

## 2.1 Introduction

In this chapter, the differential equations governing fluid motion are derived. They include the continuity, momentum, energy and concentration equations. The control volume approach to fluid analysis is used in the form of rectangular coordinate.

## 2.2 The Continuity Equation

The partial differential equations modelling the motion of a parcel of fluid is obtained by applying the conservation law of mass to a small volume of fluid flow. Consider the mass flux through each face of the fixed infinitesimal control volume shown in Fig 2.1. Let the net flux of mass entering the element be equal to the rate of change of the mass of the element; that is,

$$\dot{m}_{in} - \dot{m}_{out} = \frac{\partial}{\partial t} m_{element}$$
(2.1)

To perform this mass balance, identify  $\rho u$ ,  $\rho v$  and  $\rho w$  at the centre of the element and then treat each of these quantities as a single variable.





Figure 2.1 Mass flux through each of the six faces of a control volume of fluid (Çengel and Cimbala (2006))

From **Figure2.1** which shows the mass flux through each of the six faces; Equation (2.1) then takes the form

$$\begin{bmatrix} \rho u - \frac{\partial(\rho u)}{\partial x} \frac{dx}{2} \end{bmatrix} dy dz - \begin{bmatrix} \rho u + \frac{\partial(\rho u)}{\partial x} \frac{dx}{2} \end{bmatrix} dy dz + \begin{bmatrix} \rho v - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz - \begin{bmatrix} \rho v - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dy + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \frac{dy}{2} \end{bmatrix} dx dy + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \frac{dy}{2} \end{bmatrix} dx dy dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \frac{dy}{2} \frac{dy}{2} \end{bmatrix} dx dy dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \frac{dy}{2} \frac{dy}{2} \frac{dy}{2} \frac{dy}{2} \end{bmatrix} dx dz + \begin{bmatrix} \rho w - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \frac{dy}{2}$$

Subtracting the appropriate terms and dividing by dx dy dz gives,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = -\frac{\partial\rho}{\partial t}$$
(2.3)



Expanding and simplifying results in

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$
(2.4)

In terms of substantial derivative we can write (2.4) as

$$\frac{D\rho}{Dt} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$
(2.5)

This is the most general form of the differential continuity equation expressed in rectangular coordinates. We can introduce the gradient operator,  $\nabla$  called "del", which, in rectangular coordinates, is

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$
(2.6)

The continuity equation can then be written in the form

$$\frac{D\rho}{Dt} + \rho \nabla \cdot = 0 \tag{2.7}$$

Where  $V = u\hat{i} + v\hat{j} + w\hat{k}$  and  $\nabla \cdot V$  is called the divergence of the velocity. This form of the continuity equation does not refer to any particular coordinate system. It is the form used to express the continuity equation using various coordinate systems.

### 2.3 The Momentum Equation

#### 2.3.1 General formulation

The differential momentum equation is a vector equation and thus provides us with three scalar equations. These component equations will help us in our attempt to



determine the velocity and pressure fields. There are nine stress components of the stress tensor  $\tau_{ij}$  that act in a particular point in a fluid field. These stress tensors can be related to the velocity and the vector fields with the appropriate equations.



Figure 2. 2 x-directional surface forces due to stress tensor component of a control volume (Cengel and Cimbala (2006)).

The stress components that act at a point are displayed on a two- and threedimensional rectangular element in **Figure 2.2**. This element is considered to be an exaggerated point, a cubical point; the stress components act in the positive direction on a positive face (a normal vector point in the positive coordinate direction) and in the negative direction on a negative face (a normal vector points in the negative coordinate direction). The first subscript on a stress component denotes the face, upon which the component acts, and the second subscript denotes the direction in which it acts; the component  $\tau_{xy}$  acts in the positive y-direction on a positive x-face. A stress component that acts perpendicular to a face is referred to as normal stress; the

components  $\sigma_{xx}$ ,  $\sigma_{yy}$  and  $\sigma_{zz}$  are normal stresses. A stress component that acts tangential to a face is called a shear stress; the components  $\tau_{xy}$ ,  $\tau_{yx}$   $\tau_{zz}$   $\tau_{zx}$   $\tau_{yz}$  and  $\tau_{zy}$  are the shear fluid. To derive the differential momentum equation, consider and the stress infinitesimal fluid particle. Only forces acting on the faces are shown. The stress components are assumed to be functions of x, y, z and t; and hence the values of the stress components change from face to face since the location of each face is slightly different. The body force is shown acting in an arbitrary direction. Newton's second law applied to a fluid particle, for the x-component direction,  $\sum F_x = ma_x$ . For the particle shown, it takes the form:

$$\left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right]dydz + \left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right]dxdz + \left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right]dxdy - \left[\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right]dydz$$

$$-\left[\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right]dxdz - \left[\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right]dxdy + \rho g_{x}dxdydz = \rho \,dx \,dy \,dz \frac{Du}{Dt}, \quad (2.8)$$

where the component of the gravity vector g in the x-direction is  $g_x$  and  $\frac{Du}{Dt}$  is the xcomponent acceleration of the fluid particle. Dividing by the volume dxdydz, (2.8) simplifies to

$$\rho \frac{Du}{Dt} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x.$$
(2.9)

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Similarly, for *y*- and *z*-directions, we have

$$\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y, \qquad (2.10)$$

$$\rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z.$$
(2.11)

We can show by taking moments about the axes passing through the centre of the infinitesimal element, that

$$\tau_{xy} = \tau_{yx}, \, \tau_{xz} = \tau_{zx}, \, \tau_{yz} = \tau_{zy} \tag{2.12}$$

That is, from (2.12), we say the stress tensor is symmetric; so there are actually six independent stress components. The stress tensor may be displayed in the usual way as in (2.13)

The subscripts *i* and *j* take on numerical values 1, 2, or 3. Then  $\tau_{12}$  represents the element  $\tau_{xy}$  in the first row, second column.

$$\tau_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$
(2.13)

#### 2.3.2 The Navier – Stokes Equations

Many fluids exhibit a linear relationship between the stress components and the velocity gradients. Such fluids are called Newtonian fluids and include common fluids such as water, oil, and air. If in addition to linearity, we require that the fluid be isotropic, it is possible to relate the stress components and the velocity gradients using only two fluid properties, the viscosity  $\mu$  and the second coefficient of viscosity  $\lambda$ . The



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stress-velocity gradient relations, often referred to as the constitutive equations, are stated as follows:

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot V, \ \tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right),$$
  
$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot V, \ \tau_{xz} = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \qquad (2.14)$$
  
$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot V, \ \tau_{yz} = \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right).$$

For most gases and for monatomic gases exactly, the second coefficient of viscosity is related to the viscosity by

$$\lambda = -\frac{2}{3}\mu, \qquad (2.15)$$

a condition that is known as Stokes' hypothesis. With this relationship, the negative average of the three normal stresses is equal to the pressure, that is,

$$-\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = p.$$
 (2.16)

Using (2.14), this can be shown to always be true for a liquid in which  $\nabla \cdot V = 0$ , and with stokes' hypothesis, it is also true for a gas. If the constitutive equations are substituted into the differential momentum equations (2.9), (2.10), and (2.11), there results, using Stokes' hypothesis,

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \frac{\mu}{3} \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right), \quad (2.17)$$


$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho g_y + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \frac{\mu}{3} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right), \quad (2.18)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{\mu}{3} \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right), \quad (2.19)$$

where a homogeneous fluid is assumed. That is, the fluid properties (for example, the viscosity) are independent of position. For an incompressible flow, the continuity equation allows the equations above to be reduced to

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$
(2.20)

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho g_y + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$
(2.21)

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$
(2.22)

These are the Navier-Stokes Equations, named after Louis M. H. Navier (1785-1836) and George Stokes (1819-1903). The three differential equations together with the continuity equation give four equations and four unknowns in u, v, w and p. The viscosity and density are the fluid properties that can often be determined. With the appropriate boundary and initial conditions, approximate solutions can be obtained. Several relatively simple geometries allow for analytical solutions. Numerical solutions have also been determined for many flows of interest. Since the equations are nonlinear partial differential equations, it cannot be certain that the solution obtained can exactly be realized in the laboratory. That is: the solutions are not unique. For example, a laminar flow and a turbulent flow may have the identical

initial and boundary conditions, yet the two flows (the two solutions) are very different.

We can express the Navier-Stokes equations in vector form by multiplying (2.20), (2.21) and (2.22) by  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$  respectively, and adding recognizing that;

$$\frac{DV}{Dt} = \frac{Du}{Dt}\hat{i} + \frac{Dv}{Dt}\hat{j} + \frac{Dw}{Dt}\hat{k},$$

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k},$$

$$\nabla^2 V = \nabla^2 u\hat{i} + \nabla^2 v\hat{j} + \nabla^2 w\hat{k},$$
(2.23)

where the Laplacian:

$$\nabla^{2} = \frac{\partial^{2}}{\partial x^{2}} + \frac{\partial^{2}}{\partial y^{2}} + \frac{\partial^{2}}{\partial z^{2}}$$
(2.24)

Combining the above, the Navier-Stokes equations in (2.20), (2.21) and (2.22) take the vector form

$$\rho \frac{DV}{Dt} = -\nabla p + \rho g + \mu \nabla^2 V \qquad (2.25)$$

# 2.4 The Energy Equation

Most problems of interest in fluid mechanics do not involve temperature gradients. They do however involve flows in which temperature everywhere is constant. For such flows it is not necessary to introduce the differential energy equation. There are situations, however, for both compressible and incompressible flows, in which temperature gradients are important, and for such flows the differential energy



equation may be needed. The differential energy equation is derived assuming negligible viscous effects, an assumption that significantly simplifies the derivation. Since the shear stresses that result from viscosity are quite small for many applications, this assumption may be acceptable. These shear stresses do, however, account for the high temperatures that burn up satellites on re-entry to the atmosphere; if they are significant, they must be included in any analysis.

Consider the infinitesimal fluid element, shown in **Figure 2.1**. The heat transfer rate  $\dot{Q}$  through an area A is given by Fourier's law of heat transfer, named after Jean B.J. Fourier (1768-1830):

$$\dot{Q} = -\kappa A \frac{\partial T}{\partial n}, \qquad (2.26)$$

where *n* is the direction normal to the area, *T* is the temperature, and  $\kappa$  is the thermal conductivity, assumed to be constant. The rate of work done by a force is the magnitude of the force multiplied by the velocity in the direction of the force, that is,

$$\dot{W} = pAV, \tag{2.27}$$

where V is the velocity in the direction of the pressure force pA. The first law of thermodynamics applied to a fluid particle can be written as

$$\dot{Q} - \dot{W} = \frac{DE}{Dt},\tag{2.28}$$

where D/Dt is used since we are following a fluid particle at the instant shown.

For the particle occupying the infinitesimal element of **Figure 2.1**, the relationships above allow us to write:



$$\kappa dy dz \left( \frac{\partial T}{\partial x} \Big|_{x+dx} - \frac{\partial T}{\partial x} \Big|_{x} \right) - \frac{\partial}{\partial x} (pu) dx dy dz + \kappa dx dz \left( \frac{\partial T}{\partial y} \Big|_{y+dy} - \frac{\partial T}{\partial y} \Big|_{y} \right) - \frac{\partial}{\partial y} (pv) dx dy dz$$
$$+ \kappa dx dy \left( \frac{\partial T}{\partial z} \Big|_{z+dz} - \frac{\partial T}{\partial z} \Big|_{z} \right) - \frac{\partial}{\partial z} (pw) dx dy dz = \rho dx dy dz \frac{D}{Dt} \left( \frac{u^{2} + v^{2} + w^{2}}{2} + gz + \widetilde{u} \right), \quad (2.29)$$

where  $\tilde{u}$  is the internal energy, *E* has included kinetic, potential and internal energy, and the *z*-axis is assumed vertical. Also, since the mass of a fluid particle is constant  $\rho dx dy dz$  is outside the *D/Dt* -operator. Divide both sides by dx dy dz. The result is

$$\kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) - \frac{\partial}{\partial x} (pu) - \frac{\partial}{\partial y} (pv) - \frac{\partial}{\partial z} (pw) = \rho \frac{D}{Dt} \left( \frac{u^2 + v^2 + w^2}{2} + gz + \widetilde{u} \right). \quad (2.30)$$

This can be rearranged as follows:

$$\kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) - p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - u \frac{\partial p}{\partial x} - v \frac{\partial p}{\partial y}$$
$$- w \frac{\partial p}{\partial z} = \rho u \frac{Du}{Dt} + \rho v \frac{Dv}{Dt} + \rho w \frac{Dw}{Dt} + \rho g \frac{Dz}{Dt} + \rho \frac{D\widetilde{u}}{Dt}.$$
(2.31)

The Euler's equations are applicable for this inviscid flow. Hence, the last three terms on the left equal the first four terms on the right if we recognize that

$$\frac{Dz}{Dt} = \frac{\partial z}{\partial t} + u \frac{\partial z}{\partial x} + v \frac{\partial z}{\partial y} + w \frac{\partial z}{\partial z} = w, \qquad (2.32)$$

since x, y, z and t are all independent variables. The simplified energy equation then takes the form

$$\rho \frac{D\widetilde{u}}{Dt} = \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) - p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right).$$
(2.33)



In vector form, this is expressed as

$$\rho \frac{D\widetilde{u}}{Dt} = \kappa \nabla^2 T - p \nabla \cdot V \tag{2.34}$$

Before simplifying this equation for incompressible gas flow, it could be written in terms of enthalpy rather than internal energy. Using

$$\widetilde{u} = h - \frac{P}{\rho}.$$
(2.35)

The energy equation now becomes

$$\rho \frac{Dh}{Dt} = \kappa \nabla^2 T + \frac{Dp}{Dt}.$$
(2.36)

Two special cases can be considered. First, for a liquid flow, the continuity equation requires that  $\nabla \cdot V = 0$  and with  $\tilde{u} = c_p T$ ,  $c_p$  being the specific heat capacity at constant pressure. Equation (2.34) becomes

$$\frac{DT}{Dt} = \alpha \nabla^2 T \tag{2.37}$$

In (2.37) we have introduced the thermal diffusivity defined by

$$\alpha = \frac{\kappa}{\rho c_p} \tag{2.38}$$

If viscous effects are not negligible, the derivation would include the work input due to the shear stress components. This would add a term to the right-hand side of all the differential energy equations above; this term is called the dissipation function  $\Phi$ , which, in rectangular coordinates, is



$$\Phi = 2\mu \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 + \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right]$$
(2.39)

Therefore, the energy equation for incompressible fluid flow becomes

$$\frac{DT}{Dt} = \alpha \Delta^2 T + \Phi \tag{2.40}$$

In equation (2.40), the left hand represents the convective term whilst the right hand side are respectively, the rate of heat diffusion to the fluid particles and the rate of viscous dissipation per unit volume.

# 2.5 The Concentration Equation

Similar to the energy equation derived on the principles of energy conservation, the concentration equation is derived on the principles of species conservation in a mixture. In addition to accounting for the convection and diffusion of each species, we must allow the possibility that a species may be created or destroyed by chemical reactions occurring in the bulk medium (homogeneous reactions). Reactions on surfaces surrounding the medium (heterogeneous reactions) must be accounted for in the boundary conditions.



Figure 2. 3 A Control volume in a fluid-flow and mass-diffusion field (Lienhard IV and Lienhard (2008))

Consider, in the usual way, an arbitrary control volume, R, with a boundary, S, as shown in Fig 2.3. The control volume is fixed in space, with fluid moving through it. Species *i* may accumulate in R, it may travel in and out of R by bulk convection or by diffusion, and it may be created within R by homogeneous reactions. The rate of creation species *i* is denoted by  $\dot{r}_i$ ; and, because chemical reaction conserve mass, the net mass reaction is  $\dot{r} = \sum \dot{r}_i = 0$ . The rate of change of the mass of species *i* in R is then described by the following balance:

$$\frac{d}{dt}\int_{R}\rho_{i}dR = -\int_{S}\vec{n}_{i}.d\vec{S} + \int_{R}\dot{r}_{i}dR \qquad (2.41)$$

$$\frac{d}{dt} \int_{R} \rho_{i} dR = -\int_{S} \rho_{i} \bar{v} \cdot d\bar{S} - \int_{S} \bar{j}_{i} \cdot d\bar{S} + \int_{R} \dot{r}_{i} dR \qquad (2.42)$$

The term on the left hand side of equation (2.42) is the rate of increase of the species in the control volume and the terms on the right hand side depicts respectively, the rate of convection of the species out of the control volume; the diffusion of the species out of the control volume; and the rate of creation of the species in the control volume.

This species conservation statement is identical to the energy conservation statement except that the mass of the species has taken the place of energy and heat. The surface integral may be converted to volume integrals using Gauss' theorem and rearranged to appear as:

$$\int_{R} \left[ \frac{\partial \rho_{i}}{\partial t} + \nabla \cdot \left( \rho_{i} \vec{v} \right) + \nabla \cdot \vec{j}_{i} - \dot{r}_{i} \right] = 0$$
(2.43)

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Since the control volume is selected arbitrarily, the integrand must be identically zero. Thus, the general form of the species conservation equation becomes:

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot \left( \rho_i \vec{v} \right) = -\nabla \cdot \vec{j}_i + \dot{r}_i$$
(2.44)

The mass conservation equation for the entire mixture can be obtained by summing equation (2.44) over the species region and apply the requirement that there be no net creation of mass:

$$\sum_{i} \left[ \frac{\partial \rho_{i}}{\partial t} + \nabla \cdot \left( \rho_{i} \vec{v} \right) \right] = \sum_{i} \left[ -\nabla \cdot \vec{j}_{i} + \dot{r}_{i} \right]$$
(2.45)

So that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho \vec{v} \right) = 0 \tag{2.46}$$

Equation (2.46) applies to any mixture, including those with varying density.

# 2.5.1 Incompressible Mixtures

For an incompressible mixture,  $\nabla \cdot \vec{v} = 0$  and the second term in equation (2.46) may be written as

$$\nabla \cdot (\rho \vec{v}) = \vec{v} \cdot \nabla \rho_i + \rho_i \cdot \nabla \vec{v} = \vec{v} \cdot \nabla \rho_i \tag{2.47}$$

Comparing the resulting incompressible species equation to the incompressible energy equation:

$$\frac{\partial \rho_i}{\partial t} + \vec{v} \cdot \nabla \rho_i = -\nabla \cdot \vec{j}_i + \dot{r}_i$$
(2.48)



$$\rho c_{p} \left( \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right) = -\nabla \cdot \vec{q} + \dot{q}$$
(2.49)

In equations (2.48) and (2.49), the reaction term,  $\dot{r}_i$ , is analogous to the heat generation term,  $\dot{q}$ ; the diffusion flux,  $\vec{j}_i$ , is analogous to the heat flux,  $\vec{q}$ , and  $d\rho_i$  is analogous to  $\rho c_p dT$ .

# 2.5.2 Fick's Law

The Fick's Law states that during mass diffusion, the flux,  $\vec{j}_i$  of a dilute component, 1, into a second fluid, 2, is proportional to the gradient of its mass concentration,  $m_i$ . Thus

$$\vec{j}_i = -\rho D_{12} \nabla m_1, \qquad (2.50)$$

where the constant  $D_{12}$  is the binary diffusion coefficient.

We can use Equation (2.50) to eliminate  $\overline{j}_i$  in equation (2.48). The resulting equation may be written in different forms depending on what is assumed about the variation of the physical properties. If the product  $\rho D_{im}$  is spatially uniform, then equation (2.48) becomes:

$$\frac{D}{Dt}m_i = D_{im}\nabla^2 m_i + \frac{\dot{r}_i}{\rho}$$
(2.51)

If instead,  $\rho$  and  $D_{im}$  are both spatially uniform, then,

$$\frac{D}{Dt}\rho_i = D_{im}\nabla^2\rho_i + \dot{r}_i$$
(2.52)

We now state the equation of species conservation and its particular form in molar variables instead of the mass variables as,

$$\frac{DC}{Dt} = D\nabla^2 C + \dot{r}, \qquad (2.53)$$

where C is the species concentration, D is the mass diffusivity and  $\dot{r}$  is the rate of generation of species.



# **CHAPTER THREE**

# HYDROMAGNETIC FLOW OVER A FLAT PLATE WITH HEAT AND MASS TRANSFER IN THE PRESENCE OF RADIATION AND CONVECTIVE BOUNDARY CONDITIONS

#### 3.1 Introduction

In this chapter, the mathematical governing equations in chapter two are employed for a particular engineering problem frequently encountered in industry. Relevant assumptions are made and a numerical procedure based on Runge-Kutta algorithm is used to solve the problem. The model as indicated in chapter two is highly nonlinear and a method based on similarity analysis shall be used.

#### **3.2** Formulation of the Problem

The steady laminar two-dimensional hydrodynamic boundary layer flow with heat and mass transfer over a flat plate in a stream of cold fluid at temperature  $T_{\infty}$  in the presence of radiation and magnetic field is considered. It is assumed that the fluid property variations due to temperature and chemical species concentration are limited to fluid density. It is also assumed that the lower surface of the plate is heated by convection from a hot fluid at temperature  $T_s$  which provides the heat transfer coefficient  $h_s$ . The cold fluid on the upper side of the plate is assumed to be Newtonian, electrically conducting with constant fluid property. A uniform transverse magnetic field  $B_0$  is imposed normal to the *x*-axis, as shown in Figure 3.1. Both the induced magnetic field due to the motion of the electrically conducting fluid and the electric field due to the polarisation of charges are assumed to be negligible.

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Figure 3.1 Schematic Diagram of Problem

Let the *x*-axis be taken along the direction of the plate and *x*-axis normal to it. If u, v, T and C are the fluid *x*-component of velocity, *y*-component of velocity, temperature and concentration respectively. With these assumptions, the governing equations for the problem can be modeled as:

#### 3.2.1 The Continuity Equation

Consider the substantial derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$
(3.1)

The problem under study is steady (flow properties do not change with time), two-

dimensional (no *z*-axis) so  $\frac{\partial}{\partial t} = 0$  and  $\frac{\partial}{\partial z} = 0$ . Hence (3.1) reduces to

$$\frac{D}{Dt} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$
(3.2)



This means the velocity field,  $V = u\hat{i} + v\hat{j} + w\hat{k}$  and its divergence,  $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ , now become:

$$V = u\,\hat{i} + v\,\hat{j}\,,\,\nabla = \frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\,\hat{j}$$
(3.3)

The continuity equation given in equation (2.7) now simplifies to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.4}$$

# 3.2.2 The Momentum Equation

It is to be noted that once the flow is horizontal, gravity effects do not exist. The flow is free stream and hence pressure is atmospheric and can be ignored since atmospheric pressure is negligible. This is a parallel flow (the y-component of velocity, v, is zero) so we the y-component of momentum goes to zero. Thus, the x-component velocity, u, is the only velocity. From the Navier-Stokes equation (2.25), the momentum equation becomes:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2},$$
(3.5)

A field of magnetic induction  $B_0$  is applied transverse to the motion of an electrically conducting fluid flowing on a flat plate with velocity field, V. Charged particles moving with the fluid will experience an induced electric field,  $V \ge B_0$  which will tend to drive an electric current in the direction perpendicular to both V and  $B_0$ . Neglecting the Hall Effect, the magnitude of the current density for a weakly ionized fluid is given by the generalized Ohm's law as;

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$$j = \sigma(-V \times B_0) \tag{3.6}$$

For the purposes of this study, both V and  $\sigma$  are assumed to be uniform. In terms of the two dimensional coordinate system:

$$j = \sigma B_0 u \tag{3.7}$$

Any movement of a conducting material in a magnetic field generates electric currents  $\mathbf{j}$ , which in turn induces a magnetic field. Each unit volume of liquid having  $\mathbf{j}$  and  $\mathbf{B}_0$  experiences MHD force  $\sim \mathbf{j} \times \mathbf{B}_0$ , known as the "Lorentz force" which retards the motion of the flow:

$$j \times B_0 = \sigma B_0^2 u \tag{3.8}$$

Therefore, including a magnetic force into the flow field reduces the momentum of the flow as given equation in (3.5) to:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} (u - U_\infty)$$
(3.9)

Equation (3.9) is the modelled momentum equation for the problem under study.

Applying the stated assumptions to the energy equation (2.40) we get:

$$\rho c_{p} \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \kappa \frac{\partial^{2} T}{\partial y^{2}} + \mu \left( \frac{\partial u}{\partial y} \right)^{2}$$
(3.10)

Now, consider the radiation effect with a radiative heat flux,  $q_r$ . and assume that  $q_r$  is a function of y only. In other words, the radiative heat decays exponentially with



respect to the y-axis. The contribution of this radiative heat flux and the magnetic force to the energy equation makes equation (3.10):

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma B_0^2}{\rho c_p}\left(u - U_\infty\right)^2 - \frac{1}{\rho c_p}\frac{\partial q_p}{\partial y} \qquad (3.11)$$

Equation (3.11) is the modelled energy equation to the study.

# 3.2.3 The Concentration Equation

Continuing in same fashion, from (2.53), the species concentration equation governing the problem is:

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - \gamma (C - C_{\infty})$$
(3.12)

In equation (3.12), the rate of generation of species  $\gamma(C - C_{\infty})$  is considered destructive hence the negative, where  $\gamma$  is the reaction rate.

# 3.2.4 Associated Boundary Conditions

The associated boundary conditions for the problem are required to enable a complete solution. Due to the no-slip and convective boundary conditions, the boundary conditions are specified as follows:

$$y = 0: u = v = 0, -\kappa \frac{\partial T}{\partial y} = h_s (T_s - T), C = C_s$$
(3.13)

$$y \to \infty : u(x, y) = 0, T = T_{\infty}, C = C_{\infty},$$
(3.14)

# 3.3 Self-Similar Solutions

The similarity solution is based on the idea that the velocity, temperature and species concentration distributions at any position along the plate surface, x, will collapse if they are plotted in dimensionless form as a function of an appropriately defined similarity variable. The similarity variable is defined as the ratio of the distance from the plate surface (y) to the approximate thickness of the momentum boundary layer  $(\delta_m)$ :

$$\eta = \frac{y}{\delta_m} \tag{3.15}$$

Therefore, the partial differential equations that describe the problem in terms of x and y will collapse to ordinary differential equations in  $\eta$  for dimensionless velocity, dimensionless temperature and dimensionless species concentration.

## 3.3.1 The Similarity Variable

The growth of the velocity, thermal and concentration boundary layers in a laminar flow occur primarily due to the molecular diffusion of momentum and energy. Therefore, the momentum boundary layer thickness  $(\delta_m)$  will grow approximately according to:

$$\delta_m \approx 2\sqrt{\upsilon t} \tag{3.16}$$

where v is the kinematic viscosity and t is time, which is related to the distance from the leading edge (x) and the characteristic velocity ( $u_{char}$ ) according to:

$$t = \frac{x}{u_{char}} \tag{3.17}$$

For this study, the length is the total length of the plate (which is taken along the x-axis) while the characteristic velocity is the constant free-stream velocity far from the plate  $(U_{\infty})$ .

Substituting (3.16) into (3.17) leads to:

$$\delta_m \approx 2 \sqrt{\frac{\nu x}{U_{\infty}}} \tag{3.18}$$

Substituting (3.18) into (3.15) gives:

$$\eta = \frac{y}{2} \sqrt{\frac{U_{\infty}}{\upsilon x}}$$
(3.19)

Following the presentation of Ostrach (1953), the constant used to define the similarity parameter is adjusted slightly, thus (3.19) becomes,

$$\eta = y \sqrt{\frac{U_{\infty}}{\nu x}}$$
(3.20)

Hence (3.20) defines the similarity variable for this particular problem under investigation.

The dimensionless velocity, temperature and concentration are obtained from:

$$f' = \frac{u}{U_{\infty}} \tag{3.21}$$

$$\theta = \frac{T - T_{\infty}}{T_s - T_{\infty}} \tag{3.22}$$

$$\phi = \frac{C - C_{\infty}}{C_s - C_{\infty}} \tag{3.23}$$

At any position, x will collapse when expressed in terms of (3.20). Therefore,

$$f' = f'(x, y) = f'(\eta)$$
(3.24)

$$\theta = \theta(x, y) = \theta(\eta) \tag{3.25}$$

$$\phi = \phi(x, y) = \phi(\eta) \tag{3.26}$$

#### 3.3.2 The Stream Function

The stream function is defined such that the continuity equation, (3.4), is automatically satisfied. That is,

$$u = \left(\frac{\partial \psi}{\partial y}\right)_x$$
 and  $v = -\left(\frac{\partial \psi}{\partial x}\right)_y$  (3.27)

The stream function is related to the volumetric flow Q, between the surface of the plate and any position y according to,

$$Q = W\psi, \tag{3.28}$$

where W is the width of the plate. The volumetric flow rate is calculated from the velocity according to:

$$Q = W \int_0^y u \, dy \tag{3.29}$$

Equation (3.29) can be expressed in terms of the dimensionless variables (f' and  $\eta$ ) using Equations (3.20) and (3.21) as,



$$Q = WU_{\infty} \sqrt{\frac{ux}{U_{\infty}}} \int_0^{\eta} f' \, d\eta \tag{3.30}$$

Substituting Eq. (3.30) into Eq. (3.28) leads to:

$$\psi = U_{\infty} \sqrt{\frac{ux}{U_{\infty}}} \int_0^{\eta} f'(\eta) \, d\eta \tag{3.31}$$

The integral  $\int_0^{\eta} f'(\eta) d\eta = f(\eta)$  in Equation (3.31) can be thought of as the dimensionless form of the stream function and it must be a function of only the similarity variable  $(\eta)$ . Simplifying Eq. (3.31) leads to:

$$\psi = \sqrt{\nu U_{\infty} x} f(\eta) \tag{3.32}$$

Equation (3.32) can be rewritten as:

$$\psi = x^{\frac{1}{2}} \sqrt{\upsilon U_{\infty}} f(\eta)$$
(3.33)

Equation (3.33) is the stream function of the study.

#### 3.3.3 Transformation of the Problem

The similarity variables are substituted into the governing x-momentum, thermal energy conservation and species concentration equations as well as the boundary conditions for velocity, temperature and concentration in order to transform the three coupled partial differential equations into three coupled ordinary differential equations that can be solved more easily. The continuity equation is automatically satisfied using the stream function as defined. The transformation process involves taking the problem in terms of x and y and re-stating it in terms of  $\eta$ .

The similarity variable (3.20) is differentiated w.r.t x and y to get;

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2} x^{-\frac{3}{2}} y \sqrt{\frac{U_{\infty}}{\upsilon}}, \frac{\partial \eta}{\partial y} = x^{-\frac{1}{2}} \sqrt{\frac{U_{\infty}}{\upsilon}}$$
(3.34)

From (2.29), we express the x-velocity (u) in terms of the similarity variables as:

$$u = \frac{\partial \psi}{\partial y} = U_{\infty} f'(\eta) \tag{3.35}$$

From (2.29), we can also express the y-velocity (v) in terms of the similarity variables as:

$$v = -\frac{\partial \psi}{\partial x} = -\left(\frac{1}{2}x^{-\frac{1}{2}}\sqrt{\upsilon U_{\infty}}f(\eta) - \frac{1}{2}x^{-\frac{3}{2}}y\sqrt{\frac{U_{\infty}}{\upsilon}} \cdot x^{-\frac{1}{2}}\sqrt{\upsilon U_{\infty}}f'(\eta)\right)$$
(3.36)

Simplifying and rearranging (3.36) gives:

$$v = \frac{1}{2} U_{\infty} x^{-1} y f'(\eta) - \frac{1}{2} x^{-1/2} f(\eta)$$
(3.37)

The partial derivatives of u in (3.35) w.r.t x and y are as follows:

$$\frac{\partial u}{\partial x} = -\frac{1}{2} x^{-\frac{3}{2}} y \sqrt{\frac{U_{\infty}^3}{\upsilon}} f''(\eta)$$
(3.38)

$$\frac{\partial u}{\partial y} = x^{-\frac{1}{2}} \sqrt{\frac{U_{\infty}^{3}}{\upsilon}} f''(\eta)$$
(3.39)

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_{\infty}^2}{\upsilon} x^{-1} f'''(\eta)$$
(3.40)

The partial derivative of v in (3.37) w.r.t y gives:

$$\frac{\partial v}{\partial y} = \frac{1}{2} x^{-\frac{3}{2}} y \sqrt{\frac{U_{\infty}^3}{v}} f''(\eta)$$
(3.41)

# The Continuity Equation

The continuity equation (3.4) is satisfied using (3.38) and (3.41).

# **3.3.3.1 The Dimensionless Momentum Equation**

For the momentum equation, substituting (3.35), (3.37), (3.38), (3.39) and (3.40) in (3.9) gives,

$$U_{\infty}f'(\eta)\left(-\frac{1}{2}x^{-\frac{3}{2}}y\sqrt{\frac{U_{\infty}^{3}}{\upsilon}}f''(\eta)\right) + \left(\frac{1}{2}U_{\infty}x^{-1}yf'(\eta) - \frac{1}{2}x^{-\frac{1}{2}}f(\eta)\right) \cdot x^{-\frac{1}{2}}\sqrt{\frac{U_{\infty}^{3}}{\upsilon}}f''(\eta) = \upsilon * \frac{U_{\infty}^{2}}{\upsilon}x^{-1}f'''(\eta) - \frac{\sigma B_{\circ}^{2}}{\rho}(U_{\infty}f'(\eta) - U_{\infty})$$
(3.42)

Expanding brackets, dividing through by 4 and grouping like-terms (3.44) yields,

$$-\frac{1}{2}x^{-\frac{3}{2}}y\sqrt{\frac{U_{\infty}^{5}}{\upsilon}}f'(\eta)f''(\eta) + \frac{1}{2}x^{-\frac{3}{2}}y\sqrt{\frac{U_{\infty}^{5}}{\upsilon}}f'(\eta)f''(\eta) - \frac{1}{2}U_{\infty}^{2}x^{-1}f(\eta)f''(\eta) = U_{\infty}^{2}x^{-1}f'''(\eta) - \frac{\sigma B_{0}^{2}U_{\infty}}{\rho}(f'(\eta) - 1)$$
(3.43)

Simplifying and dividing through by  $U_*^2 x^{-1}$  (3.43) gives,

$$-\frac{1}{2}f(\eta)f''(\eta) = f'''(\eta) - \frac{\sigma B_{\circ}^{2} x}{\rho U_{\infty}}(f'(\eta) - 1)$$
(3.44)

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But the coefficient  $\frac{\sigma B_{\circ}^2 x}{\rho U_{\infty}}$  in (3.44) is the local Magnetic field parameter ( $Ha_x$ ).

Rearranging (3.44) gives,

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) - Ha_x(f'(\eta) - 1) = 0$$
(3.45)

Equation (3.45) represents the dimensionless Momentum Equation which is a Third-Order Non-Linear Ordinary Differential Equation.

# **3.3.3.2 Dimensionless Energy Equation**

From (3.24) we can write,

$$T = (T_s - T_{\infty})\theta(\eta) + T_{\infty}$$
(3.46)

Finding the partial derivatives of (3.46) w.r.t x and y yields,

$$\frac{\partial T}{\partial x} = -\frac{1}{2} x^{-\frac{3}{2}} y \sqrt{\frac{U_{\infty}}{\upsilon}} (T_s - T_{\infty}) \theta'(\eta)$$
(3.47)

$$\frac{\partial T}{\partial y} = x^{-\frac{1}{2}} \sqrt{\frac{U_{\infty}}{\upsilon}} (T_s - T_{\infty}) \theta'(\eta)$$
(3.48)

$$\frac{\partial^2 T}{\partial y^2} = \frac{U_{\infty} x^{-1}}{\upsilon} \left( T_s - T_{\infty} \right) \theta''(\eta)$$
(3.49)

For simplicity and comparison, the radiative heat flux term in the energy equation in (3.11) is analyzed by utilizing the Rosseland diffusion approximation (Sparrow and Cess 1961) for an optically thick boundary layer as follows:



$$q_r = -\frac{4\sigma^*}{3K'}\frac{\partial T^4}{\partial y} \tag{3.50}$$

where K' and  $\sigma^*$  are the mean absorption coefficient and the Stefan-Boltzmann constant respectively. This approximation is valid at points optically far from the bounding surface, and it is good only for intensive absorption, that is, for an optically thick boundary layer (Hossain et al. 2001).

We assume that the temperature differences within the flow such as the term  $T^4$  may be expressed as a linear function of temperature. Hence, expanding  $T^4$  in a Taylor series about  $T_{\infty}$  and neglecting higher order terms, we get;

$$T^{4} \cong 4T_{\infty}^{3}T - 3T_{\infty}^{4} \tag{3.51}$$

Multiplying through (3.51) by  $\frac{\partial}{\partial y}$  gives,

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$$\frac{\partial T^4}{\partial y} \cong 4T_{\infty}^3 \frac{\partial T}{\partial y} \tag{3.52}$$

Equation (3.52) simplifies (3.50):

$$q_r = -\frac{4\sigma^*}{3K'} * 4T_{\infty}^3 \frac{\partial T}{\partial y}$$
(3.53)

Differentiating (3.53) with respect to y gives,

$$\frac{\partial q_r}{\partial y} = -\frac{4\sigma^*}{3K'} * 4T_{\infty}^3 \frac{\partial^2 T}{\partial y^2}$$
(3.54)

For the energy equation, substitute equations (3.35), (3.37), and (3.46)-(3.48) into (3.11) to get,

$$U_{\infty}f'(\eta)\left(-\frac{1}{2}x^{-\frac{3}{2}}y\sqrt{\frac{U_{\infty}}{\upsilon}}(T_{s}-T_{\infty})\theta'(\eta)\right)+$$

$$\left(\frac{1}{2}U_{\infty}x^{-1}yf'(\eta)-\frac{1}{2}x^{-\frac{1}{2}}\sqrt{\upsilon U_{\infty}}f(\eta)\right)\cdot x^{-\frac{1}{2}}y\sqrt{\frac{U_{\infty}}{\upsilon}}(T_{s}-T_{\infty})\theta'(\eta)=$$

$$\alpha*\frac{U_{\infty}x^{-1}}{\upsilon}(T_{s}-T_{\infty})\theta''(\eta)-\frac{\upsilon}{c_{p}}\left(x^{-\frac{1}{2}}\sqrt{\frac{U_{\infty}^{3}}{\upsilon}}f''(\eta)\right)^{2}+\frac{\sigma B_{\circ}^{2}}{\rho c_{p}}(U_{\infty}f'(\eta)-U_{\infty})^{2}-$$

$$-\frac{4\sigma^{*}}{3K'}*4T_{\infty}^{3}\left(\frac{U_{\infty}x^{-1}}{\rho c_{p}\upsilon}(T_{s}-T_{\infty})\theta''(\eta)\right)$$
(3.55)

Expanding and grouping like-terms in (3.55) give,

$$-\frac{1}{2}x^{-\frac{3}{2}}y\sqrt{\frac{U_{\infty}^{3}}{\upsilon}}(T_{s}-T_{\infty})f'(\eta)\theta'(\eta) + \frac{1}{2}x^{-\frac{3}{2}}y\sqrt{\frac{U_{\infty}^{3}}{\upsilon}}(T_{s}-T_{\infty})f'(\eta)\theta'(\eta) - \frac{1}{2}U_{\infty}x^{-1}(T_{s}-T_{\infty})f(\eta)\theta'(\eta) = \frac{\alpha}{\upsilon}*U_{\infty}x^{-1}(T_{s}-T_{\infty})\theta''(\eta) - \frac{U_{\infty}^{3}x^{-1}}{c_{p}}f''^{2}(\eta) + (3.56)$$

$$\frac{\sigma B_{\circ}^{2}U_{\infty}^{2}}{\rho c_{p}}(f'(\eta)-1)^{2} + \frac{4\sigma^{*}T_{\infty}^{3}}{\kappa K'}*\frac{4}{3}*\frac{\kappa}{\rho c_{p}}\left(\frac{U_{\infty}x^{-1}}{\upsilon}(T_{s}-T_{\infty})\theta''(\eta)\right)$$

Dividing through by  $\frac{\alpha}{\upsilon} * U_{\infty} x^{-1} (T_s - T_{\infty})$  and simplifying in (3.56) yield,

$$-\frac{1}{2} \cdot \frac{\upsilon}{\alpha} \cdot f(\eta) \theta'(\eta) = \theta''(\eta) - \frac{\nu U_{\infty}^{2}}{\alpha c_{p}(T_{s} - T_{\infty})} f''^{2}(\eta) + \frac{\sigma B_{\circ}^{2} x}{\rho U_{\infty}} \cdot \frac{\nu U_{\infty}^{2}}{\alpha c_{p}(T_{s} - T_{\infty})} (f'(\eta) - 1)^{2}$$
$$-\frac{4\sigma^{*} T_{\infty}^{3}}{\kappa K'} * \frac{4}{3} \theta''(\eta)$$
(3.57)



We rearrange and simplify further in (3.57) to get,

$$\theta''(\eta) + \frac{4\sigma^* T_{\infty}^3}{\kappa K'} * \frac{4}{3} \theta''(\eta) + \frac{1}{2} \cdot \frac{\upsilon}{\alpha} \cdot f(\eta) \theta'(\eta) + \frac{\mu U_{\infty}^2}{\kappa (T_s - T_{\infty})} f''^2(\eta) + \frac{\sigma B_{\omega}^2 x}{\rho U_{\infty}} \cdot \frac{\mu U_{\infty}^2}{\kappa (T_s - T_{\infty})} (f'(\eta) - 1)^2 = 0$$

$$(3.58)$$

Here the dimensionless Parameters of the flow are:  $Pr = \frac{v}{\alpha}$  is the Prandtl number;

 $Ra = \frac{4\sigma^*T^{3}_{\infty}}{\kappa K'}$  is the thermal radiation parameter;  $Ha_x = \frac{\sigma B_0^2 x}{\rho U_{\infty}}$  is the local magnetic

field parameter; and  $Br = \frac{\mu U_{\infty}^2}{\kappa (T_w - T_{\infty})}$  is the Brinkmann number. Equation (3.58)

finally gives,

$$\theta''(\eta) + \frac{4}{3} Ra\theta''(\eta) + \frac{1}{2} \Pr f(\eta) \theta'(\eta) + Br f''^{2}(\eta) + Br Ha_{x} (f'(\eta) - 1)^{2} = 0 \qquad 3.59$$

This can be rewritten as:

$$\left(1 + \frac{4}{3}Ra\right)\theta''(\eta) + \frac{1}{2}\Pr f(\eta)\theta'(\eta) + Brf''^{2}(\eta) + BrHa_{x}(f'(\eta) - 1)^{2} = 0$$
(3.60)

Equation (3.60) is the dimensionless thermal boundary layer equation. This is a second order nonlinear ordinary differential equation.

## 3.3.3.3 Dimensionless Concentration Equation

From (3.25) we can write:

$$C = (C_s - C_{\infty})\phi(\eta) + C_{\infty} \tag{3.61}$$



Finding the partial derivatives of (3.62) w.r.t. x and y yields,

$$\frac{\partial C}{\partial x} = -\frac{1}{2} x^{-\frac{3}{2}} y \sqrt{\frac{U_{\infty}}{\upsilon}} (C_s - C_{\infty}) \phi'(\eta)$$
(3.62)

$$\frac{\partial C}{\partial y} = x^{-\frac{1}{2}} \sqrt{\frac{U_{\infty}}{\upsilon}} (C_s - C_{\infty}) \phi'(\eta)$$
(3.63)

$$\frac{\partial^2 C}{\partial y^2} = \frac{U_{\infty} x^{-1}}{\upsilon} (C_s - C_{\infty}) \phi''(\eta)$$
(3.64)

Substituting Equations (3.61)-(3.64) into (3.12) gives,

$$U_{\infty}f'(\eta)\left(-\frac{1}{2}x^{-\frac{3}{2}}y\sqrt{\frac{U_{\infty}}{\upsilon}}(C_{s}-C_{\infty})\phi'(\eta)\right)+ \left(\frac{1}{2}U_{\infty}x^{-1}yf'(\eta)-\frac{1}{2}x^{-\frac{1}{2}}\sqrt{\upsilon U_{\infty}}f(\eta)\right)\cdot x^{-\frac{1}{2}}\sqrt{\frac{U_{\infty}}{\upsilon}}(C_{s}-C_{\infty})\phi'(\eta)= (3.65)$$
$$D*\frac{U_{\infty}x^{-1}}{\upsilon}(C_{s}-C_{\infty})\phi''(\eta)-\gamma(C_{s}-C_{\infty})\phi(\eta)$$

Expanding and grouping like terms in (3.65) give,

$$-\frac{1}{2}x^{-\frac{3}{2}}y\sqrt{\frac{U_{\infty}^{3}}{\upsilon}}(C_{s}-C_{\infty})f'(\eta)\phi'(\eta)+\frac{1}{2}x^{-\frac{3}{2}}y\sqrt{\frac{U_{\infty}^{3}}{\upsilon}}(C_{s}-C_{\infty})f'(\eta)\phi'(\eta)-\frac{1}{2}U_{\infty}x^{-1}(C_{s}-C_{\infty})f(\eta)\phi'(\eta)=\frac{DU_{\infty}x^{-1}}{\upsilon}(C_{s}-C_{\infty})\phi''(\eta)-\gamma(C_{s}-C_{\infty})\phi(\eta)$$
(3.66)

Simplifying and dividing through by  $\frac{DU_{\infty}x^{-1}}{\upsilon}(C_s - C_{\infty})$  in (3.66) gives,

$$-\frac{1}{2}\frac{\upsilon}{D}f(\eta)\phi'(\eta) = \phi''(\eta) - \frac{\upsilon}{D} \cdot \frac{\gamma x}{U_{\infty}}\phi(\eta)$$
(3.67)

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The dimensionless parameters are:  $Sc = \frac{v}{D}$  is the Schmidt number, and  $\beta_x = \frac{\gamma x}{U_{\infty}}$  is the local reaction rate parameter.

Simplifying further and rearranging in (3.67) yield,

$$\phi''(\eta) + \frac{1}{2} Scf(\eta)\phi'(\eta) - Sc\beta_x\phi(\eta) = 0$$
(3.68)

Equation (3.68) is our dimensionless species concentration boundary layer equation. This is a second order nonlinear ordinary differential equation.

## 3.3.3.4 Dimensionless Boundary Conditions

The corresponding dimensionless boundary conditions are obtained by substituting equations (3.22), (3.25), (3.35), (3.37) and (3.47) into equation (3.16).

For the convective boundary condition, substitute the dimensionless temperature gradient (3.47) in  $-\kappa \frac{\partial T}{\partial y} = h_s (T_s - T)$ .

That is,

$$-\kappa \frac{\partial T}{\partial y}\Big|_{y=0} = -\kappa \sqrt{\frac{U_{\infty}}{\upsilon x}} (T_s - T_{\infty}) \theta'(0) = h_s [T_s - T]$$
(3.69)

Simplifying further by substituting (3.24) for T gives,

$$-\kappa \sqrt{\frac{U_{\infty}}{\iota x}} (T_s - T_{\infty}) \theta'(0) = h_s [T_s - (T_s - T_{\infty}) \theta(0) - T_{\infty}]$$
(3.70)



Rearranging for the dimensionless temperature gradient in (3.70) yields,

$$\theta'(0) = -\frac{h_s}{\kappa} \sqrt{\frac{Ux}{U_{\infty}}} \left[-\theta(0) + 1\right]$$
(3.71)

The dimensionless number in (3.71) is:  $Bi_x = \frac{h_s}{\kappa} \sqrt{\frac{Ux}{U_{\infty}}}$  represents the local Biot

Hence, the convective boundary condition in dimensionless form is expressed as:

$$\theta'(0) = Bi_x[\theta(0) - 1] \tag{3.72}$$

Continuing in same fashion will lead to all the dimensionless conditions at no-slip and free stream.

We organize the self-similar solution of our governing equations to the problem with the associated boundary conditions:

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) - Ha_{x}(f'(\eta) - 1) = 0,$$

$$\left[1 + \frac{4}{3}Ra\right]\theta''(\eta) + \frac{1}{2}\Pr f(\eta)\theta'(\eta) + Brf''^{2}(\eta) + BrHa_{x}(f'(\eta) - 1)^{2} = 0,$$
(3.73)

$$\phi''(\eta) + \frac{1}{2} \operatorname{Scf}(\eta) \phi'(\eta) - \operatorname{Sc}\beta_x \phi(\eta) = 0.$$

number.



The corresponding boundary conditions in (3) now become

$$f(0) = 0, f'(0) = 0, \theta'(0) = Bi_x[\theta(0) - 1], \phi(0) = 1,$$
  
$$f'(\infty) = 1, \theta(\infty) = 0, \phi(\infty) = 0.$$
 (3.74)

In the above equations, primes denote the order of differentiation with respect to the similarity variable. Obviously the local parameters  $Ha_x$ ,  $Bi_x$  and  $\beta_x$  in (3.73) and (3.74) are functions of x. In order to have a similarity solution, all the parameters must be constants and we therefore assume that  $h_s = ax^{-\frac{1}{2}} \sigma = bx^{-1} \gamma = cx^{-1}$ , where a, b and c are constants.

#### 3.3.4 Dimensionless Fluxes

#### 3.3.4.1 The Skin Friction Coefficient

The fact that the function  $f(\eta)$  gives all information about the flow in the boundary layer must be emphasized. The shear stress can be obtained from it, using Newton's law of viscous shear:

$$\tau_{s} = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \mu \frac{\partial}{\partial y} \left( u_{\infty} f' \right)\Big|_{y=0} = \mu u_{\infty} \left( \frac{\partial f'}{\partial \eta} \frac{\partial \eta}{\partial y} \right)_{y=0}$$
(3.75)

From (3.34) we can rewrite (3.75) as,

$$\tau_{s} = \mu u_{\infty} \frac{\sqrt{u_{\infty}}}{\sqrt{\iota x}} \frac{\partial^{2} f}{\partial \eta^{2}} \bigg|_{\eta=0}$$
(3.76)



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We rewrite (3.76):

$$\tau_s = \mu u_\infty \frac{\sqrt{u_\infty}}{\sqrt{ux}} f''(0) \tag{3.77}$$

The local skin friction coefficient or local skin drag coefficient is defined as,

$$C_f = \frac{2\tau_s}{\rho u_p^2} \tag{3.78}$$

Substituting Equation (3.77) into (3.78) we get,

$$C_{f} = \frac{2\mu u_{\infty}}{\rho u_{\infty}^{2}} \frac{\sqrt{u_{\infty}}}{\sqrt{\upsilon x}} f''(0) = \frac{2\upsilon}{u_{\infty}} \frac{\sqrt{u_{\infty}}}{\sqrt{\upsilon x}} f''(0)$$
(3.79)

We simplify (3.79) to get,

$$C_{f} = 2\sqrt{\frac{\upsilon}{u_{\infty}x}} f''(0) = 2 \operatorname{Re}_{x}^{-\frac{1}{2}} f''(0)$$
(3.80)

The constant in (3.80) is adjusted to get the local skin friction as,

$$\operatorname{Re}_{x}^{1/2} C_{f} = f''(0) \tag{3.81}$$

# 3.3.4.2 The Rate of Heat Transfer Coefficient

The rate of conduction of heat transfer coefficients are usually expressed in terms of the Nusselt number:

$$Nu = \frac{x q_s}{\kappa (T_s - T_\infty)},\tag{3.82}$$



where  $q_s$  is the heat flux at the surface of the plate. In the context of our problem, we define it to be the sum of the convective heat flux and radiative heat flux:

$$q_{s} = -\kappa \frac{\partial T}{\partial y}\Big|_{y=0} - \frac{4\sigma^{*}}{3K'} \frac{\partial T^{4}}{\partial y}\Big|_{y=0}$$
(3.83)

From (3.51) we can rewrite (3.83) as,

$$q_{s} = -\kappa \frac{\partial T}{\partial y} \bigg|_{y=0} - \frac{4\sigma^{*}}{3K'} \cdot 4T_{\infty}^{3} \frac{\partial T}{\partial y} \bigg|_{y=0}$$
(3.84)

By factorization, (3.84) becomes,

$$q_{s} = -\kappa \frac{\partial T}{\partial y} \bigg|_{y=0} \left( 1 + \frac{4\sigma^{*}T_{\infty}^{3}}{\kappa K'} \cdot \frac{4}{3} \right)$$
(3.85)

We put (3.47) into (3.85) to get,

$$q_{s} = -\kappa x^{-\frac{1}{2}} \sqrt{\frac{U_{\infty}}{\upsilon}} (T_{s} - T_{\infty}) \theta'(0) \left(1 + \frac{4\sigma^{*}T_{\infty}^{3}}{\kappa K'} \cdot \frac{4}{3}\right)$$
(3.86)

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We rearrange (3.86) to get,

$$q_{s} = -\kappa x^{-1} \sqrt{\frac{U_{\infty} x}{\upsilon}} (T_{s} - T_{\infty}) \theta'(0) \left(1 + \frac{4\sigma^{*} T_{\infty}^{3}}{\kappa K'} \cdot \frac{4}{3}\right)$$
(3.87)

We simplify (3.87) to get,

$$q_{s} = -\operatorname{Re}_{x}^{\frac{1}{2}} \kappa x^{-1} (T_{s} - T_{\infty}) \theta'(0) \left(1 + \frac{4}{3} Ra\right)$$
(3.88)

Now (3.88) in (3.82) gives,

$$Nu = \frac{-x \cdot \kappa \ x^{-1} \cdot \operatorname{Re}_{x}^{\frac{1}{2}}(T_{s} - T_{\infty}) \cdot \theta'(0) \cdot \left(1 + \frac{4}{3} \operatorname{Ra}\right)}{\kappa(T_{s} - T_{\infty})}$$
(3.89)

Simplifying (3.89) gives,

$$Nu = -\operatorname{Re}_{x}^{\frac{1}{2}} \cdot \left(1 + \frac{4}{3} \operatorname{Ra}\right) \cdot \theta'(0)$$
(3.90)

Rearranging (3.90) gives,

$$\operatorname{Re}_{x}^{-\frac{1}{2}} Nu = -\left(1 + \frac{4}{3}Ra\right)\theta'(0)$$
 (3.91)

# 3.3.4.3 The Rate of Mass Transfer Coefficient

The coefficient of mass transfer is generally specified by the Sherwood number:

$$Sh = \frac{x q_m}{D(C_s - C_\infty)} \tag{3.92}$$

where  $q_m$  is the mass diffusion flux.

In the context of this problem, the Fick's law is defined as:

$$q_m = -D\frac{\partial C}{\partial y} \tag{3.93}$$

From (3.62), we can write (3.93) as:

$$q_{m} = -Dx^{-\frac{1}{2}} \sqrt{\frac{U_{\infty}}{\upsilon}} (C_{s} - C_{\infty}) \phi'(0)$$
(3.94)

We simplify (3.94) to get:

$$q_{m} = -Dx^{-1} \sqrt{\frac{U_{\infty}x}{\upsilon}} (C_{s} - C_{\infty}) \phi'(0)$$
 (3.95)

We simplify (3.95) further to get,

$$q_{m} = -Dx^{-1} \cdot \operatorname{Re}_{x}^{\frac{1}{2}}(C_{s} - C_{\infty})\phi'(0)$$
(3.96)

We then put (3.96) into (3.92):

$$Sh = \frac{-x \cdot Dx^{-1} \cdot \operatorname{Re}_{x}^{1/2}(C_{s} - C_{\infty})\phi'(0)}{D(C_{s} - C_{\infty})}$$
(3.97)

We simplify (3.97) to get,

$$Sh = -\operatorname{Re}_{r}^{\frac{1}{2}} \phi'(0)$$
 (3.98)

We rearrange (3.98):

$$\operatorname{Re}_{*}^{-\frac{1}{2}}Sh = -\phi'(0) \tag{3.99}$$

# **CHAPTER FOUR**

# **RESULTS AND DISCUSSIONS**

#### 4.1 Introduction

This chapter explores various findings obtained through the mathematical analysis of the dimensionless coupled governing equations in (3.73) and the associated boundary conditions (3.74). We develop most effective numerical shooting technique with fourth-order Runge-Kutta integration algorithm. To select a representative value for the infinity the similarity value can assume  $(\eta_{\infty})$ , we begin with some initial guess value and solve the problem with some particular set of parameters to obtain f''(0),  $\theta'(0)$  and  $\phi'(0)$ . The solution process is repeated with another larger value of  $\eta_{\infty}$  until two successive values of f''(0),  $\theta'(0)$  and  $\phi'(0)$  differ only after desired digit signifying the limit of the boundary along  $\eta$ . The last value of  $\eta_{\infty}$  is chosen as appropriate value for that particular simultaneous equation of first order for seven unknowns following the method of superposition. To solve this system we require seven initial conditions whilst we have only two initial conditions f'(0) and f(0) on f; and one initial condition each on  $\theta$  and  $\phi$ . This means that there are three initial conditions, f''(0),  $\theta'(0)$  and  $\phi'(0)$  which are not prescribed. Now, we employ numerical shooting technique where these two ending boundary conditions are utilized to produce two unknown initial conditions at  $\eta = 0$ . In this calculation, the step size  $\Delta \eta = 0.001$  was used while obtaining the numerical solution with  $\eta_{max}=10$  and six-decimal (10<sup>-6</sup>) accuracy as the criterion for convergence. The numerical procedure was carried out using a Maple 16 software package. A representative set of numerical results are displayed graphically and discussed quantitatively to show some interesting aspects of some pertinent



controlling parameters of the flow on the dimensionless axial velocity profiles, temperature profiles, concentration profiles, local skin friction coefficient, rate of heat transfer and the rate of mass transfer. The discussions of the results are also around 1 **4.2** Numerical Results

The results of the study were compared to that of Aziz (2009) and Makinde (2012) in **Table 4.1** to justify the accuracy of the method used. In the absence of chemical species concentration effects, our work reduces to the work reported by Makinde (2012) and in the absence of magnetic field, viscous dissipation and chemical species concentration our work reduces to that of Aziz (2009). From the comparison, the results were observed to be consistent with published results in literature and thus, validate the accuracy of the numerical procedure.

Table 4.1 Comparison with Aziz (2009) and Makinde (2012) for  $Ha_x=0$ , Br =0, Pr=0.72

•	Aziz (2009)		Makinde (2	2012)	Present Study		
$\overline{Bi_x}$	- heta'(0)	$\theta(0)$	- heta'(0)	$\theta(0)$	$-\theta'(0)$	θ(0)	
0.05	0.0428	0.1447	0.04276	0.14466	0.042767	0.144661	
0.60	0.1981	0.6699	0.19805	0.66991	0.198051	0.669916	
1.00	0.2282	0.7718	0.22817	0.77182	0.228178	0.771822	
5.00	0.2791	0.9441	0.27913	0.94417	0.279131	0.944174	
20.00	0.2913	0.9854	0.29132	0.98543	0.291329	0.985434	

Effects of Parameters on Skin Friction and the Rate of Heat and Mass Transfer

In **Table 4.2**, it is observed that the rate of heat transfer  $-\theta'(0)$  increases with increasing Prandtl number and the local Biot number. This is due to the fact that high Prandtl and Biot numbers contribute to enhancing the convective heat exchange at the

surface of the plate, thus, increasing the rate at which heat is transferred at the surface of the plate. At the same time, the Lorenz force induced by increasing the magnetic field intensity increased the skin friction at the surface. Though this force is a retarding one, it enhances the rate at which the chemical species  $\phi'(0)$  is transported. Meanwhile this retarding force was distractive to the rate of heat transfer,  $-\theta'(0)$ . Just as the magnetic field parameter, increasing the thermal radiation parameter and the Brinkmann numbers reduced the rate of heat transfer due to viscous dissipation. Furthermore, increasing the Schmidt number and the reaction rate parameter increases the rate of mass transfer for obvious reasons.

1'a	ble 4.2	Results	of skin	friction	coefficient,	Nusselt	and S	Sherwood	numbers	tor
vai	rious va	alues of	controlli	ing para	meters					

Pr	Sc	Hax	Ra	Br	$\beta_x$	Bi <sub>x</sub>	<i>f</i> "(0)	$-\theta'(0)$	$-\phi'(0)$
0.71	0.24	0.1	0.1	0.1	0.1	0.1	0.451835	0.068283	0.248586
4.0	0.24	0.1	0.1	0.1	0.1	0.1	0.451835	0.081449	0.248586
7.1	0.24	0.1	0.1	0.1	0.1	0.1	0.451835	0.084597	0.248586
0.72	1.24	0.1	0.1	0.1	0.1	0.1	0.451835	0.068415	0.494321
0.72	1.78	0.1	0.1	0.1	0.1	0.1	0.451835	0.068415	0.575143
0.72	2.64	0.1	0.1	0.1	0.1	0.1	0.451835	0.068415	0.679036
0.72	0.24	0.5	0.1	0.1	0.1	0.1	0.770792	0.064224	0.261862
0.72	0.24	1.0	0.1	0.1	0.1	0.1	1.044009	0.060326	0.269658
0.72	0.24	1.5	0.1	0.1	0.1	0.1	1.260190	0.057136	0.274372
0.72	0.24	0.1	0.5	0.1	0.1	0.1	0.451835	0.066984	0.248586
0.72	0.24	0.1	1.0	0.1	0.1	0.1	0.451835	0.065314	0.248586
0.72	0.24	0.1	1.5	0.1	0.1	0.1	0.451835	0.063856	0.248586
0.72	0.24	0.1	0.1	0.5	0.1	0.1	0.451835	0.042658	0.248586
0.72	0.24	0.1	0.1	1.0	0.1	0.1	0.451835	0.010461	0.248586
0.72	0.24	0.1	0.1	1.5	0.1	0.1	0.451835	-0.021753	0.248586
0.72	0.24	0.1	0.1	0.1	0.5	0.1	0.451835	0.068415	0.384277
0.72	0.24	0.1	0.1	0.1	1.0	0.1	0.451835	0.068415	0.511822
0.72	0.24	0.1	0.1	0.1	1.5	0.1	0.451835	0.068415	0.615465
0.72	0.24	0.1	0.1	0.1	0.1	1	0.451835	0.209665	0.248586
0.72	0.24	0.1	0.1	0.1	0.1	2	0.451835	0.236829	0.248586
0.72	0.24	0.1	0.1	0.1	0.1	3	0.451835	0.247518	0.248586

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### 4.3 Graphical Results

# 4.3.1 Effects of Parameter Variation on the Velocity Profiles

The effect of varying the controlling parameters on the velocity boundary layer is depicted in **Figure 4.1**. Generally, the fluid velocity is lowest at the plate surface and increases to the free stream value satisfying the far field boundary condition. A consistent decrease in the longitudinal velocity informed by increasing magnetic field intensity with all profiles tending asymptotically to the free stream value away from the plate is observed. In practice, this phenomenon is due to the fact that increasing the magnetic field strength increases the Lorenz force which causes a greater opposition to fluid transport.







### 4.3.2 Effects of Parameter Variation of Temperature Profiles

**Figures 4.2 – 4.6** show the effects of the magnetic field parameter, Biot number, radiation parameter, Brinkmann number and Prandtl number respectively on the temperature profiles. It is observed that increasing the magnetic field intensity increases the fluid temperature which in turn, increases the thermal boundary layer (Figure 4.2). This can be attributed to the effect of Ohmic heating on the flow system. An increase in the Biot number is observed to increase the increase the temperature of the fluid due to the convective heat exchange between the hot fluid at the lower surface of the plate and the cold fluid at the upper surface of the plate (Figure 4.3). Increasing the thermal radiation parameter causes an increase in the fluid temperature within the boundary layer which in turn, increases the thermal boundary layer (Figure 4.4). In Figure 4.5, it was observed that the same trend occurs for the Brinkmann number due to viscous dissipation. Meanwhile, increasing the Prandtl number decreases the fluid temperature within the boundary layer (Figure 4.6). When the Prandtl number is high, the fluid velocity decreases, which implies lower thermal diffusivity and hence, decrease in fluid temperature (Figure 4.6).





Figure 4.2 Temperature profiles for varying values of magnetic parameter  $(Ha_x)$ 



Figure 4.3 Temperature profiles for varying values of Biot number  $(Bi_x)$ 





Figure 4.4 Temperature profiles for varying values of radiation parameter (Ra)



Figure 4.5 Temperature profiles for varying values of Brinkmann number (Br)

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Figure 4.6 Temperature profiles for varying values of Prandtl number (Pr)

### 4.3.3 Effects of Parameter Variation on the Concentration Profiles

**Figures 4.7 – 4.9** show the effects of the Magnetic field parameter, reaction rate parameter and the Schmidt number respectively on the concentration boundary layer. It is observed that increasing the magnetic parameter reduces the species concentration boundary layer thickness (Figure 4.7). The same is true for increasing reaction rate parameter (Figure 4.8) and the Schmidt number (Figure 4.9) since increasing the Schmidt number in particular implies momentum diffusivity dominates mass species diffusivity. Though increasing the reaction rate parameter implies increasing rate of reaction over momentum, it is interesting to note that, the concentration boundary layer reduces. This can be due to the fact that the reaction rate in this study is destructive and hence has adverse effect on the concentration boundary layer thickness.



Figure 4.7 Concentration profiles for varying values of magnetic parameter  $(Ha_x)$ 



Figure 4.8 Concentration profiles for varying values of reaction rate parameter  $(\beta_x)$ 



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# **CHAPTER FIVE**

# **CONCLUSIONS AND RECOMMENDATIONS**

#### 5.1 Introduction

This chapter presents the summary, conclusion and recommendation from the study.

# 5.2 Summary of Findings

The research primarily investigated the combined effects of radiation and viscous dissipation on MHD flow over a flat surface with heat and mass transfer and convective boundary conditions. The effects of controlling parameters on the skin friction, the rate of heat and mass transfers, and velocity, temperature, and concentration profiles were examined.

The equations governing fluid flow in general was derived and based on some assumptions, a mathematical model was developed for the problem under study and the associated boundary conditions. These models included the momentum, energy and species concentration equations which were nonlinear partial differential equations of higher order whose solution was not readily available. To make these coupled partial differential equations solvable numerically, similarity analysis was employed which reduced the problem to a set of nonlinear ordinary differential equations. These nonlinear ordinary differential equations which revealed the dimensionless parameters were solved by the Runge-Kutta integration along with the Newton Raphson algorithm using Maple 16 software.



With respect to thermal boundary layer; increasing viscous dissipation, convective heating and radiation was observed to increase the temperature profile but when the momentum diffusivity dominated, thermal diffusivity reduces. Meanwhile the induced Lorenz force reduces the velocity as well as the thickness of the species concentration. Increasing the concentration parameters also reduced the concentration boundary layer due to dominant momentum diffusion over mass species diffusion.

The findings also indicate that increasing viscous dissipation and radiation have adverse effect on the rate of heat transfer whereas increasing convective heat transfer makes the heat transfer process enhanced. On the other hand, the rate of mass transfer is enhanced by an increase in the Lorenz force and the species diffusion parameters.

The results pointed to the high increase in the skin friction as a result of intensifying the magnetic field strength.

### 5.3 Conclusion

Chemically reacting MHD flow over a flat surface in the presence of radiation and viscous dissipation with convective boundary conditions has been studied. Numerical results have been compared to earlier results published in the literature and consistency was achieved. Our results revealed that:

i. A consistent decrease in the longitudinal velocity within the boundary layer accompanies a rise in the magnetic field intensity with all profiles tending asymptotically to the free stream value away from the plate.

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- The thermal boundary layer increases with increasing values of the magnetic field parameter, Biot number, radiation parameter and Brinkmann number. Meanwhile, increasing the Prandtl number reduces it.
- iii. The concentration boundary layer decreases with increasing Magnetic field parameter, reaction rate parameter and the Schmidt number.
- iv. The skin friction at the surface increases for the increase in the magnetic field parameter.
- v. The rate of heat transfer at the surface increases with increasing values of the Prandtl number and the Biot number; whereas a decrease is observed for increasing the magnetic field parameter, the radiation parameter and the Brinkmann number
- vi. The rate of mass transfer at the surface increases with increasing values of the magnetic field parameter, the reaction rate parameter and the Schmidt number

## 5.4 **Recommendations**

i. Investigation of the effects of cooling and heating on MHD heat transfer revealed that for the convective heat flux boundary condition, heat transfer (and associated enhancement) is higher when flow temperature is higher.
When it comes to the convective surface temperature boundary condition, the dominant parameter that affects the heat transfer is the surface temperature, radiation and viscous dissipation. As the surface temperature increases, the heat transfer and the associated enhancement increases. These facts should be taken into account for the practical application of MHD in heat transfer device.

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- ii. Textile manufacturing involves a crucial energy-intensive drying stage at the end of the process to remove moisture left from dye setting. It is recommended that, in the process of optimising heat and mass transfer in the drying process, thermal radiation should be controlled.
- iii. In the design of plate fin Exchangers, it is recommended that convective heat transfer parameter among other parameters, must be given considerate attention, since it enhances the heat transfer process.
- iv. It is recommended that, in the process of heat transfer, viscous dissipation parameters must be controlled as it increases temperature.

### 5.5 Suggestion for further study

- i. The contemporary trend in the field of fluid flow, heat and mass transfer design is the second law of thermodynamics and its related concept of entropy generation minimisation (Sahin, 1998). It is therefore recommended to future researchers to include entropy analysis to provide better analysis.
- ii. It has been found that energy flux can be generated not only by temperature gradients but also by concentration gradients. The heat transfer caused by a concentration gradient is termed as diffusion thermo (Dufour) effect. On the other hand, mass transfer created by temperature gradients is called thermal - diffusion (Soret) effect. Generally, in heat and mass transfer process, the Soret and Dufour effects are neglected because they are smaller order of magnitude than the effects described by Fourier's and Fick's laws. The Soret effect has been utilized for isotope separation

and in mixture between gases of very light molecular weight and of medium molecular weight. Further research is recommended to include Soret and Dufour effects in the present study for a better analysis.

iii. It is also recommended that further research is done to investigate into this problem by varying the orientation of the flat plate. Particularly, an inclined plate will make this work more general.

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# **APPENDIX I**

# LIST OF PUBLICATIONS FROM THE RESEARCH

- Arthur, E.M. and Seini, Y.I. (2014). MHD thermal stagnation point flow towards a stretching porous surface, *Mathematical Theory and Modeling*, 4(5), 163-169.
- E.J Christian, Y.I Seini, Arthur E.M. (2014). MHD boundary Layer stagnation point flow with Radiation and Chemical Reaction towards a Heated Shrinking Porous Surface, *International Journal of Physical Sciences*, 9(14), 320-328.
- 3. Imoro, R., Arthur, E.M. and Seini, Y.I. (2014). Heat and mass transfer over a vertical surface with convective boundary conditions in the presence of viscous dissipation and n<sup>th</sup> order chemical reaction, *International Journal of Computational and Applied Mathematics*, 9 (2), 101-118. ISSN 1819-4966
- 4. Arthur, E. M. and Seini, I. Y. (2014). Hydromagnetic stagnation point flow over a porous stretching surface in the presence of radiation and viscous dissipation, *Applied and Computational Mathematics*.
- Arthur, E. M., Seini, I. Y. and Seidu, A. (2014). On Chemically Reacting Hydromagnetic Flow over a Flat Surface in the Presence of Radiation with Viscous Dissipation and Convective Boundary Conditions. *American Journal* of Applied Mathematics, Vol. 2, No. 5, pp. 179-185. doi: 10.11648/j.ajam.20140205.15

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# **APPENDIX II**

## **MAPLE CODE FOR NUMERICAL RESULTS**

> $Pr := 0.72 : M := 0.1 : Sc := 0.24 : \beta := 0.1 : Ra := 0.1 : Br := 0.1$ Bi := 0.1 :	:
$fcns := \{F(y), \Theta(y), \phi(y)\}:$	
$sys1 := diff(F(y), y$3) + \frac{1}{2} \cdot F(y) \cdot diff(F(y), y$2) - M$	
$\cdot (diff(F(y), y) - 1) = 0, \left(1 + \frac{4}{3} \cdot (Ra)\right) \cdot diff(\theta(y), y \ge 2)$	
$+\frac{1}{2} \cdot Pr \cdot F(y) \cdot diff(\theta(y), y) + Br \cdot (diff(F(y), y))^{2} + Br$	
$\cdot M \cdot (diff(F(y), y) - 1)^2 = 0, diff(\phi(y), y \ge 2) + \frac{1}{2} \cdot Sc \cdot F(y)$	
$\begin{aligned} & diff(\phi(y), y) - Sc \cdot \beta \cdot \phi(y) = 0, D(F)(0) = 0, F(0) = 0, \\ & D(\theta)(0) = Bi \cdot (\theta(0) - 1), \phi(0) = 1, D(F)(10) = 1, \theta(10) = 0, \\ & \phi(10) = 0: \end{aligned}$	
$\frac{d^3}{dy^3} F(y) + \frac{1}{2} F(y) \left( \frac{d^2}{dy^2} F(y) \right) - 0.1 \left( \frac{d}{dy} F(y) \right) + 0.1 = 0,$	
$1.13333333\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}\theta(y)\right) + 0.360000000F(y)\left(\frac{\mathrm{d}}{\mathrm{d}y}\theta(y)\right)$	
$+ 0.1 \left(\frac{d^2}{dy^2} F(y)\right)^2 + 0.01 \left(\frac{d}{dy} F(y) - 1\right)^2 = 0, \frac{d^2}{dy^2} \phi(y)$	
+ 0.120000000F(y) $\left(\frac{d}{dy}\phi(y)\right)$ - 0.024 $\phi(y) = 0, D(F)(0)$	
= 0, $F(0) = 0$ , $D(\theta)(0) = 0.1 \theta(0) - 0.1$ , $\phi(0) = 1$ , $D(F)(10)$	
$= 1, \theta(10) = 0, \phi(10) = 0$	

 $p := dsolve(\{sys1, D(F)(0) = 0, F(0) = 0, D(\theta)(0) = Bi \cdot (\theta(0) - 1), \phi(0) = 1, D(F)(10) = 1, \theta(10) = 0, \phi(10) = 0\}, fcns, type = numeric, method = bvp, abserr = 1e-6)$ 

 $proc(x_bvp)$  ... end proc

dsoll := dsolve({sys1}, numeric, output = operator)

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$$[y = \text{proc}(y) \dots \text{ end proc}, F = \text{proc}(y) \dots \text{ end proc}, D(F) = \text{proc}(y) \dots$$
  
end proc,  $D^{(2)}(F) = \text{proc}(y) \dots$  end proc,  $\phi = \text{proc}(y) \dots$   
end proc,  $D(\phi) = \text{proc}(y) \dots$  end proc,  $\theta = \text{proc}(y) \dots$  end proc,  
 $D(\theta) = \text{proc}(y) \dots$  end proc]  
$$dsoll(0);$$
$$[y = 0, F(0) = 0, D(F)(0) = 0, D^{(2)}(F)(0) = 0.99999999999999999995 \text{(D)}(\phi)(0) = -0.24858613612211488 \text{(C)}(0) = 0.31584661550448467.4$$

 $D(\theta)(0) = -0.0684153384495514994$ 



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# **APPENDIX III**

# MAPLE CODE FOR GRAPHICAL RESULTS

with (plots):
Pr := 0.72: 
$$M := 0.1: Sc := 0.24: \beta := 0.1: Ra := 0.1: Br := 0.1:$$
Bi := 0.1:
fcns := { $F(y), \theta(y), \phi(y)$ }:
sys := diff ( $F(y), y$ \$3) +  $\frac{1}{2} \cdot F(y) \cdot diff (F(y), y$ \$2) - M
 $\cdot (diff (F(y), y) - 1) = 0, \left(1 + \frac{4}{3} \cdot (Ra)\right) \cdot diff (\theta(y), y$ \$2)
+  $\frac{1}{2} \cdot Pr \cdot F(y) \cdot diff (\theta(y), y) + Br \cdot (diff (F(y), y$ \$2))<sup>2</sup> + Br
 $\cdot M \cdot (diff (F(y), y) - 1)^2 = 0, diff (\phi(y), y$ \$2) +  $\frac{1}{2} \cdot Sc \cdot F(y)$ 
 $\cdot diff (\phi(y), y) - Sc \cdot \beta \cdot \phi(y) = 0:$ 

 $p1 := dsolve(\{sys, D(F)(0) = 0, F(0) = 0, D(\theta)(0) = Bi \cdot (\theta(0) - 1), \phi(0) = 1, D(F)(10) = 1, \theta(10) = 0, \phi(10) = 0\}, fcns, type = numeric, method = bvp, abserr = 1e-6)$ 

 $\mathbf{proc}(x\_bvp)$  ... end proc

$$plf := odeplot(pl, [y, F'(y)], 0..10, numpoints = 50, labels = ["y", "velocity"], style = patch, symbol = asterisk, color = black):$$

 $plt := odeplot(pl, [y, \theta(y)], 0..10, numpoints = 50, labels = ["y", "Temperature"], style = patch, symbol = asterisk, color = black):$ 

 $plc := odeplot(pl, [y, \phi(y)], 0..10, numpoints = 50, labels = ["y", "concentration"], style = patch, symbol = asterisk, color = black):$ 

with(plots):

 $Pr := 1 : M := 0.1 : Sc := 0.24 : \beta := 0.1 : Ra := 0.1 : Br := 0.1 : Bi :: 0.1 : Bi := 0.1 : Bi := 0.1 : Bi := 0.1 : Bi := 0.1 : Bi :: 0.1 : Bi := 0.1 : Bi := 0.1 : Bi :: 0.1 : Bi :: 0.1 : Bi := 0.1 : Bi :: 0.1 :: 0.1 : Bi :: 0.1 ::$ 



 $proc(x_bvp)$  ... end proc

- p2f := odeplot(p2, [y, F'(y)], 0..10, numpoints = 50, labels = ["y", "velocity"], style = point, symbol = circle, color = black):
- $p2t := odeplot(p2, [y, \theta(y)], 0..10, numpoints = 50, labels = ["y", "Temperature"], style = point, symbol = circle, color = black):$
- $p2c := odeplot(p2, [y, \phi(y)], 0..10, numpoints = 50, labels = ["y", "concentration"], style = point, symbol = circle, color = black):$

 $\begin{aligned} & \text{with}(plots): \\ & Pr := 4: M := 0.1: Sc := 0.24: \beta := 0.1: Ra := 0.1: Br := 0.1: Bi \\ & := 0.1: \\ & \text{fcns} := \{F(y), \theta(y), \phi(y)\}: \\ & \text{sys} := diff(F(y), y\$3) + \frac{1}{2} \cdot F(y) \cdot diff(F(y), y\$2) - M \\ & \cdot (diff(F(y), y) - 1) = 0, \left(1 + \frac{4}{3} \cdot (Ra)\right) \cdot diff(\theta(y), y\$2) \\ & + \frac{1}{2} \cdot Pr \cdot F(y) \cdot diff(\theta(y), y) + Br \cdot (diff(F(y), y\$2))^2 + Br \\ & \cdot M \cdot (diff(F(y), y) - 1)^2 = 0, diff(\phi(y), y\$2) + \frac{1}{2} \cdot Sc \cdot F(y) \\ & \cdot diff(\phi(y), y) - Sc \cdot \beta \cdot \phi(y) = 0: \end{aligned}$ 

 $p3 := dsolve(\{sys, D(F)(0) = 0, F(0) = 0, D(\theta)(0) = Bi \cdot (\theta(0) - 1), \phi(0) = 1, D(F)(10) = 1, \theta(10) = 0, \phi(10) = 0\}, fcns, type = numeric, method = bvp, abserr = 1e-6\}$ 

proc(x, bvp) ... end proc

- p3f := odeplot(p3, [y, F'(y)], 0..10, numpoints = 50, labels = ["y", "velocity"], style = point, symbol = point, color = black):
- $p3t := odeplot (p3, [y, \theta(y)], 0..10, numpoints = 50, labels = ["y", "Temperature"], style = point, symbol = point, color = black):$
- $p3c := odeplot(p3, [y, \phi(y)], 0..10, numpoints = 50, labels = ["y", "concentration"], style = point, symbol = point, color = black):$

#### with(plots):

 $Pr := 7.1: M := 0.1: Sc := 0.24: \beta := 0.1: Ra := 0.1: Br := 0.1: Bi := 0.1:$ Bi := 0.1:

$$fcns := \{F(y), \theta(y), \phi(y)\}:$$

$$sys := diff(F(y), y^{3}) + \frac{1}{2} \cdot F(y) \cdot diff(F(y), y^{2}) - M$$

$$\cdot (diff(F(y), y) - 1) = 0, \left(1 + \frac{4}{3} \cdot (Ra)\right) \cdot diff(\theta(y), y^{2})$$

$$+ \frac{1}{2} \cdot Pr \cdot F(y) \cdot diff(\theta(y), y) + Br \cdot (diff(F(y), y^{2}))^{2} + Br$$

$$\cdot M \cdot (diff(F(y), y) - 1)^{2} = 0, diff(\phi(y), y^{2}) + \frac{1}{2} \cdot Sc \cdot F(y)$$

$$\cdot diff(\phi(y), y) - Sc \cdot \beta \cdot \phi(y) = 0:$$

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 $p4 := dsolve(\{sys, D(F)(0) = 0, F(0) = 0, D(\theta)(0) = Bi \cdot (\theta(0) - 1), \phi(0) = 1, D(F)(10) = 1, \theta(10) = 0, \phi(10) = 0\}, fcns, type = numeric, method = bvp, abserr = 1e-6\}$ 

 $proc(x_bvp)$  ... end proc

- p4f := odeplot(p4, [y, F'(y)], 0..10, numpoints = 50, labels = ["y", "velocity"], style = point, symbol = cross, color = black):
- $p4t := odeplot (p4, [y, \theta(y)], 0..10, numpoints = 50, labels = ["y", "Temperature"], style = point, symbol = cross, color = black):$
- >  $p4c := odeplot(p4, [y, \phi(y)], 0..10, numpoints = 50, labels = ["y", "concentration"], style = point, symbol = cross, color = black):$

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