UNIVERSITY FOR DEVELOPMENT STUDIES, TAMALE

HEAT AND MASS TRANSFER WITH VISCOUS DISSIPATION AND CONVECTIVE BOUNDARY CONDITIONS

BY

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DECLARATION

Student

I hereby declare that this thesis is the result of my own original work and that no part of it has been presented for another degree in this University or elsewhere:

Candidate's Signature:…………………………… Date: 27/01/2015

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Supervisor

I hereby declare that the preparation and presentation of the thesis was supervised in accordance with the guidelines on supervision of thesis laid down by the University for Development Studies.

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ABSTRACT

This study investigates the effect of chemical reaction and viscous dissipation on heat and mass transfer with convective boundary conditions. A system of partial differential equations describing the problem has been modelled and the technique of similarity analysis employed to transform the model into ordinary differential equations. The reduced system was solved using the Newton-Raphson shooting method alongside with the Forth-order Runge-Kutta algorithm. The results are presented graphically and in tabular form for various controlling parameters. Among others, the results obtained revealed that: The velocity increases with the increase in Eckert, thermal and concentration Grashof numbers. It also decreases with an increase in Schmidt number. The temperature reduces with increasing Prandlt and increases with Eckert, convective heat transfer parameter, thermal and concentration Grashof numbers.The concentration boundary layer decreases with increase in reaction rate parameter, Schmidt, convective heat transfer parameter and thermal and solutal Grashof numbers; and increases slightly with increasing order of chemical reaction.

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DEDICATION

I dedicate this dissertation to my lovely son Festus Berigah Junior.

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NOMENCLATURE

b_f	Body force
Bi	Biot number
C	Fluid concentration
C_w	Species concentration at the surface
C_∞	Species concentration in the free stream
C_f	Skin friction
C_h	Heat transfer coefficient
C_p	Specific heat at constant pressure
D	Molecular diffusivity
$\mathrm{D_{AB}}$	Molecular diffusivity for species A diffusing into species B
Ec	Eckert number
F_s	Surface force
g	Gravitational acceleration
Gr	Grashof number
Gc	Grashof number for molecular diffusion
Gr	Grashof number for thermal diffusion
h_f	Heat transfer coefficient
k	Fluid thermal conductivity
L	Reference length
m_w	Mass flux
n	Order of chemical reaction

Nu	Nusselt number
P	Pressure
Pr	Prandtl number
q_w	Heat flux
S	Control surface
Sc	Schmidt number
Sh	Sherwood number
T	Fluid temperature
T_∞	Free stream temperature
T_w	Surface temperature
u	x- direction velocity
U	Free stream velocity
v	y - direction velocity
V	Control volume
(x,y)	Coordinates

Greek symbols

α	Fluid thermal diffusivity
η	Similarity variable
ρ	Fluid density
ρ_∞	Reference density
ρ_A	Density of species A
ρ_B	Density of species B
υ	Kinematic viscosity
ψ	Stream function
β_T	Coefficient of thermal expansion
β_C	Coefficient of solutal expansion
τ_{ii}	Normal stress
τ_{ij}	Shear stress
τ_w	Wall shear stress
θ	Dimensionless temperature
ϕ	Dimensionless concentration
f	Dimensionless stream function
Bi_x	Convective heat transfer parameter
Φ	Viscous dissipation parameter
μ	Coefficient of viscosity
γ	Rate of chemical reaction

CHAPTER ONE

INTRODUCTION AND BACKGROUND OF THE STUDIES

1.0 Introduction

Engineers all over the world are largely concerned with fashioning the material world through physical, chemical and biological changes to achieve desired predetermined results. Most of these changes require the transfer of energy, momentum or chemical species from one substance, phase or location to another. The design of the processes effecting these changes calls for efficient transfer of these quantities (Vijay, 2008). Heat and mass transfer continue to receive considerable attention because of their numerous industrial applications in many fields of engineering such as mechanical, civil, chemical and process engineering.

In thermal power plants, boilers and condensers are designed in such a way that the desired rate of heat transfer is achieved. A Bessemer converter for making steel from pig iron must be designed so that it provides sufficient opportunity for the carbon to be oxidized quickly enough for the process to be economical. An artificial kidney must have sufficient capacity to remove toxins from the blood. Similarly, knowledge of transfer processes is required in the design of smoke-stacks to keep the concentration of pollutants at ground level within acceptable limits (Vijay, 2008). Due to its diverse applications in Thermodynamics, Material Science, Diffusion Theory, Fluid Mechanics, and Radiation Theory, Heat Transfer is considered the “heart” of Thermal Science.

Heat transfer is energy in transit, and occurs as a result of temperature differences. Temperature difference is thought of as a driving force that causes heat to flow. The three basic modes of heat transfer are convection, conduction and radiation, which may occur separately, or simultaneously. The subject matter of this research is solely based on the individual mechanism of convection. The number of heat transfer applications in which convection is a dominant phenomenon is large and better understanding of this phenomenon has even increased the number of applications and has led to a number of sophisticated industrial and environmental designs.

1.1 Modes of Heat Transfer

There are three basic modes by which heat can be transferred. These include conduction, radiation, and convection.

i. Conduction is an exchange of energy by direct interaction between molecules of a substance having temperature differences. It mostly occurs in solids and has a strong basis in the molecular kinetic theory of Physics.

ii. Radiation is the transfer of thermal energy in the form of electromagnetic waves. Like electromagnetic radiation (light, X- rays, microwaves), thermal radiation travels at the speed of light, passing most easily through a vacuum or a nearly transparent gasses. Liquids containing gases, such as carbon dioxide, water vapour, and glasses transmit only a portion of incident radiation, while most of solids are essentially opaque to radiation.

iii. Convection is the transfer of heat between a solid surface and fluid due to combined mechanisms of diffusion at the surface and bulk fluid flow within boundary layer and it remains the basic mode of heat transfer among fluids.

1.1.1 Heat and Mass Transfer by Convection

Convection refers to the movement of molecules within fluid and it is one of the major modes of heat and mass transfer in fluids. A fluid flow that transports heat may also transport a chemical species. In fluids, convective heat and mass transfer takes place through diffusion: the random motion of individual particles in the fluid and advection: matter or heat transported by the larger-scale motion of currents in fluid. In the context of heat and mass transfer, the term "convection" is used to refer to the sum of advective and diffusive transfers. Mass diffusion is analogous to heat conduction and occurs whenever there is a gradient in the concentration of a species. Mass convection is essentially identical to heat convection.

Fluid flows from high pressure location to points of low pressure. A moving fluid carries energy and the faster a fluid moves, the greater the convective heat and mass transfer. In the absence of any bulk fluid motion, heat transfer between a solid surface and the adjacent fluid is by pure conduction. The presence of bulk motion of fluid enhances the heat transfer between a solid surface and the fluid.

Convection is called natural or free convection if the fluid motion is caused by buoyancy force that is induced by density difference due to the variation of temperature or concentration in the fluid. In contrast, convection is called forced convection if the fluid is forced to flow over the surface by external means such as by a fan, pump, or the wind. Basically, any forced convection heat transfer process comprises natural convection heat transfer since temperature difference results in density difference. However, due to its small contribution, compared to forced convection, natural convection is sometimes neglected. Natural convection on a surface depends on the geometry of the surface and its orientation. It also depends on the variation of temperature on the surface and the thermophysical properties of the fluid.

In convective heat transfer, the heat flux is given by:

$$q(w) = h(T_w - T_\infty) \tag{1.1}$$

where q is the heat flux, h is the heat transfer coefficient, T_w and T_∞ are the wall and far stream temperature respectively.

1.2 Newton's Law of Viscosity and Newtonian Fluids

According to Newton's law of viscosity for laminar flow, the shear stress is directly proportional to the strain rate or the velocity gradient.

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}, \tag{1.2}$$

where τ_{xy} is the shear stress, μ is the constant of proportionality representing the dynamic viscosity of the fluid and $\frac{\partial u}{\partial y}$ is the velocity gradient. The shear stress is maximum at the surface of the plate in direct contact with the fluid, due to no slip condition. Fluids obeying the Newton's law of viscosity are termed as Newtonian fluids.

1.2.1 Fourier's Law

The Fourier's law of heat conduction relates heat flow with temperature difference and conductivity of the medium. Assuming that, the temperature T varies in the x direction, it can be written mathematically as:

$$q = -k \frac{dT}{dx}, \tag{1.3}$$

where q is the heat energy through unit area in unit time, k is the material transport property, called the conductivity of the medium. It is valid for all common solids, liquids and gases. The minus sign is to ensure that heat flow is positive in the direction of decreasing temperature.

1.2.2 Fick's Law

Consider a mixture of two fluid species, A and B with respective densities ρ_A and ρ_B. Suppose that ρ_A varies with ρ_B in the x direction. Then, there will be a diffusion mass transfer of species A in the direction of its decreasing density defined through the following relation

$$J_{AX} = -D_{AB}\frac{d\rho_A}{dx} \tag{1.4}$$

where J_{AX} is the mass flux of species A in the direction x and D_{AB} is called the molecular diffusion coefficient, which varies with temperature, pressure and the mixture composition.

1.3 Dimensionless Numbers in Convective Heat and Mass Transfer

These dimensionless numbers are the measure of relative importance of different forces or the transport phenomenon involving fluid flow. In these dimensionless numbers, different properties of the flow are lumped together to represent their cumulative effect.

1.3.1 The Eckert Number (Ec)

The Eckert number (Ec) is a dimensionless quantity useful in fluid mechanic. It is the ratio of the kinetic energy to the enthalpy (or the dynamic temperature to the temperature) driving force for heat transfer.

$$Ec = \frac{u}{C_p \Delta T} \tag{1.5}$$

where u is the fluid velocity , C_p is the specific heat at constant pressure and ΔT is the driving force for heat transfer (e.g. wall temperature minus free stream temperature). The

Eckert number is a key parameter in determining the viscous dissipation of energy in a low speed flow.

1.3.2 The Grashof Number (Gr)

The Grashof number is a dimensionless quantity used in analysing the velocity distribution in free convection systems. It is defined as the ratio of the buoyancy force to the viscous force. The Grashof number is analogous to the Reynolds number in forced convection.

$$Gr = \frac{\beta \Delta T g L^3 \rho^2}{\mu^2} \tag{1.6}$$

where β is the volumetric expansion coefficient, ρ is the density evaluated at the mean temperature, g is the gravitational constant, ΔT is the temperature difference, L is the distance between regions of high temperature and low temperature and μ is the viscosity of the convecting fluid.

1.3.3 The Prandtl Number (Pr)

The Prandtl number is defined as a measure of the ratio of the viscous diffusivity to the thermal diffusivity

$$\Pr = \frac{\mu c_p}{k} \tag{1.7}$$

where Pr is the Prandtl number, C_p is the specific heat at constant pressure, μ is the coefficient of viscosity and k is the thermal conductivity.

In heat transfer problems, the Prandtl number controls the relative thickness of the momentum and thermal boundary layers. When Pr is small, it means that heat diffuses very quickly compared to the velocity (momentum). When both the thermal and viscous diffusivities are equal, the Prandtl number is unity. The momentum and thermal boundary layers are equal in that case.

1.3.4 The Schmidt Number (Sc)

The Schmidt number is defined as the ratio of the kinematic viscosity to the molecular diffusivity.

$$Sc = \frac{\upsilon}{D} \tag{1.8}$$

where D is the molecular or chemical diffusivity and υ is the kinematic viscosity or viscous diffusivity.

1.3.5 Skin Friction Coefficient (C_f)

The dimensionless shear stress at the surface is defined as the skin friction, given by

$$C_f = \frac{\tau_w}{\rho\mu^2} \tag{1.9}$$

where τ_w is the shear stress, ρ is the density and μ is the coefficient of viscosity. The overall skin friction coefficient, $\overline{C_f}$ is based on the average of the shear stress τ_w over the length L of the plate.

1.3.6 The Nusselt Number (Nu)

The Nusselt Number is the measure of the ratio of magnitude of the convective heat transfer rate to the magnitude of heat transfer rate that would exist when there was pure conduction.

$$Nu = \frac{h_f(T_w - T)}{k(T_w - T)/l} \tag{1.10}$$

The convective heat transfer from the surface will depend upon the magnitude of $h_f(T_w - T)$, where h_f is the heat transfer coefficient and T_w and T are the temperatures of wall and fluid respectively. Also, if there is no flow, the heat transfer will purely be due to conduction. The Fourier's law states that the quantity $k(T_w - T)/l$ is the measure of the heat transfer rate, where k is the thermal conductivity and l is the characteristic length.

1.3.7 The Biot Number (B)

The Biot number is defined as the ratio of temperature gradient inside the body to the overall temperature gradient in the fluid. It is similar to the nusselt number and given by the expression;

$$B = \frac{hL}{k}, \tag{1.11}$$

where L is the characteristic length, h is the heat transfer coefficient and k is the thermal conductivity. This number comes into play when a solid body is cooled. If the biot number is much less than 1, the temperature of the body is uniforn at any given instant.

1.3.8 Sherwood Number (Sh)

The Sherwood number is defined as the dimensionless mass flux at the surface

$$Sh = \frac{m_w x}{D(T_w - T_\infty)}, \tag{1.12}$$

where m_w is the mass flux at the surface and D is the diffusion coefficient, and T_w and T_∞ are the temperatures of wall and fluid respectively.

1.4 Background of the Study

Convective heat transfer is a common phenomenon in engineering systems due to its diverse applications in electronic cooling, heat exchanger designs and thermal systems. It has wide applications in mechanical, geothermal, and chemical sciences. Many industrial and technological setups such as nuclear reactors, food processing, and polymer production experience not only temperature difference but concentration. The chemical concentration variation ultimately affects the rate of heat and mass transfer. Natural convection still remains the most common method used in industrial cooling. Numerous research results on heat and mass transfer have been reported in the literature. This section shall elucidate some previous findings on the subject of heat and mass transfer.

1.4.1 Convective Heat and Mass Transfer

The problem of convective heat and mass transfer has been investigated by many researchers in recent times. For instance, Aziz, (2009) analysed the thermal boundary layer flow over a flat plate in a uniform free stream with a convective surface boundary conditions. This problem was an extension of the works of Bataller, (2008) who investigated the Blasius and Sakiadis flows, both under convective surface boundary conditions and in the presence of thermal radiation. Makinde, (2011) reported similarity solutions for natural convection from a moving vertical plate with internal heat generation and convective boundary conditions and observed that internal heat generation prevented the flow of heat from one surface to the other unless the local Grashof number was strong enough to convert the internally generated heat in the fluid.

Ishak, (2010) obtained similarity solutions for steady laminar boundary layer flow over a permeable plate with convective boundary conditions. Recently, Nor *et al*., (2012) analysed the boundary layer flow over a stretching sheet with convective boundary conditions with slip effects. Fazlina and Anuar, (2012) investigated the mixed convection boundary layer flow towards a vertical plate with convective surface boundary conditions. Olanrewaju *et al.*,(2011) investigated the effects of internal heat generation, thermal radiation and buoyancy forces on the boundary layer over a vertical plate with convective surface boundary conditions and concluded that the combined effect of increasing the Prandtl number and the Grashof number tend to reduce the thermal boundary layer thickness along the plate whilst increasing the Biot number, the internal heat generation parameter and the radiation absorption parameter enhances thermal diffusion. Sadia and Hossain, (2012) further investigated the problem of mixed convection boundary layer flow over a vertical flat plate with radiative heat transfer. Aiyesimi *et al.*, (2013) then performed a computational analysis of the effect of mass and radiative heat transfer on free convective boundary layer flow over vertical surfaces and made some interesting observations which have direct implications to industrial practice.

1.4.2 Free and Forced Convection Flow

Mohammad *et al.*,(2013) analysed the heat and mass transfer in MHD free convection flow over an inclined plate with hall current. Chamkha, (2004) investigated unsteady MHD convective heat and mass transfer past a semi-infinite vertical permeable moving plate with heat absorption. Makinde, (2010) produced Similarity solution of hydromagnetic heat and mass transfer over a vertical plate with convective surface boundary conditions. Gangadhar *et al.*, (2012) discussed similarity solution of hydromagnetic heat and mass transfer over a vertical plate with convective surface boundary condition and chemical reaction. Ibrahim and Makinde, (2010) analysed the

MHD boundary layer flow of heat and mass transfer over a moving plate with suction and in the presence of chemical reaction.

Ibrahim and Makinde,(2010a, b; 2011a, b) and Seini and Makinde, (2013) have made significant contributions to the subject of heat and mass transfer by investigating Hydromagnetic flow with Dufour and Soret effects past a vertical plate in porous medium and obtain very interesting results. Seini, (2013) also presented Flow over unsteady stretching surface with chemical reaction and non-uniform heat source. Arthur and Seini, (2014) analyzed the MHD thermal stagnation point flow towards a stretching porous surface. AbdelKhalek,(2009) examined MHD free convection with mass transfer from a moving permeable vertical surface and produced interesting results using the perturbation techniques. Makinde, (2010) studied similarity solution of MHD heat and mass transfer over a moving vertical plate and convective surface boundary conditions. He concluded that an increase in magnetic field intensity causes a decrease in the momentum boundary layer thickness whiles both thermal and concentration boundary layer thicknesses increase. The local Nusselt number and the local Sherwood number increases while the magnitude of the local Skin friction coefficient decreases with an increase in the convective heat exchange at both sides of the plate.

1.4.3 Viscous Dissipation in Transport Processes

Deformation and flow of materials require energy. This mechanical energy is dissipated during the flow and converted to internal energy (heat) of the material. The increase in internal energy expresses itself in temperature rise. In a viscous fluid flow, the viscosity of the fluid takes energy from the motion of the fluid and transforms it to internal energy of the fluid thereby changing the temperature distribution by playing a role like an energy source. This process is partially irreversible and is referred to as dissipation, or viscous dissipation. Viscous dissipation plays an important role in natural convection in various devices which are subjected to large deceleration in geological processes, polymer processing and in oil products transportation through ducts. Jena *et al*., (2014) recently studied numerical solution of boundary layer MHD flow with viscous dissipation and concluded that the dimensionless temperature profile as well as the thermal boundary layer thickness quickly reduces whilst the rate of heat transfer increases as the Prandtl number increased. Jai, (2012) examined the viscous dissipation and chemical reaction effects on flow past a stretching porous surface in a porous medium.

Singh, (2012) analysed the effects of variable fluid properties and viscous dissipation on mixed convection flow past a vertical plate in porous medium. Kazi *et al*., (2013) studied the problem of viscous dissipation on MHD natural convection flow along a vertical wavy surface. Pantokratoras, (2004) investigated the effect of viscous dissipation in natural convection along a heated vertical plate and observed that viscous dissipation assisted the upward flow and opposes the downward flow. Abdullah *et al*., (2007) later examined the combined effect of conduction and viscous dissipation on magnetohydrodynamic free convection flow along a vertical flat plate and concluded that the velocity of the fluid within the boundary layer decreases with increasing magnetic parameter, Prandtl number and conjugate conduction parameter while it increases slightly for the increasing viscous dissipation parameter.

Gangadhar, (2012) presented a similarity solution for natural convection over a moving vertical plate with internal heat generation and viscous dissipation. He found among others that the internal heat generation prevented the flow of heat from the left surface to the right surface of the plate unless the local Grashof number was strong enough to convert the internally generated heat in the fluid as observed by Makinde, (2011). Ibrahim and Bhashar, (2013) produced similarity solution of heat and mass transfer for natural convection over a moving vertical plate with internal heat generation and convective boundary conditions in the presence of thermal radiation, viscous dissipation and chemical reaction and their results agreed with the results obtained by Gangadhar, (2012).

Prasanna *et al*., (2012) solved numerically MHD boundary layer flow of heat and mass transfer over a moving vertical plate in a porous medium with suction and viscous dissipation using the fourth order Runge-Kutta method along with shooting techniques and observed that the momentum boundary layer thickness decreased, while both thermal and concentration boundary layer thicknesses increased with increase in the magnetic field intensity. Kishan *et al*., (2010) also studied the effects of viscous dissipation on MHD flow with heat and mass transfer over a stretching surface with heat source, thermal stratification and chemical reaction.

1.4.4 Heat and Mass Transfer with Chemical Reaction

A chemically reacting flow is a fluid flow in which a chemical reaction is also occurring. Such flows occur in a wide range of fields including combustion, chemical engineering, biological processes and pollution abatement. In many industrial processes involving flow and mass transfer over a flat surface such as, manufacturing of ceramics, polymer production, drying, evaporation at the surface of a water body and electric power industry, the diffusing species can be generated or absorbed due to some kind of chemical reaction with the ambient fluid, which can greatly affect the flow and hence the properties and quality of the final product (Makinde, 2011). The effect of a chemical reaction depends on its order and whether the reaction is heterogeneous or homogeneous. A reaction is said to be of nth order, if the rate of reaction is proportional to the nth power of the concentration itself. A homogeneous reaction is one in which all the reactants are in the same phase and occurs uniformly throughout the given phase, whereas a heterogeneous reaction takes place in a restricted region or within the boundary of a phase.

Das *et al*., (1994) studied the effect of homogeneous first-order chemical reaction on the flow past an impulsively started infinite vertical plate with uniform heat flux and mass transfer. Muthucumaraswamy and Ganesan, (2001) investigated the Effect of chemical reaction and injection on flow characteristics in an unsteady upward motion of an isothermal plate. Prasad *et al*., (2003) studied the influence of reaction rate on the transfer of chemically reactive species in a laminar, non-Newtonian fluid immersed in porous medium over a stretching sheet. They concluded that the effect of chemical reaction was more effective for zero and first-order reactions than second and third order reactions.

The MHD flow of uniformly stretching vertical permeable surface in the presence of heat generation/absorption and chemical reaction was reported by Chamkha, (2003). Chamkha *et al*., (2012) studied mass transfer with chemical reaction in MHD mixed convective flow along a vertical stretching sheet. Mahdy, (2010) also investigated the effect of chemical reaction and heat generation or absorption on double-diffusive convection from a vertical truncated cone in porous media with variable viscosity. Rushi *et al.*, (2013) studied the chemically reacting dusty viscoelastic fluid flow in an irregular channel with convective boundary condition while Prakash *et al*., (2011) examined the influence of chemical reaction on unsteady MHD mixed convective flow over a moving vertical porous plate. Jai *et al*., (2009) investigated the chemical reaction effects on the heat and mass transfer flow over a vertical stretching surface in a porous medium with constant suction and variable permeability.

Cortel, (2007) examined the flow and mass diffusion of chemical species with first and higher order reactions of two electrically conducting viscoelastic fluids over porous stretching sheets with magnetic fields. He observed that increasing the order of the chemical reaction decreases the concentration boundary layer when the reaction rate was negative and the opposite trend was true for the case of a positive reaction rate. Makinde, (2011) examined the MHD mixed-convection interaction with thermal radiation and n^{th} order chemical reaction past a vertical porous plate embedded in a porous medium and observed among others that the chemical species concentration increases with the reaction order and decreases with the Schmidt number and chemical reaction parameter.

Ibrahim and Makinde, (2010) analyzed chemically reacting MHD boundary layer flow of heat and mass transfer over a moving vertical plate with suction. Gangadhar *et al*., (2012) discussed similarity solutions of hydromagnetic heat and mass transfer over a vertical plate with convective surface boundary conditions and chemical reaction and concluded that the local skin-friction coefficient, the local heat and mass transfer rates at the plate surface increased with increasing intensity of magnetic field, buoyancy force, convective heat exchange parameter and the chemical reaction rate parameter.

1.5 Statement of the Problem

Heat and mass transfer is a common phenomenon in many engineering systems. It forms an integral part of every human activity and occurs in respiration, blood circulation, metabolic processes, etc. The application of heat and mass transfer processes extend beyond human activities to include modern day applications such as in the cooling of nuclear reactors and electronic equipment and heating or sterilization in food industry. The effects of excessive heating can be destructive and can cause irreversible damage to human cells and to electronic equipment if proper control measures are not observed. Manufacturing industries dealing with the production of mobile phones, laptops etc are faced with the challenge of cooling their products efficiently whilst maintaining the desired properties. The combined effect of heat and mass transfer with convective boundary conditions is of great relevance in achieving the desired product characteristics. This study therefore investigates the problem of heat and mass transfer with viscous dissipation and n^{th} order chemical reaction.

1.6 Objectives of the Study

The general objective of this research is to analyse the heat and mass transfer over a vertical surface with convective boundary conditions. In particular, the specific objectives of this work include to:

i. model the governing equations for heat and mass transfer over a vertical surface with convective boundary conditions.

ii. transform the partial differential equations modelling the problem of heat and mass transfer to ordinary differential equations using similarity analysis.

iii. analyse the effect of chemical reaction on heat and mass transfer parameters as the order increases.

iv. examine the effect of viscous dissipation parameter on heat and mass transfer.

1.7 Significance of the Study

This study would be of interest to many professionals including:

i. engineers in the design of effective and efficient heat exchanger components.

ii. biomedical engineers who produce lasers for medical applications in which the cooling rate is properly controlled to avoid irreversible damages to cells.

iii. manufacturing industries to cool their finished products efficiently.

iv. Add up to existing literature and serve as a reference material for future researchers.

1.8 Organisation of the Thesis

This thesis is organized into five chapters. The chronological development of the topic is outlined briefly in Chapter one with appropriate references from literature. Chapter two presents the derivation of differential equations governing fluid motion. In Chapter three, heat and mass transfer over a vertical surface with convective boundary conditions is discussed in the presence of viscous dissipation and n^{th} order chemical reaction. Appropriate equations for the dimensionless velocity, temperature and concentration are also derived. Chapter four presents the results obtained and subsequently discussed. Chapter five also presents the conclusion and recommendations made.

CHAPTER TWO

DERIVATION OF THE MATHEMATICAL MODELS AND NUMERICAL METHODS

2.0 Introduction

This chapter presents the derivation of the equations necessary to study all fluid flow problems. The subject of computational fluid dynamics is best described in the form of partial differential equations as the characteristics of a moving fluid will depend on multiple flow quantities. The basic equations of continuity, momentum, energy and concentration shall be derived based on some fundamental laws of physics:

2.1 The Continuity Equation

The principle of mass conservation stipulates that the mass of a system remains constant. The mass contained within a control volume (V) though may not be constant, since the fluid moves across the bounding surface known as the control surface (S) as illustrated in Figure 2.1. To simplify the problem, a *steady, two-dimensional flow* in the x and y directions of a Cartesian coordinate system is assumed. A unit depth is therefore assigned to the z direction, thereby providing a differential control volume of extent $(\partial x \partial y.1)$

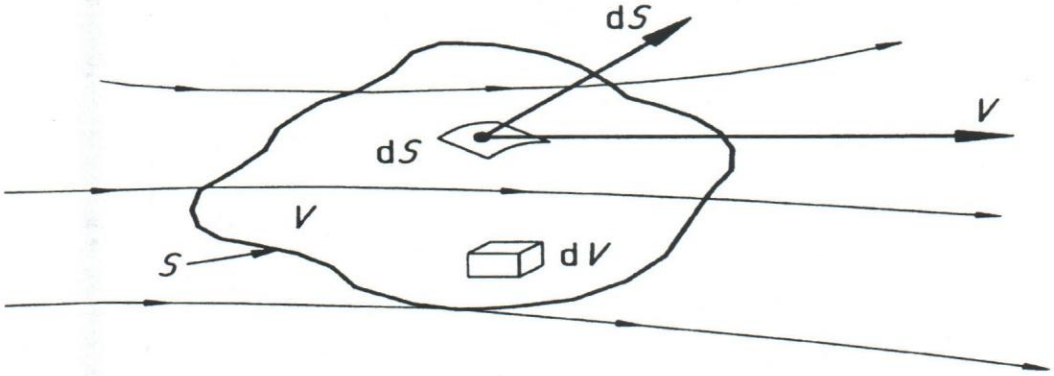

Figure 2. 1 Finite control volume fixed in space

Consider an infinitesimal control volume in two dimensions at $P(x, y)$ within a chemically homogeneous fluid as shown in Figure 2.2. The side parallel to the x and y-axes are respectively δx and δy in length. If the density is denoted by ρ and the velocity by $\underline{V}$, the mass contained in the control volume is $\rho(\delta x.\delta y.1)$. The law of conservation of mass for the control volume states that the rate of change of mass contained in the control volume plus the net efflux of mass (diffusive as well as convective) across the control surface should be zero provided that there are no sources of mass present in the control volume. The rate of change of mass contained in the control volume is $(\partial\rho/\partial t).(\delta x.\delta y.1)$.

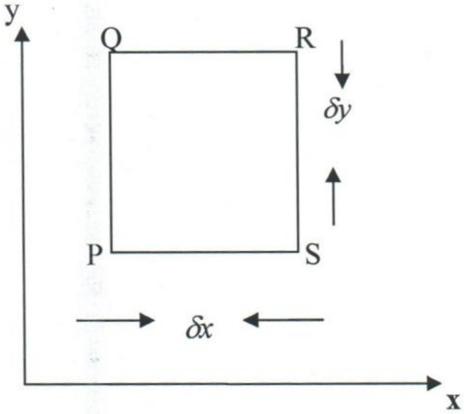

Figure 2. 2 An infinitesimal control-volume

Mass entering the face PQ is given by $\rho u(\delta y.1)$ where u is the x component of the velocity. Mass leaving the face RS is by Taylor expansion, $\rho u(\delta y.1)+[\partial(\rho u)/\partial x].\delta x(\delta y.1)$. In evaluating the net efflux, the first quantity is considered as negative and the second as positive, so that the net efflux in the x- direction is $[\partial(\rho u)/\partial x].(\delta x\delta y.)$. Similarly, the net efflux in the y - direction is $[\partial(\rho v)/\partial y].(\delta x\delta y)$. This is the convective component of efflux. The diffusive component is zero because of the uniform chemical composition.

Thus the net efflux from the control volume is

$$\left(\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}\right).\partial x\partial y \tag{2.1}$$

The mass conservation equation then becomes

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \tag{2.2}$$

Equation (2.2) is the general statement for the continuity of flow and is valid for all flow categories whether steady or non-steady, compressible or incompressible. The last two terms can be expanded into;

$$u(\partial\rho/\partial x)+v(\partial\rho/\partial y)+\rho(\partial u/\partial x+\partial v/\partial y) \tag{2.3}$$

The first two terms of equation (2.3) combine with $\partial\rho/\partial t$ to give $D\rho/Dt$ and equation (2.2) becomes;

$$D\rho/Dt+\rho(\partial u/\partial x+\partial v/\partial y)=0 \tag{2.4}$$

For three dimensions, this can be generalized to

$$D\rho/Dt+\rho\nabla.\underline{V}=0 \tag{2.5}$$

For incompressible fluid, $D\rho/Dt=0$, and the continuity equation becomes

$$\nabla.\underline{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{2.6}$$

For two dimensional steady flows, the continuity equation becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.7}$$

2.2 The Momentum Equation

The second fundamental law that is pertinent to the flow of a viscous fluid is *Newton's second law of motion*. For a differential control volume in the fluid, this requirement states that the sum of all forces acting on the control volume must equal the net rate at which momentum leaves the control volume, Figure 2.3.

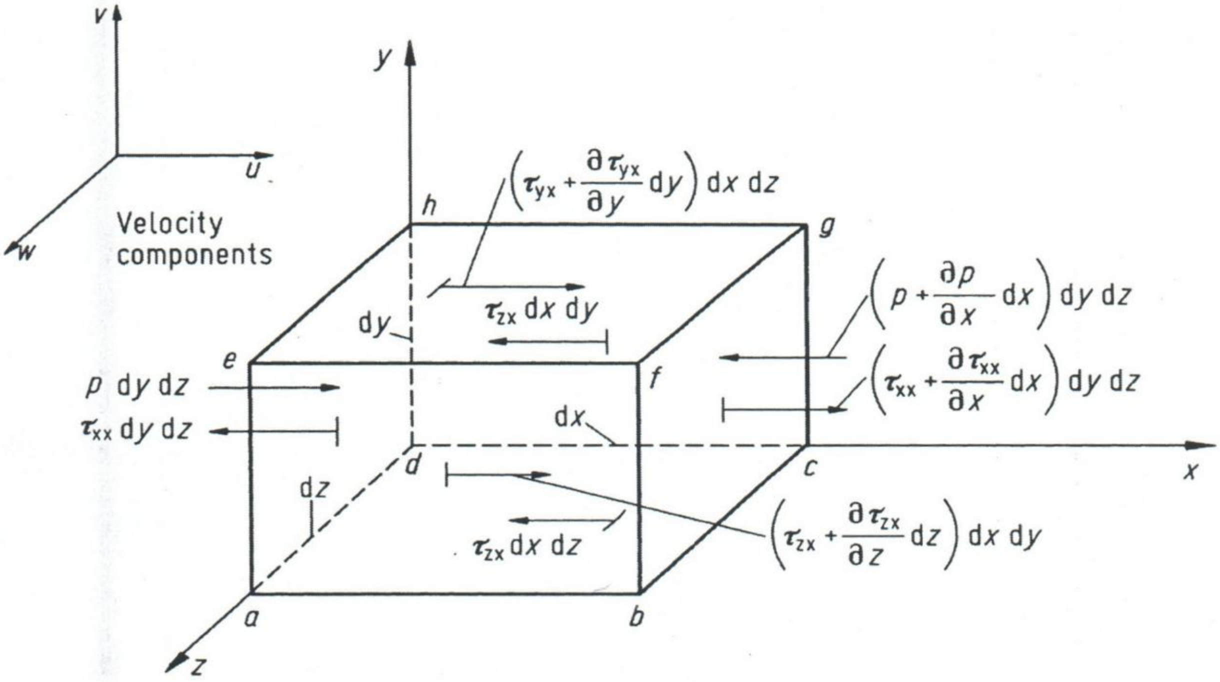

Figure 2. 3 Infinitesimal Fluid Element with *x* direction forces

Consider only the *x*-components of forces as shown in Figure 2.3. The Newton's 2nd law requires two kinds of forces to act on the fluid: *body forces*, which are proportional to the volume, and *surface forces*, which are proportional to area. Gravitational, centrifugal, magnetic, and/or electric fields may contribute to the total body force. The *x* components of these forces per unit volume of fluid are designated as *X*. The surface forces, *Fs,* are due to the fluid static pressure as well as to *viscous stresses*. At any point in the fluid, the viscous stress (a force per unit area) may be resolved into two perpendicular components, which include a *normal stress* τ_{ii} and a *shear stress* τ_{ij}.

The surface forces in the *x*-direction exerted on the fluid element are sketched in Figure 2.3. On face *abcd*, the only force in the *x*-direction is that due to shear stress, $\tau_{xy}\partial x$. Face *efgh* is a distance ∂y above face *abcd*; hence the shear force in the *x*-direction on face *efgh* is $\lfloor \tau_{yx} + (\partial \tau_{yx}/\partial y)\partial y \rfloor \partial x$. On face *adhe*, the only forces in the *x*-direction are the pressure force $p\partial y$, which always acts in the direction into the fluid element, and $\tau_{xx}\partial y$, which is in the negative *x*-direction. In contrast, on face *bcgf*, the pressure force $[p + (\partial p/\partial x)\partial x]\partial y$ presses inward on the fluid element (in the negative *x*-direction) with a shear force equal to $[\tau_{xx} + (\partial \tau_{xx}/\partial x)\partial x]\partial y$. The net surface force in the *x*-direction is;

$$\left[p - \left(p + \frac{\partial p}{\partial x}\partial x\right)\right]\partial y + \left[\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\partial x\right) - \tau_{xx}\right]\partial y + \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\partial y\right) - \tau_{yx}\right]\partial x \tag{2.8}$$

Simplifying, the *net* surface force for the x- direction becomes:

$$F_{s,x} = \left(\frac{\partial \tau_{xx}}{\partial x} - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right)\partial x \partial y \tag{2.9}$$

The net force for the x-direction become

$$F_x = \left(\frac{\partial \tau_{xx}}{\partial x} - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right)\partial x \partial y + X \tag{2.10}$$

To use Newton's second law, the fluid momentum fluxes for the control volume must also be evaluated. For example, the mass flux through the *x* surface (in the *y-z* plane) is (ρu), the corresponding *x*-momentum flux is $(\rho u)u$. Similarly, the *x*-momentum flux due to mass flow through the y surface (in the x-z plane) is $(\rho v)u$. These fluxes may change in each of the coordinate directions, and the *net* rate at which x momentum leaves the control volume is:

$$\frac{\partial[(\rho u)u]}{\partial x}\partial x(\partial y)+\frac{\partial[(\rho v)u]}{\partial y}\partial y(\partial x) \tag{2.11}$$

Equating the rate of change in the x momentum of the fluid to the sum of the forces in the x direction, gives:

$$\frac{\partial[(\rho u)u]}{\partial x}+\frac{\partial[(\rho v)u]}{\partial y}=\frac{\partial\tau_{xx}}{\partial x}-\frac{\partial p}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+X \tag{2.12}$$

This expression may be put in a more convenient form by expanding the derivatives on the left-hand side:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}(\tau_{xx}-p)+\frac{\partial\tau_{yx}}{\partial y}+X \tag{2.13}$$

The two terms on the left-hand side of equation (2.13) represent the *net* rate of momentum flow from the control volume. The terms on the right-hand side account for the net viscous, pressure and body forces. These equations must be satisfied at each point in the fluid. The stresses are associated with the deformation of the fluid and are a function of the fluid viscosity and velocity gradients. Moreover, the magnitude of a stress is proportional to the *rate* at which the deformation occurs. The deformation rate is, in turn, related to the fluid viscosity and to the velocity gradients in the flow. For a *Newtonian fluid,* the stresses are proportional to the velocity gradients, where the proportionality constant is the fluid viscosity. For two dimensional flows, Stokes viscosity law gives the following results:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \tau_{xy} = \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \tau_{yy} = 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \qquad (2.14)$$

Substituting equations (2.13) and (2.14) into equations (2.12), the *x*-momentum equation becomes;

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left\{\mu\left[2\frac{\partial u}{\partial x} - \frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]\right\} + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + X \qquad (2.15)$$

Rearranging the right-hand side of each expression and substituting from the continuity equation, the *x*- momentum equation becomes

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) + X \qquad (2.16a)$$

In a similar fashion the y- momentum and z- momentum components are obtained respectively as;

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) + Y \qquad (2.16b)$$

$$\rho\left(u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right) + Z \qquad (2.16c)$$

2.3 The Energy Equation

Consider the element of control volume shown in Figure 2.4. The energy associated with mass is in the form of kinetic energy and internal energy. If e represents the internal energy per unit mass, then the total energy contained within the control volume is $\rho\left(e+\frac{1}{2}\underline{V}^2\right)(\partial x\partial y.1)$ and its rate of change with time is $\frac{\partial}{\partial t}\left(\rho e+\frac{\rho}{2}\underline{V}^2\right)\partial x\partial y$ with $\underline{V}^2 = u^2 + v^2$

Energy efflux consists of two parts, one convective and the other diffusive. Consider first the convective efflux. The rate at which mass crosses the surface PS is $\rho u(\partial y.1)$. Therefore, the rate of energy converted across PS is $\left(e+\frac{1}{2}\underline{V}^2\right)\rho u\partial y$. By Taylor expansion, the energy converted out across RQ is;

$$\left\{\left(e+\frac{1}{2}\underline{V}^2\right)\rho u+\frac{\partial}{\partial x}\left[\left(e+\frac{1}{2}\underline{V}^2\right)\rho u\right]\partial x\right\}(\partial y.1) \tag{2.17}$$

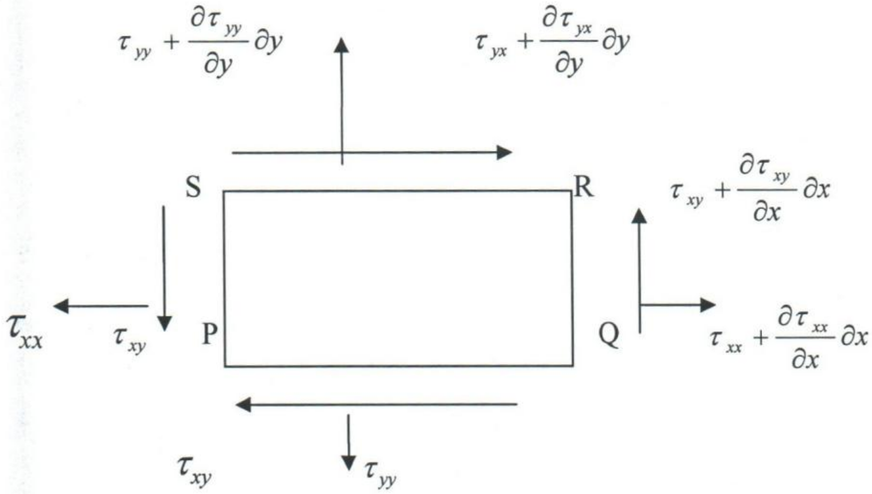

Figure 2. 4 Stress on an infinitesimal volume

Therefore, the net convective efflux of energy across this pair of faces is

$$\frac{\partial}{\partial x}\left[\left(e+\frac{1}{2}\underline{V}^2\right)\rho u\right]\partial x\partial y \tag{2.18}$$

Similarly the efflux across the other pair of faces PQ and RS can be obtained.

The total convective efflux, therefore, is

$$\frac{\partial}{\partial x}\left[\left(e+\frac{1}{2}\underline{V}^2\right)\rho u\right]\partial x\partial y+\frac{\partial}{\partial y}\left[\left(e+\frac{1}{2}\underline{V}^2\right)\rho v\right]\partial x\partial y \tag{2.19}$$

Consider next the diffusive flux of energy. This is determined from Fourier's law of heat conduction. Conduction flux across PS into the control volume is given by $-k\dfrac{\partial T}{\partial x}.(\partial y.1)$. Again, by Taylor's expansion, the flux across RQ out of the control volume is given by

$$-\left[k\frac{\partial T}{\partial x}+\frac{\partial}{\partial x}\left\{k\frac{\partial T}{\partial x}\right\}\partial x\right]\partial y.1 \tag{2.20}$$

For a constant k, the net efflux across this pair of faces is, $-k\dfrac{\partial^2 T}{\partial x^2}\partial x\partial y$ and across the whole control volume is

$$\left[\frac{\partial\left(e+\frac{1}{2}V^2\right)\rho u}{\partial x}+\frac{\partial\left(e+\frac{1}{2}V^2\right)\rho v}{\partial y}-k\left\{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right\}\right]\partial x\partial y \tag{2.21}$$

which in two-dimensional vector notation becomes

$$\left\{\nabla.\left[\rho V\left(e+1/2V^2\right)\right]-k\nabla^2 T\right\}\partial x\partial y \tag{2.22}$$

The rate at which the body force bf does work is given by

$$\rho\left[f_x u+f_x v\right]\partial x\partial y=\rho f.V\partial x\partial y \tag{2.23}$$

The rate at which the surface forces do work can be calculated. Work done by forces on the surface PS of the control volume is $-(\tau_{xx}.\partial y).u-(\tau_{xy}.\partial y).v$ and the work done by the forces on the surface RQ is $\left[\tau_{xx}.u+\dfrac{\partial\tau_{xx}u}{\partial x}\partial x+\tau_{xy}v+\dfrac{\partial\tau_{xy}v}{\partial x}\partial x\right]\partial y$ (2.24)

Therefore, the net work done by the forces acting on these two surfaces is

$$\left[\frac{\partial\tau_{xx}}{\partial x}u+\frac{\partial\tau_{xy}}{\partial x}v\right]\partial x\partial y \tag{2.25}$$

The work done by the forces on the other two surfaces is similarly determined and adding the results of the current to the previous, the total work done on the control volume is

$$\left[\frac{\partial\tau_{yy}}{\partial y}v+\frac{\partial\tau_{xy}}{\partial x}v+\frac{\partial\tau_{xx}}{\partial x}u+\frac{\partial\tau_{yx}}{\partial y}u\right]\partial x\partial y \tag{2.26}$$

The total energy becomes:

$$\begin{aligned}&\partial\rho\frac{\left(e+\frac{1}{2}V^2\right)}{\partial t}+\frac{\partial\left(e+\frac{1}{2}V^2\right)\rho u}{\partial x}+\frac{\partial\left(e+\frac{1}{2}V^2\right)\rho v}{\partial y}-k\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}\right)\\&=\rho\left[f_xu+f_yv+\frac{\partial\left(\tau_{xx}u+\tau_{xy}v\right)}{\partial x}+\frac{\partial\left(\tau_{yy}v+\tau_{yx}u\right)}{\partial y}\right]\end{aligned} \tag{2.27}$$

The thermal energy equation is obtained by subtracting the mechanical energy equation obtained by multiplying the momentum equation by the velocity, thus multiplying the x momentum equation by u, and the y component by v.

$$\begin{aligned}&\rho u\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\rho v\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\rho\left[f_xu+f_yv\right]+\\&u\left[\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}\right]+v\left[\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}\right]\end{aligned} \tag{2.28}$$

Subtracting equation (2.28) from the total energy equation (2.27), the thermal energy equation becomes:

$$\rho\left[\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}\right]-k\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}\right)=\tau_{xx}\frac{\partial u}{\partial x}+\tau_{xy}\frac{\partial v}{\partial x}+\tau_{yx}\frac{\partial u}{\partial y}+\tau_{yy}\frac{\partial v}{\partial y} \tag{2.29}$$

Substituting Stokes Viscosity law equation 2.14 into equation 2.29 which yields;

$$\rho\left[\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}\right]=k\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}\right)-p\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]+\mu\Phi \tag{2.30}$$

where Φ is the viscous dissipative term.

Substituting $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

$$\rho\left[\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}\right]=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\Phi \tag{2.31}$$

where $\partial e = C_v \partial T$ and $C_v = C_p$ for an incompressible fluid.

The generalised thermal energy equation may then be expressed as

$$\rho C_p\left[\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\Phi \tag{2.32}$$

Where $$\Phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]^2-\frac{2}{3}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]^2 \tag{2.33}$$

Collectively, the terms on the right-hand side of equation (2.33) account for the rate at which mechanical work is irreversibly converted to thermal energy due to viscous effects in the fluid.

2.4 The Concentration Equation

If the viscous fluid consists of a binary mixture in which there are species concentration gradients (Figure 2.5), there will be relative transport of the species, and species conservation must be satisfied at each point in the fluid. The pertinent form of the conservation equation may be obtained by identifying the processes that affect the *transport* and *generation* of the species for a differential control volume in the fluid.

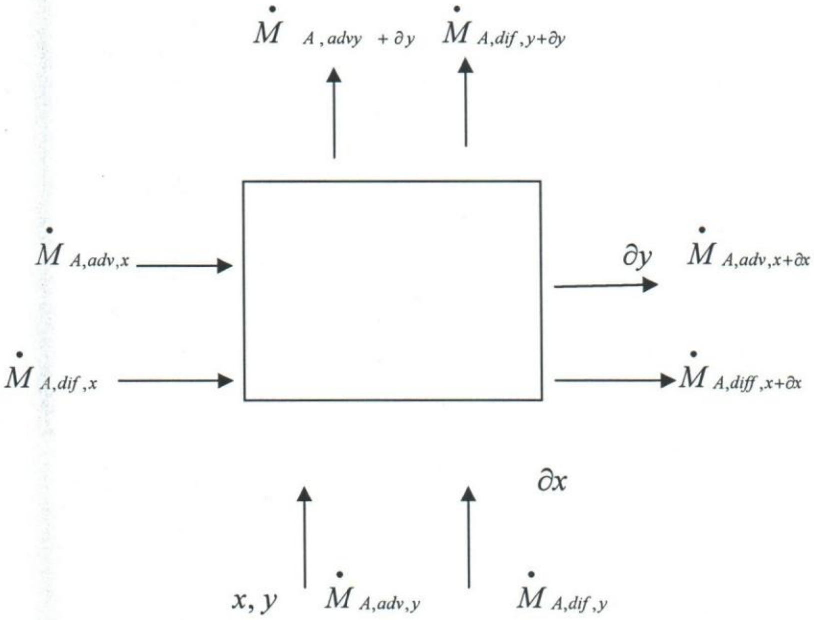

Figure 2. 5 Species conservation in two-dimensional flow of a viscous fluid with mass transfer

Figure 2.5 describes the flow of species by advection and diffusion in a two dimensional flow of a viscous fluid. Consider the control volume of Figure 2.5. Species A may be transported by *advection* (with the mean velocity of the mixture) and by *diffusion* (relative to the mean motion) in each of the coordinate directions. The concentration may also be affected by chemical reactions, and the rate at which the mass of species A is generated per unit volume due to such reactions is designated as $\dot{n}_A$. The *net* rate at which species A *enters* the control volume due to *advection* in the *x*-direction is;

$$\dot{M}_{A,adv,x} - \dot{M}_{A,advx+\partial x} = (\rho_A u) - \left[(\rho_A u) + \frac{\partial(\rho_A u)}{\partial x}\partial x\right]\partial y = \frac{\partial(\rho_A u)}{\partial x}\partial x\partial y \qquad (2.34)$$

Similarly, multiplying both sides of Fick's law (1.4) by the molecular weight MA (kg/kmol) of species A to evaluate the diffusion flux, the *net* rate at which species A *enters* the control volume due to *diffusion* in the *x*-direction is determined as:

$$\dot{M}_{A,dif,x} - \dot{M}_{A,dif,x+\partial x} = \left(-D_{AB}\frac{\partial \rho_A}{\partial x}\right)\partial y - \left[\left(-D_{AB}\frac{\partial \rho_A}{\partial x}\right) + \frac{\partial}{\partial x}\left(-D_{AB}\frac{\partial \rho_A}{\partial x}\partial x\right)\right]\partial y$$

$$= \frac{\partial}{\partial x}\left(D_{AB}\frac{\partial \rho_A}{\partial x}\right)\partial x \partial y \qquad (2.35)$$

Expressions similar to equations 2.34 and 2.35 may be formulated for the *y*-direction. Referring to Figure 2.5, the species conservation requirement is

$$\dot{M}_{A,adv,x} - \dot{M}_{A,advx+\partial x} + \dot{M}_{A,adv,y} - \dot{M}_{A,advy+\partial y} + \dot{M}_{A,dif,x} - \dot{M}_{A,dif,x+\partial x} + \dot{M}_{A,dif,y} - \dot{M}_{A,dif,x+\partial y} - \dot{M}_{A,d} = 0$$

Substituting from equations (2.34) and (2.35), as well as from similar forms for the *y*-direction, it follows that,

$$\frac{\partial(\rho_A u)}{\partial x} + \frac{\partial(\rho_A v)}{\partial y} = \frac{\partial}{\partial x}\left(D_{AB}\frac{\partial \rho_A}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{AB}\frac{\partial \rho_A}{\partial y}\right) + \dot{n}_A \qquad (2.36)$$

A more useful form of this equation may be obtained by expanding the terms on the left-hand side and substituting from the overall continuity equation for an incompressible fluid. Equation 2.36 then reduces to

$$u\frac{\partial \rho_A}{\partial x} + v\frac{\partial \rho_A}{\partial y} = \frac{\partial}{\partial x}\left(D_{AB}\frac{\partial \rho_A}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{AB}\frac{\partial \rho_A}{\partial y}\right) + \dot{n}_A \qquad (2.37)$$

or in Molar form, to

$$u\frac{\partial C_A}{\partial x} + v\frac{\partial C_A}{\partial y} = \frac{\partial}{\partial x}\left(D_{AB}\frac{\partial C_A}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{AB}\frac{\partial C_A}{\partial y}\right) + \dot{N}_A \qquad (2.38)$$

2.5 Newton-Raphson Method

The Newton-Raphson method or simply the Newton method is a well-known and most powerful method used for finding the root of the equation $f(x) = 0$ (Singerisu, 2002). The Newton method can be derived by considering the Taylor's series expansion of the function $f(x)$ about an arbitrary point x_1 as:

$$f(x) = f(x_1) + (x - x_1)f'(x_1) + \tfrac{1}{2}(x - x_1)^2 f''(x_1) + \ldots \tag{2.39}$$

Where the function, f and it's derivatives, $f', f'', \ldots$ on the right-hand side of equation (3.83) are evaluated at x_1. By considering only the first two terms in the expansion, we have;

$$f(x) \approx f(x_1) + (x - x_1)f'(x_1) \tag{2.40}$$

In order to find the root of $f(x) = 0$, we set $f(x)$ equal to zero in (3.84) to obtain

$$f(x_1) + (x - x_1)f'(x_1) = 0 \tag{2.41}$$

Since the higher order derivative terms were neglected in the approximation of $f(x)$ in (2.39), the solution of equation (2.41) yields a next approximation to the root (instead of the exact root) as:

$$x = x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{2.42}$$

Where x_2 denotes an improved approximation to the root. This iterative procedure can be generated as $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$; i = 1, 2, ...

The problem now becomes fairly straightforward. We begin with an initial guess of x_i. Then solve for $f(x_i)$ and $f'(x_i)$. From these values x_{i+1} is calculated. The process is repeated until convergence is obtained in x.

An advantage of using the Newton's method is that it converges very fast in most cases and also requires one initial guess (x_1). However, in some problems, such as those involving transcendental functions, the differentiation of the function $f(x)$ may not be available in explicit form. Transcendental functions are not involved in this research work hence the Newton method is an efficient method for this present study.

2.6 Runge-Kutta Method

The Runge-Kutta methods are iterative ways to calculate the solution of a differential equation. Starting from an initial condition, the solution is calculated forward step by step. The most common method is the fourth-order Runge-Kutta method, often simply referred to as the Runge-Kutta method. The Runge-Kutta method is a good choice for solving differential equations because it is quite accurate, stable and easy to program.

There are several versions of the fourth-order Runge-Kutta Method (RK4). The method is based on computing y_{k+1} as follows:

$$y_{k+1} = y_k + \frac{h}{6}(f_1 + 2f_2 + 2f_3 + f_4) \tag{2.43}$$

Where

$$\begin{aligned} f_1 &= f(t_k, y_k), \\ f_2 &= f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2}f_1\right), \\ f_3 &= f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2}f_2\right), \\ f_4 &= f(t_k + h, y_k + hf_3), \end{aligned} \tag{2.44}$$

Thus, we begin the algorithm by first calculating f_1 with t_k, and y_k as the initial inputs. Thereafter, the step size is increased by $h/2$ and f_2, f_3 and f_4 are subsequently calculated. With these values, the new value for the variable y can be obtained.

The advantage of the RK4 method is obvious; no formulas for the higher derivatives need to be computed nor do they have to be in the program.

Equations (2.7), (2.26), (2.33) and (2.38) are the generalised continuity, momentum, energy and concentration equations respectively. These equations shall be revisited in the next chapter and will be applied to solving a practical problem of relevance by incorporating into these equations the assumptions for the development of the model.

CHAPTER THREE

HEAT AND MASS TRANSFER OVER A VERTICAL SURFACE WITH CONVECTIVE BOUNDARY CONDITIONS IN THE PRESENCE OF VISCOUS DISSIPATION AND nTH ORDER CHEMICAL REACTION

3.0 Introduction

In this chapter, the incompressible flow of a viscous fluid over a vertical surface is considered. The effect of viscous dissipation and nth order chemical reaction is imbedded in the heat and mass transfer problem. Boundary layer approximations are employed to represent the problem in the form of continuity, momentum, energy and concentration equations. Since thermal buoyancy force and buoyancy force due to concentration difference is considered, the flow affect the velocity, temperature and concentration distributions. The quantities of physical interest like the skin-friction coefficient, and the rate of heat and mass transfers are computed for different values of the physical parameters. The Newton -Raphson shooting method alongside the fourth order Runge-Kutta method is used to solve the resulting coupled differential equation.

3.1 Developing the Mathematical Model

Consider a steady, laminar, incompressible, convection flow with heat and mass transfer over a vertical plate in a stream of cold fluid at temperature T_∞. The left surface of the plate is assumed to be heated by convection from a hot fluid at temperature T_f, which provides a heat transfer coefficient, h_f. The cold fluid at the right side of the plate is assumed to be Newtonian, and its property variations due to temperature and chemical species concentration are limited to fluid density. It is also assumed that it is a viscous dissipative and chemically homogeneous fluid. The concentration of diffusing species is very small in comparison to other chemical species. The concentration of species far from the surface, C_∞ is infinitesimally very small and hence the Soret and Dufour effects are neglected. An nth order homogeneous chemical reaction occurs in the flow and all physical properties are assumed to be constant. The density variation and the effects of buoyancy are taken into account in the momentum equation. In addition, there is no applied electric field.

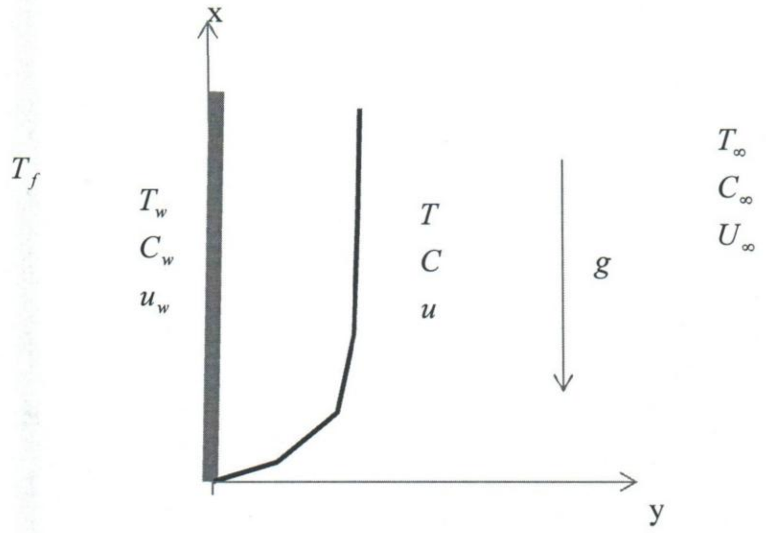

Figure 3. 1 Flow Configurations and Coordinate System

Let the x-axis be taken along the direction of the plate (which is oriented vertically) and y-axis normal to it. If u, v, T and C are the fluid x-component velocity, y-component velocity, temperature and concentration respectively, then under the Boussinesq and boundary-layer approximations, and based on the above assumptions the continuity, momentum, energy and mass transfer equations for the problem under consideration can be derived.

3.1.1 The Continuity Equation

For incompressible and steady fluid flow, the continuity equation for a 2-dimensional flow situation is obtained as denoted by equation (2.7).

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

3.1.2 The Momentum Equation

The generalised X- momentum equation is as derived in equation (2.16a) and restated as;

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+X,$$

It is assumed that the only body force acting on the fluid is the gravitational force, hence the body force bf per unit volume, X is ρg and the net force on the x-direction becomes

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\rho g \qquad (3.1)$$

The density of a mixture is a function of its temperature and mass fractions of its species. It can be expanded using a Taylor's series near the vicinity of a reference point (T_∞, C_∞) of a single chemically reacting element given by:

$$\rho=\rho_\infty+\frac{\partial\rho}{\partial T}(T-T_\infty)+\frac{\partial\rho}{\partial C}(C-C_\infty), \qquad (3.2)$$

where ρ_∞ is the density at the reference point. By definition, the coefficient of thermal expansion, β_T, and composition coefficient of volume expansion, β_C are respectively:

$$\beta_T=-\frac{1}{\rho_\infty}\left(\frac{\partial\rho}{\partial T}\right)_p \quad \text{and} \quad \beta_C=-\frac{1}{\rho_\infty}\left(\frac{\partial\rho}{\partial C}\right)_p \qquad (3.3)$$

and neglecting the higher order terms in the Taylor's expansion, one obtains:

$$\rho = \rho_\infty - \rho_\infty \beta_T (T - T_\infty) - \rho_\infty \beta_C (C - C_\infty) \tag{3.4}$$

Which is valid only if $\beta_T (T - T_\infty)$ and $\beta_C (C - C_\infty) << 1$

Substituting equation (3.4) into the momentum equation results in:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\rho_\infty g + \rho_\infty g\beta_T (T - T_\infty) + \rho_\infty g\beta_C (C - C_\infty) - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(3.5)

Dividing equation (3.5) by ρ reduces it to:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g + g\beta_T (T - T_\infty) + g\beta_C (C - C_\infty) - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.6}$$

Given that; $\frac{\partial p}{\partial x} = \frac{\partial p_\infty}{\partial x} = -\rho_\infty g$ (3.7)

Substituting equation (3.7) into (3.6) yields:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta_T (T - T_\infty) + g\beta_C (C - C_\infty) + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.8}$$

Furthermore, in a boundary layer approximation, $\frac{\partial^2 u}{\partial x^2} << \frac{\partial^2 u}{\partial y^2}$,

Therefore the Momentum equation for this present work becomes:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} + g\beta_T (T - T_\infty) + g\beta_C (C - C_\infty), \tag{3.9}$$

where υ is the kinematic viscosity and g is gravitational acceleration.

3.1.3 The Energy Equation

The generalised energy equation was derived as given by equation (2.32) and restated here for emphasis.

$$\rho C_p\left[\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\Phi$$

where $\Phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]^2-\frac{2}{3}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]^2$

For a steady flow, $\frac{\partial T}{\partial t}=0$ hence, equation (2.32) reduces to:

$$\rho C_p\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\Phi \qquad (3.10)$$

Equation 2.33 has a number of boundary layer approximations, that is:

- $\frac{\partial u}{\partial x}$ is generally $<<$ $\frac{\partial u}{\partial y}$
- v is generally << u hence $\frac{\partial v}{\partial x}\approx\frac{\partial v}{\partial y}\approx 0$
- Furthermore, $\frac{\partial^2 T}{\partial x^2}<<\frac{\partial^2 T}{\partial y^2}$, so the boundary layer form is

$$\Phi=\left(\frac{\partial u}{\partial y}\right)^2. \qquad (3.11)$$

Therefore, equation 3.10 becomes:

$$\rho C_p\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]=k\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

or $u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\upsilon}{C_p}\left(\frac{\partial u}{\partial y}\right)^2$, where $\alpha=\frac{k}{\rho C_p}$ and $\mu=\upsilon\rho$

Thus, the energy equation describing the problem is:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\upsilon}{c_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3.12}$$

where T_∞ is the free stream temperature, α is the thermal diffusivity and C_p is the specific heat at constant pressure.

3 .1.4 The Concentration Equation

The generalised concentration equation was derived in equation (2.38) given by:

$$u\frac{\partial C_A}{\partial x}+v\frac{\partial C_A}{\partial y}=\frac{\partial}{\partial x}\left(D_{AB}\frac{\partial C_A}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_{AB}\frac{\partial C_A}{\partial y}\right)+\dot{N}_A,$$

In boundary layer approximation, $\frac{\partial^2 C}{\partial x^2} << \frac{\partial^2 C}{\partial y^2}$, so the boundary layer form becomes

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}+\dot{N}_A \tag{3.13}$$

Assuming that the chemical reaction leads to the destruction of species A , then the molar destruction rate can be defined as

$$\dot{N}_A=-\gamma C^n \quad \text{where } C=C-C_\infty, \tag{3.14}$$

The index n represents the order of the reaction, γ is the plate surface rate of chemical reaction.

Substituting (3.14) into (3.13) gives the Concentration equation below;

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-\gamma\left(C-C_\infty\right)^n \tag{3.15}$$

3 .2 Boundary and Initial Conditions

The governing equations discussed above generally require initial and boundary conditions. The boundary of the calculation domain may be either solid or fluid and the computation domain comprises usually only a part of the whole flow field. At present no adequate mathematical theory is available to ensure a correct boundary condition for the full Navier-Stokes equations in general. For mass and momentum equations all velocity components are specified, for the energy and concentration equations, the values of the dependent variables such as the temperature and the concentration at the wall and upstream are also specified. Because it is a solid wall, it is assumed that the fluid cannot flow across the surface and a no-slip condition is specified on the boundary.

The vertical component of velocity and the horizontal velocity are set to zero. The concentration at the wall is set to be C_w. It is also assumed that the left surface of the plate is heated by convection from a hot fluid at temperature T_f which provides a heat transfer coefficient, h_f. Hence the wall surface temperature, concentration and velocity is

$$-k\frac{\partial T}{\partial y} = h_f\left[T_f - T(x,0)\right],\ \ C_w(x,0) = C_w \text{ and } u(x,0) = 0,\ v(x,0) = 0 \qquad (3.16)$$

The free stream velocity, temperature and concentration are as follows

$$u(x,\infty) = U,\ \ T(x,\infty) = T_\infty,\ \ C(x,\infty) = C_\infty \qquad (3.17)$$

where h_f is the plate heat transfer coefficient, C_w is the species concentration at the plate surface and k is the thermal conductivity coefficient.

3.3 Similarity Solution

The idea of similarity solution is based on the fact that the solution of a problem $u(x,y)$ will collapse on the same curve if defined by a function U if $u(x,y)=U(y/f(x))$. The function $f(x)$ may be found by substitution in the partial differential equation to obtain an ordinary differential equation for U. The similarity variable η is defined as the ratio of the distance from the plate surface (y) to the approximate thickness of the momentum boundary layer δ_m

$$\eta=\frac{y}{\delta_m} \tag{3.18}$$

3.4 Boundary Layer

Fluids flowing past solid bodies adhere to them. So a region of variable velocity must be built up between the body and the free fluid stream. This region is called the boundary layer. The growth of the velocity, thermal and concentration boundary layer in a laminar flow is due to the molecular diffusion of momentum and energy. The momentum boundary layer thickness δ_m will grow according to:

$$\delta_m=2\sqrt{\upsilon t} \tag{3.19}$$

where υ is the kinematic viscosity and t is the time, which is related to the distance from the leading edge x and the characteristic velocity v_{char} according to;

$$t=\frac{x}{v_{chr}} \tag{3.20}$$

The length of the plate is taking along the x-axis whilst the characteristic velocity is the free stream velocity far from the plate denoted by U_o

Substituting (3.20) into (3.19) leads to:

$$\delta_m = 2\sqrt{\frac{\upsilon x}{U_o}} \tag{3.21}$$

Substituting 3.21 into 3.18 gives

$$\eta = \frac{y}{2}\sqrt{\frac{U_o}{\upsilon x}} \tag{3.22}$$

From Ostrach (1953) equation (3.19) is modified slightly to become:

$$\eta = y\sqrt{\frac{U_0}{\upsilon x}} \tag{3.23}$$

Equation (3.23) defines the similarity variable for this research work.

The dimensionless velocity, temperature and concentration are thus given by:

$$f' = \frac{U}{U_0},\ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty},\ \ \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty} \tag{3.24}$$

3.5 Stream Function

In analyzing fluid flow, the idea of a streamline which is an imaginary curve in the fluid across which at a given instant, there is no flow must be considered. Thus the velocity of every particle of fluid along the streamline is tangential to it at that moment. The concept of the streamline is very useful, especially in ideal flow, because it enables the fluid flow to be conceived as occurring in patterns of streamlines (John et al, 2001). These patterns may be described mathematically so that the whole system of analysis may be based on it. It requires a mathematical definition of a streamline. Consider a two dimensional case, the velocity and displacement vectors of a fluid at a point together with their orthogonal components are as shown.

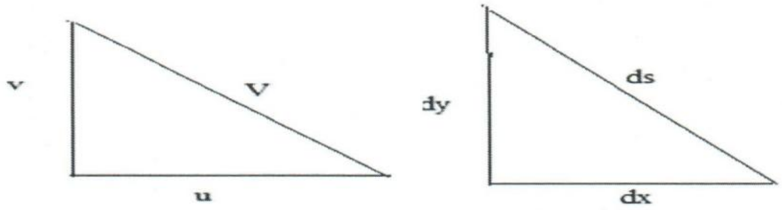

Figure 3. 2 Velocity and displacement Vectors

By definition of a stream line, $ds // V$, it follows that $dy // v$ and $dx // u$

Thus the velocity triangle and the displacement triangle are similar and therefore

$$\frac{dx}{u} = \frac{dy}{v}$$

This constitutes the equation of a streamline. The flow per unit depth between streamlines a and b is given by:

$$Q_a = \psi_a \text{ and similarly } Q_b = \psi_b$$

$$d\psi = \psi_b - \psi_a = Q_c$$

So that

$$d\psi = udy - vdx \qquad (3.25)$$

and ψ which is called the stream function is given by:

$$\psi = \int udy - \int vdx$$

Thus the stream functions depends upon position coordinates $\psi = f(x, y)$ and hence, the total derivative;

$$d\psi = \frac{d\psi}{dx}dx + \frac{d\psi}{dy}dy \tag{3.26}$$

Comparing equation 3.26 and 3.25, the relationship between the stream function and the velocity components are obtained as:

$$u = \frac{d\psi}{dy} \text{ and } v = -\frac{d\psi}{dx} \tag{3.27}$$

The stream function is related to the volumetric flow Q, between the surface of the plate and any position y according to:

$$Q = W\psi \tag{3.28}$$

where W is the width of the plate. The volumetric flow rate is obtained from the velocity as:

$$Q = W\int_0^y u dy \tag{3.29}$$

Equation (3.29) can be expressed in terms of the dimensionless variables using equation (3.23) and (3.24) as:

$$Q = WU_0\sqrt{\frac{\upsilon x}{U_0}}\int_0^\eta f'd\eta \tag{3.30}$$

Substitute (3.30) into (3.28) yields:

$$\psi = U_0\sqrt{\frac{\upsilon x}{U_0}}\int_0^\eta f'd\eta \tag{3.31}$$

The integral $\int_0^\eta f'(\eta)d\eta = f(\eta)$ and it is thought of as the dimensionless form of the stream function and must be a function of the similarity variable η

Simplifying further yields:

$$\psi = \sqrt{\upsilon U_0 x f(\eta)} \tag{3.32}$$

3.6 Procedure of the Analysis of the Model

The stream function, ψ, satisfies the continuity equation (2.7) automatically if defined as

$$u = \frac{\partial \psi}{\partial y}, \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x},$$

The system of governing equations are second order partial differential equations and can be transformed to ordinary differential equation if equation 3.23 and 3.32 are substituted into equations (2.7), (3.9), (3.12) and (3.15).

For continuity of flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

From Chain rule $u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} . \frac{\partial \eta}{\partial y},$ (3.33)

$$\frac{\partial \psi}{\partial \eta} = (\upsilon x U_0)^{1/2} f'(\eta) \quad \text{and} \quad \frac{\partial \eta}{\partial y} = \left(\frac{U_0}{\upsilon x}\right)^{\frac{1}{2}} \tag{3.34}$$

Therefore, $u = \frac{\partial \psi}{\partial \eta} . \frac{\partial \eta}{\partial y} = (\upsilon x U_0)^{1/2} f'(\eta) . \left(\frac{U_0}{\upsilon x}\right)^{\frac{1}{2}}$

$$u = U_0 f'(\eta) \tag{3.35}$$

by the chain rule, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} . \frac{\partial \eta}{\partial x}$

$$\frac{\partial \eta}{\partial x} = -\tfrac{1}{2} x^{-\frac{3}{2}} y \left(\frac{u_0}{\upsilon}\right)^{1/2} \quad \text{and} \quad \frac{\partial u}{\partial \eta} = U_0 f''(\eta) \tag{3.36}$$

$$\therefore \frac{\partial u}{\partial x} = -\frac{1}{2} x^{-\frac{3}{2}} y \left(\frac{u_o}{\upsilon}\right)^{1/2} . U_0 f''(\eta) = -\frac{1}{2} x^{-\frac{3}{2}} y U_O \left(\frac{u_o}{\upsilon}\right)^{1/2} f''(\eta) \tag{3.37}$$

Similarly, $\frac{\partial \psi}{\partial x}$ is obtained in the same manner using $\psi = \sqrt{vxU_0}\, f(\eta)$

$$\frac{\partial \psi}{\partial x} = -\tfrac{1}{2} x^{-1} y U_0\, f'(\eta) + \tfrac{1}{2} x^{-\frac{1}{2}} (\upsilon U_0)^{\frac{1}{2}} . f(\eta)$$

$v = -\frac{\partial \psi}{\partial x}$, hence

$$\text{v} = \tfrac{1}{2} x^{-1} y U_0\, f'(\eta) \text{ - } \tfrac{1}{2} x^{-\frac{1}{2}} (\upsilon U_0)^{\frac{1}{2}} . f(\eta) \qquad (3.38)$$

Applying the product rule on v and simplifying, $\frac{\partial v}{\partial y}$ is obtained as

$$\frac{\partial v}{\partial y} = \tfrac{1}{2} y U_0 x^{-\frac{3}{2}} \left(\frac{u_0}{\upsilon}\right)^{\frac{1}{2}} f''(\eta) + \tfrac{1}{2} x^{-1} U_0 f'(\eta) - \tfrac{1}{2} x^{-1} U_0 f'(\eta) - 0 = \tfrac{1}{2} y U_0 x^{-\frac{3}{2}} \left(\frac{u_0}{\upsilon}\right)^{\frac{1}{2}} f''(\eta)$$

$$\frac{\partial v}{\partial y} = \tfrac{1}{2} y U_0 x^{-\frac{3}{2}} \left(\frac{u_0}{\upsilon}\right)^{\frac{1}{2}} f''(\eta) \ = \tfrac{1}{2} x^{-\frac{3}{2}} y U_0 . \left(\frac{u_0}{\upsilon}\right)^{\frac{1}{2}} f''(\eta) \qquad (3.39)$$

Substituting (3.37) and (3.39) into (2.7)

$$\tfrac{1}{2} x^{-\frac{3}{2}} y U_0 . \left(\frac{u_0}{\upsilon}\right)^{\frac{1}{2}} f''(\eta) - \frac{1}{2} x^{-\frac{3}{2}} y U_O \left(\frac{u_o}{\upsilon}\right)^{\frac{1}{2}} f''(\eta) = 0$$

$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ hence the continuity equation is satisfied

Since the continuity equation is satisfied, the momentum equation can be obtained by substituting in the dimensionless quantities into the governing equation (3.9) therefore:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon \frac{\partial^2 u}{\partial y^2} + g\beta_T (T - T_\infty) + g\beta_C (C - C_\infty)$$

$$u\frac{\partial u}{\partial x} = U_0 f'(\eta) . -\frac{1}{2} x^{-\frac{3}{2}} y U_O \left(\frac{u_o}{\upsilon}\right)^{\frac{1}{2}} f''(\eta) = -\frac{1}{2} x^{-\frac{3}{2}} y U_O{}^2 \left(\frac{u_o}{\upsilon}\right)^{\frac{1}{2}} f''(\eta) f'(\eta) \qquad (3.40)$$

$\frac{\partial u}{\partial y}$ is evaluated using the relation; $\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \eta}.\frac{\partial \eta}{\partial y}$

Recall $\frac{\partial u}{\partial \eta}=U_0 f''(\eta)$ as in (3.29) and $\frac{\partial \eta}{\partial y}=x^{-1/2}\left(\frac{U_0}{\upsilon}\right)^{1/2}$ as in (3.22)

$$\frac{\partial u}{\partial y}= U_0 f''(\eta)\ .x^{-1/2}\left(\frac{U_0}{\upsilon}\right)^{1/2}=x^{-1/2}U_{0\cdot}\left(\frac{U_o}{\upsilon}\right)^{1/2} f''(\eta). \tag{3.41}$$

$v\frac{\partial u}{\partial y}$, is obtained by multiplying (3.38) by (3.41),

$$v\frac{\partial u}{\partial y}=\left(\frac{1}{2}x^{-1}yU_o f'(\eta)-\frac{1}{2}x^{-1/2}(\upsilon U_0)^{1/2}.f(\eta)\right)\bullet x^{-1/2}U_{0\cdot}\left(\frac{U_o}{\upsilon}\right)^{1/2} f''(\eta).$$

$$v\frac{\partial u}{\partial y}=\frac{1}{2}x^{-3/2}yU_o^{\,2}\left(\frac{U_o}{\upsilon}\right)^{1/2} f''(\eta).f'(\eta)-\frac{1}{2}x^{-1}U_0^{\,2}f''(\eta)f(\eta) \tag{3.42}$$

$\frac{\partial^2 u}{\partial y^2}$ is obtained by differentiating $\frac{\partial u}{\partial y}$ with respect to y which is given as

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(x^{-1/2}.U_o\left(\frac{U_o}{\upsilon}\right)^{1/2} f''(\eta)\right)=x^{-1/2}.U_o\left(\frac{U_o}{\upsilon}\right)^{1/2} f'''(\eta).\bullet x^{-1/2}\left(\frac{U_o}{\upsilon}\right)^{1/2}$$

$$\frac{\partial^2 u}{\partial y^2}=x^{-1}.U_o\left(\frac{U_o}{\upsilon}\right)f''(\eta) \tag{3.43}$$

Multiplying equation (3.43) by υ results in:

$$\upsilon\frac{\partial^2 u}{\partial y^2}=\upsilon..x^{-1}.U_o\left(\frac{U_o}{\upsilon}\right)f''(\eta)=\left(\frac{U_o^{\,2}}{x}\right)f''(\eta) \tag{3.44}$$

Using the similarity variable $\theta(\eta)=\dfrac{T-T_\infty}{T_w-T_\infty}$, $T-T_\infty$ is made the subject which is given as $\quad T-T_\infty=\theta(\eta)(T_w-T_\infty)$

(3.45) Multiplying (3.45) by $g\beta_T$ results in:

$$g\beta_T(T-T_\infty)=g\beta_T\theta(\eta)(T_w-T_\infty) \tag{3.46}$$

Similarly, using the similarity variable $\phi(\eta)=\dfrac{C-C_\infty}{C_w-C_\infty}$, $C-C_\infty$ is made the subject which is given as $\quad C-C_\infty=\phi(\eta)(C_w-C_\infty)$

(3.47) Multiplying (3.40) by $g\beta_c$ gives;

$$g\beta_c(C-C_\infty)=g\beta_c\phi(\eta)(C_w-C_\infty) \tag{3.48}$$

Substituting (3.40) ,(3.42), (3.44), (3.46) and (3.48) into (3.9)

$$-\frac{1}{2}x^{-\frac{3}{2}}yU_O{}^2\left(\frac{u_o}{\upsilon}\right)^{\frac{1}{2}}f''(\eta)f'(\eta)+$$

$$\frac{1}{2}x^{-3/2}yU_o{}^2\left(\frac{U_o}{\upsilon}\right)^{1/2}f''(\eta).f'(\eta)-\frac{1}{2}x^{-1}U_0{}^2f''(\eta)f(\eta)=\left(\frac{U_o{}^2}{x}\right)f'''(\eta)$$

$$+g\beta_T\theta(\eta)(T_w-T_\infty)+g\beta_c\phi(\eta)(C_w-C_\infty)$$

$$=\frac{1}{2}x^{-1}U_o^2f''(\eta)f(\eta)+\left(\frac{U_o{}^2}{x}\right)f'''(\eta)+g\beta_T\theta(\eta)(T_w-T_\infty)+g\beta_c\phi(\eta)(C_w-C_\infty)$$

$$=f'''(\eta)+\frac{1}{2}f''f+\frac{g\beta_T\theta(\eta)(T_w-T_\infty)x}{U_o^2}+\frac{g\beta_c\phi(\eta)(C_w-C_\infty)x}{U_o{}^2}$$

$$=f'''+\frac{1}{2}f''f+Gr_x\theta+Gc_x\phi \tag{3.49}$$

where $Gr_x = \dfrac{g\beta_T\theta(\eta)(T_w - T_\infty)x}{U_o^2}$ and $Gc_x = \dfrac{g\beta_c\phi(\eta)(C_w - C_\infty)x}{U_o^{\ 2}}$ (3.50)

From the dimensionless quantity $\theta(\eta) = \dfrac{T - T_\infty}{T_w - T_\infty}$, T is made the subject yielding:

$$T = \theta(\eta)\,(T_w - T_\infty) + T_\infty \tag{3.51}$$

Differentiating (3.39) with respect to x using the product rule gives;

$$\frac{\partial T}{\partial x} = -\frac{1}{2}x^{-3/2}y\left(\frac{U_o}{\upsilon}\right)^{1/2}\theta'(\eta)(T_w - T_\infty) \tag{3.52}$$

To find $u\dfrac{\partial T}{\partial x}$, recall that $u = U_0 f'(\eta)$ (3.53)

$$u\frac{\partial T}{\partial x} = -\frac{1}{2}x^{-3/2}y\left(\frac{U_o}{\upsilon}\right)^{1/2}\theta'(\eta)(T_w - T_\infty).U_o f'$$

Hence, $u\dfrac{\partial T}{\partial x} = -\dfrac{1}{2}x^{-3/2}yU_o\left(\dfrac{U_o}{\upsilon}\right)^{1/2}(T_w - T_\infty).\, f'\,\theta'$ (3.54)

Similarly differentiating T with respect to y gives,

$$\frac{\partial T}{\partial y} = x^{-1/2}\left(\frac{U_o}{\upsilon}\right)^{1/2},\ \theta'(\eta)(T_w - T_\infty) \tag{3.55}$$

Multiplying (3.55) by (3.38) gives,

$$v\frac{\partial T}{\partial y} = \frac{1}{2}x^{-3/2}yU_o\left(\frac{U_o}{\upsilon}\right)^{1/2}(T_w - T_\infty)f'(\eta)\theta'(\eta) - \frac{1}{2}x^{-1}U_o\,(T_w - T_\infty)f(\eta)\theta'(\eta) \tag{3.56}$$

Differentiating $\dfrac{\partial T}{\partial y}$ with respect to y and multiplying the results by α gives;

$$\alpha\frac{\partial^2 T}{\partial y^2} = \alpha x^{-1}.\left(\frac{U_o}{\upsilon}\right)(T_w - T_\infty)\theta'' \tag{3.57}$$

Recall $\frac{\partial u}{\partial y} = x^{-1/2}U_{0}\cdot\left(\frac{U_o}{\upsilon}\right)^{1/2} f''(\eta)$. as in (3.41), Squaring $\frac{\partial u}{\partial y}$ and multiplying the results by $\frac{\upsilon}{C_p}$ give $\frac{\upsilon}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 = \frac{U_o{}^2}{C_p}\left(\frac{U_o}{x}\right) f''^2(\eta)$. (3.58)

Substituting (3.54) , (3.56) , (3.57) and (3.58) into (3.12) and simplifying gives:

$$-\frac{1}{2}x^{-3/2}yU_o\left(\frac{U_o}{\upsilon}\right)^{1/2}(T_w - T_\infty)\,.\,f'\;\theta' + \frac{1}{2}x^{-3/2}yU_o\left(\frac{U_o}{\upsilon}\right)^{1/2}(T_w - T_\infty)\,.\,f'\;\theta' -$$

$$\frac{1}{2}x^{-1}U_o(T_w - T_\infty).\,f\;\theta' = \alpha x^{-1}.\left(\frac{U_o}{\upsilon}\right)(T_w - T_\infty)\theta'' + \frac{U_o{}^2}{C_p}\left(\frac{U_o}{x}\right)f''^2(\eta). \qquad (3.59)$$

$$\frac{1}{2}x^{-1}U_o(T_w - T_\infty)\,f\;\theta' + \alpha x^{-1}.\left(\frac{U_o}{\upsilon}\right)(T_w - T_\infty)\theta'' + \frac{U_o{}^2}{C_p}\left(\frac{U_o}{x}\right)f''^2(\eta) = 0$$

$$\theta'' + \frac{\upsilon}{2\alpha}f\theta' + \frac{\upsilon}{\alpha}\left(\frac{U_o{}^2}{C_p(T_w - T\infty)}\right)f''^2 = 0$$

Therefore the Energy equation is transformed into a dimensionless form as:

$$\theta'' + {}^1\!/_2\,P_r f\theta' + P_r E_c f''^2 = 0 \qquad (3.60)$$

where $P_r = \frac{\upsilon}{\alpha}$ *and* $E_c = \left(\frac{U_o{}^2}{C_p(T_w - T_\infty)}\right)$ (3.61)

From the dimensionless quantity $\phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}$ C is made the subject as:

$$C = \phi(\eta)(C_w - C_\infty) + C_\infty \qquad (3.62)$$

Differentiating (3.62) with respect to x using the product rule gives:

$$\frac{\partial C}{\partial x} = -\frac{1}{2}x^{-3/2}y\left(\frac{U_o}{\upsilon}\right)^{1/2}\phi'(\eta)(C_w - C_\infty) \qquad (3.63)$$

$u\dfrac{\partial C}{\partial x}$ is also evaluated by multiplying (3.63) by (3.35) yielding

$$u\frac{\partial C}{\partial x} = -\frac{1}{2}x^{-3/2}y\left(\frac{U_o}{\upsilon}\right)^{1/2}\phi'(\eta)(C_w - C_\infty).U_o f'$$

Hence $u\dfrac{\partial C}{\partial x} = -\dfrac{1}{2}x^{-3/2}yU_o\left(\dfrac{U_o}{\upsilon}\right)^{1/2}(C_w - C_\infty).\, f'\,\phi'$ (3.64)

In the same manner C is differentiated with respect to y to obtain,

$$\frac{\partial C}{\partial y} = x^{-1/2}\left(\frac{U_o}{\upsilon}\right)^{1/2}(C_w - C_\infty)\,\phi' \qquad (3.65)$$

$v\dfrac{\partial C}{\partial y}$, is obtained by multiplying (3.65) by (3.38), thus:

$$v\frac{\partial C}{\partial y} = x^{-1/2}\left(\frac{U_o}{\upsilon}\right)^{1/2}\phi'(\eta)(C_w - C_\infty)\times\left(\frac{1}{2}x^{-1}yU_o f'(\eta) - \frac{1}{2}x^{-1/2}(\upsilon U_0)^{1/2}f(\eta)\right)$$

Hence:

$$v\frac{\partial C}{\partial y} = \frac{1}{2}x^{-3/2}yU_o\left(\frac{U_o}{\upsilon}\right)^{1/2}(C_w - C_\infty)\,\phi'\,f' - \frac{1}{2}x^{-1}U_o\,(C_w - C_\infty)\,\phi'.\,f \qquad (3.66)$$

Differentiating $\dfrac{\partial C}{\partial y}$ with respect to y and multiplying the results by D gives

$$D\frac{\partial^2 C}{\partial y^2} = D\;x^{-1}\left(\frac{U_o}{\upsilon}\right)(C_w - C_\infty)\,\phi'' \qquad (3.67)$$

Using the dimensionless quantity $\phi(\eta) = \dfrac{C - C_\infty}{C_w - C_\infty}$, $(C - C_\infty)$ is made the subject for a generalised order of reaction n to obtain;

$$(C - C_\infty)^n = [\phi(C_w - C_\infty)]^n \qquad (3.68)$$

Given a reaction rate parameter, γ, results in

$$\gamma(C-C_\infty)^n=\gamma[\phi(C_w-C_\infty)]^n \tag{3.69}$$

Substituting equation (3.64), (3.66), (3.67) (3.69) into (3.15)

$$-\frac{1}{2}x^{-3/2}yU_o\left(\frac{U_o}{\upsilon}\right)^{1/2}\ (C_w-C_\infty)\ .\ f'\ \ \phi'+\ \frac{1}{2}x^{-3/2}yU_o\left(\frac{U_o}{\upsilon}\right)^{1/2}\ (C_w-C_\infty)\ \phi'\ \ f'\ -$$

$$\frac{1}{2}x^{-1}U_o\ \ (C_w-C_\infty)\,\phi'\,.\,f=Dx^{-1}.\left(\frac{U_o}{\upsilon}\right)(C_w-C_\infty)\,\phi''-\gamma[\phi(C_w-C_\infty)]^n$$

$$=Dx^{-1}.\left(\frac{U_o}{\upsilon}\right)(C_w-C_\infty)\,\phi''-\gamma[\phi(C_w-C_\infty)]^n+\frac{1}{2}x^{-1}U_o\,(C_w-C_\infty)\,\phi'\,.\,f \tag{3.70}$$

Furthermore simplifying (3.70) result in;

$$\phi''+\frac{1}{2}\frac{\upsilon}{D}f\phi'-\frac{\upsilon}{D}\frac{\gamma x}{U_o}\bullet\phi^n.\,(C_w-C_\infty)^{n-1}=0$$

Hence the Concentration equation (3.15) is transformed into a dimensionless form as:

$$\phi''+\frac{1}{2}S_c f\phi'-S_c\beta_x\phi^n.=0 \tag{3.71}$$

where;

$$S_c=\frac{\upsilon}{D}\ \text{and}\ \beta_x=\frac{\gamma x}{U_o}\,(C_w-C_\infty)^{n-1} \tag{3.72}$$

3.7 Transformed Boundary Conditions

The boundary conditions (3.16) and (3.17) are also in partial differential form and hence the need to transform them into ordinary differential equation by using the similarity variables in (3.23) and (3.24)

Given $u(x,0)=0$ as the first boundary condition, Referring to equation (3.35), $u = U_{0\cdot} f'(\eta)$ and $\eta = y\sqrt{\frac{U_0}{\nu x}}$. when y = 0, $\eta = 0$,

This implies $U_{0\cdot} f'(0) = 0$,

Hence $f'(0) = 0$ (3.73)

In solving the second boundary condition which is given as $v(x,0) = 0$, recall that

$$v = \frac{1}{2}x^{-1} y U_o f'(\eta) - \frac{1}{2}x^{-1/2}(\upsilon U_0)^{1/2} f(\eta)$$

as in (3.38). When y = 0, $\eta = 0$ and v = 0, Substituting yields,

$$0 = \frac{1}{2}x^{-1}(0) U_o f'(0) - \frac{1}{2}x^{-1/2}(\upsilon U_0)^{1/2} f(0),$$

Hence $f(0) = 0$ (3.74)

Proceeding to the third boundary condition given as

$$-k\frac{\partial T}{\partial y} = h_f\left[T_f - T(x,0)\right]$$

Referring to equation (3.55)

$$\frac{\partial T}{\partial y} = x^{-1/2}\left(\frac{U_o}{\upsilon}\right)^{1/2} \theta'(\eta)(T_w - T_\infty)$$

Multiply both sides of (3.55) by - k and equating the results to $h_f\left[T_f - T(x,0)\right]$

$$-k\frac{\partial T}{\partial y} = -k.x^{-1/2}\left(\frac{U_o}{\upsilon}\right)^{1/2} \theta'(\eta)(T_w - T_\infty)$$

$$-k..x^{-1/2}\left(\frac{U_o}{\upsilon}\right)^{1/2}\theta'(\eta)(T_w - T_\infty) = h_f\left[T_f - T(x,0)\right]$$

$$\theta'(\eta) = \frac{h_f}{k}\bullet\frac{1}{x^{-1/2}\left(\frac{U_o}{\upsilon}\right)^{1/2}}\bullet\frac{T_f - T(x,0)}{T_w - T_\infty} \quad ,$$

$$\theta'(\eta) = \frac{h_f}{k}\left(\frac{\upsilon x}{u_o}\right)^{\frac{1}{2}}\bullet\frac{(T - T_\infty)-(T_w - T_\infty)}{T_w - T_\infty}$$

$$\theta'(\eta) = \frac{h_f}{k}\left(\frac{\upsilon x}{u_o}\right)^{\frac{1}{2}}\bullet\frac{(T - T_\infty)}{T_w - T_\infty} - \frac{(T_w - T_\infty)}{T_w - T_\infty} \quad \text{but} \quad \theta(0) = \frac{(T - T_\infty)}{T_w - T_\infty} \quad \text{and}$$

$$Bi_x = \frac{h_f}{k}\left(\frac{\upsilon x}{u_o}\right)^{\frac{1}{2}} \tag{3.75}$$

Hence:

$$\theta'(0) = B_{ix}\left[\theta(0) - 1\right] \tag{3.76}$$

Giving, $C_w(x,0) = C_w$ as the fourth boundary condition

When y = 0, $\eta = 0$ and substituting into $\phi(\eta) = \dfrac{C - C_\infty}{C_w - C_\infty}$

$$\phi(0) = \frac{C_w(x,0) - C_\infty}{C_w - C_\infty} = \frac{C_w - C_\infty}{C_w - C_\infty} = 1,$$

Therefore,

$$\phi(0) = 1 \tag{3.77}$$

Evaluating the fifth boundary condition, $u(x,\infty) = u,$

as y $\to \infty, \eta \to \infty$ and $u = U_\infty f'(\infty) = u$

$$f'(\infty)=\frac{u}{u_\infty}=1 \qquad \text{Therefore } f'(\infty)=1 \tag{3.78}$$

Solving for the sixth boundary condition, $T(x,\infty)=T_\infty$ using $\theta(\eta)=\dfrac{T-T_\infty}{T_w-T_\infty}$

as $y\to\infty,\eta\to\infty$ and $T\to T_\infty$, $\theta(\infty)=\dfrac{T_\infty-T_\infty}{T_w-T_\infty}=0$

Hence,

$$\theta(\infty)=0 \tag{3.79}$$

Solving for the seventh boundary condition, $C(x,\infty)=C_\infty$ using $\phi(\eta)=\dfrac{C-C_\infty}{C_w-C_\infty}$

as $y\to\infty,\eta\to\infty$ and $C\to C_\infty$, therefore $\phi(\infty)=\dfrac{C_\infty-C_\infty}{C_w-C_\infty}=0$

$$\phi(\infty)=0 \tag{3.80}$$

It is noticed that the local parameters B_{ix}, Gr_x, Gc_x and B_x in equation (3.75), (3.50) and (3.72) respectively are functions of x. However in order to have similarity solution all parameters must be constant and we therefore assume $h_f=ax^{-1/2}$ $B_T=bx^{-1}$ $B_c=cx^{-1}$ and $\gamma=dx^{-1}$ where a, b, c and d are constants.

In the next section, the numerical solution and the software package used in the analysis of the coupled ordinary differential equations are outlined.

3. 8 Numerical Solution

Many of the problems facing applied mathematicians and physicist is the difficulties faced in solving nonlinear equations with variable coefficients and nonlinear boundary conditions. Consequently, solutions are approximated using numerical techniques, analytical techniques or a combination of both. The task of this process is to discretize the model equations and approximate them by sets of linear algebraic ones, which are solved by suitable algorithms.

The numerical technique chosen for the solution of the coupled ordinary differential equations (3.35), (3.47) and (3.58) together with the associated transformed boundary conditions (3.60) to (3.66), is the standard Newton-Raphson shooting method alongside with the fourth-order Runge-Kutta as outlined in chapter two of equation (2.42), (2.43) and (2.44) integration algorithm. η_∞ is selected to represent the similarity variable at infinity. We then begin with some initial guess value and solve the problem with some particular set of parameters to obtain $f''(0)$, $\theta'(0)$ and $\phi'(0)$. This process is repeated with another larger value of η_∞ until two successive values of $f''(0)$, $\theta'(0)$ and $\phi'(0)$ differ only with the desired digit and this signifies the limit of the boundary along η. The method of superposition is used to choose the last value of η_∞ and this serves as the appropriate value for that particular simultaneous equation of first order for seven unknowns.

To solve this system we require seven initial conditions whilst we have only two initial conditions $f'(0)$ and $f(0)$ on f; and one initial condition each on θ and ϕ. This means that there are three unknown initial conditions, $f''(0)$, $\theta'(0)$ and $\phi'(0)$ which are not prescribed. Next, the Newton Raphson shooting technique is employed to produce two unknown initial conditions at $\eta = 0$ using the two ending boundary conditions. In this calculation, the step size $\Delta\eta = 0.001$ was used while obtaining the numerical solution with $\eta_{\max} = 10$ and six-decimal (10^{-6}) accuracy as the criterion for convergence. The numerical procedure was carried out using a Maple 16 software package. From the process of numerical computation, the plate surface temperature, the local skin-friction coefficient, the local Nusselt number and the local Sherwood number, which are respectively proportional to $f''(0)$, $-\theta'(0)$, and $-\phi'(0)$ are also sorted out and their numerical values presented in a tabular form.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.0 Introduction

This chapter presents the results of the solution of the problem analyzed in chapter three. Both numerical and graphical results are presented and discussed for various control parameters. The results are also compared with previously published data to validate the numerical procedure.

4.1 Validation of Results

In order to benchmark the numerical results, the plate surface temperature θ(0) and the local heat transfer rate at the plate surface - θ'(0) in the absence of internal heat generation, viscous dissipation and nth order chemical reaction was compared with the works of Makinde and Olanrewaju (2010; 2012) for varying parameters of convective heat transfer parameter (Bi), thermal Grashof number (Grx) and Prandtl number (Pr). It is clear from the results that the present study is consistent with those reported by these authors.

Table 4. 1 Comparison of results with Makinde and Olanrewaju (2010, 2012)

Controlling parameters			**Makinde and Olanrewaju(2010)**		**Makinde and Olanrewaju (2012)**		**Present Study**	
Bix	Grx	Pr	$-\theta'(0)$	$\theta(0)$	$-\theta'(0)$	$\theta(0)$	$-\theta'(0)$	$\theta(0)$
0.1	0.1	0.72	0.075077	0.249228	0.075077	0.249228	0.075077	0.249228
1.0	0.1	0.72	0.237506	0.762494	0.237506	0.762494	0.237506	0.762494
10	**0.1**	0.72	0.305596	0.969440	0.305596	0.969440	0.305596	0.969440
0.1	**0.5**	0.72	0.076138	0.238623	0.076138	0.238623	0.076138	0.238623
0.1	**1.0**	**0.72**	0.077045	0.229552	0.077045	0.229552	0.077045	0.2295515
0.1	0.1	**3.00**	0.083046	0.169540	0.0830460	0.169540	0.0830460	0.169540
0.1	0.1	**7.10**	0.086721	0.132788	0.0867212	0.132788	0.0867212	0.132788

The results of varying parameter values on the local Skin friction coefficient, the local Nusselt number and the local Sherwood number are shown in Table 4.2. The Prandtl parameter was taken to be 0.71(air), 4.0 and 7.1(water), which correspond to the common fluids used in industries; the values of Schmidt parameter (*Sc*) were chosen to be 0.24, 0.60, 0.78, and 2.62 representing the diffusing chemical species of most common interest in air that is, H_2, H_2O, NH_3, and Propyl benzene respectively. Attention was focused on positive values of the buoyancy parameters that is, Grashof number $Grx > 0$ (which corresponds to the cooling problem) and solutal Grashof number $Gcx > 0$ (which indicates that the concentration of the chemical species in the free stream region is less than the concentration at the boundary surface). The values for the Convective heat transfer parameter as well as the chemical reaction parameter were also chosen to be greater than zero. It is observed that increasing the Prandtl number (*Pr*) reduces the local Skin friction coefficient together with the Sherwood number and the Nusselt number.

The Skin friction coefficient and the Nusselt number are reduced whilst the Sherwood number increases with increasing Schmidt number (Sc). Increasing the buoyancy forces (*Gr, Gc*) increases the local Skin friction coefficient and the Sherwood number and reduces the Nusselt number. The Skin friction coefficient and the Nusselt number as well as the Sherwood number increases for increasing values of the convective heat transfer parameter (Bi).

Furthermore, the Nusselt number and the Sherwood number are increased whilst the Skin friction coefficient is reduced for increasing values of the order of chemical reaction (n). The Eckert number (Ec) increases the Skin friction and the Sherwood number whilst the Nusselt number is reduced. Increasing the reaction rate parameter reduces the Skin friction coefficient and the Nusselt number whilst increasing the Sherwood number. The Eckert number (Ec) increases the Skin friction and the Sherwood number whilst the Nusselt number is reduced.

Table 4. 2 Numerical results of skin friction coefficient, Nusselt number and the Sherwood number

Pr	Sc	Gr	Gc	Ec	β	n	Bi	$-f''(0)$	$-\theta'(0)$	$-\phi'(0)$
0.71	0.24	0.1	0.1	0.1	0.1	1	0.1	0.539343	0.071929	0.255722
4.0	0.24	0.1	0.1	0.1	0.1	1	0.1	0.527355	0.073771	0.254717
7.1	**0.24**	0.1	0.1	0.1	0.1	1	0.1	0.526772	0.071826	0.254605
0.71	**0.60**	0.1	0.1	0.1	0.1	1	0.1	0.507188	0.071935	0.371435
0.71	**0.78**	0.1	0.1	0.1	0.1	1	0.1	0.498585	0.071929	0.413099
0.71	**2.62**	**0.1**	0.1	0.1	0.1	1	0.1	0.463067	0.071868	0.463067
0.71	0.24	**0.5**	0.1	0.1	0.1	1	0.1	0.681540	0.071760	0.263087
0.71	0.24	**1.0**	0.1	0.1	0.1	1	0.1	0.851384	0.071218	0.270995
0.71	0.24	**1.4**	**0.1**	0.1	0.1	1	0.1	0.986547	0.070574	0.276758
0.71	0.24	0.1	**0.5**	0.1	0.1	1	0.1	1.079843	0.069706	0.284628
0.71	0.24	0.1	**1.0**	0.1	0.1	1	0.1	1.643945	0.064596	0.307937
0.71	0.24	0.1	**1.5**	**0.1**	0.1	1	0.1	2.147562	0.058561	0.325423
0.71	0.24	0.1	0.1	**0.5**	0.1	1	0.1	0.569874	0.052712	0.257477
0.71	0.24	0.1	0.1	**1.0**	0.1	1	0.1	0.610857	0.025826	0.259745
0.71	0.24	0.1	0.1	**1.5**	**0.1**	1	0.1	0.655511	0.004786	0.262116
0.71	0.24	0.1	0.1	0.1	**0.5**	1	0.1	0.516806	0.071935	0.387433
0.71	0.24	0.1	0.1	0.1	**1.0**	**1**	0.1	0.499240	0.071926	0.513307
0.71	0.24	0.1	0.1	0.1	0.1	**2**	0.1	0.543097	0.071925	0.243557
0.71	0.24	0.1	0.1	0.1	0.1	**3**	**0.1**	0.544584	0.071924	0.237357
0.71	0.24	0.1	0.1	0.1	0.1	1	**0.5**	0.582216	0.186420	0.257990
0.71	0.24	0.1	0.1	0.1	0.1	1	**1.0**	0.599122	0.233423	0.258863
0.71	0.1	0.1	0.1	0.1	0.1	1	**1.5**	0.606753	0.254978	0.259254

4.2 Graphical Results

4.2.1 Effects of Parameter Variation on Velocity Profiles

The effects of parameter variation on the velocity boundary layer are shown in Figures 4.1 to 4.4. It is observed in Figure 4.1 that increasing values of the Schmidt number tend to reduce the velocity profile slightly. In Figures 4.1 and 4.3, the velocity profiles for increasing thermal and solutal Grashof numbers increases just as that of increasing the Eckert number in Figure 4.4. This happens because increases in these parameters cause an increase in buoyancy forces and hence increase in velocity. Adding, the higher fluid velocity ensures better convection and distribution of temperature and concentration, respectively which is seen as lowering of fluid temperature and species concentration. It is therefore noted that, increasing buoyancy forces will lead to a better flow kinematics.

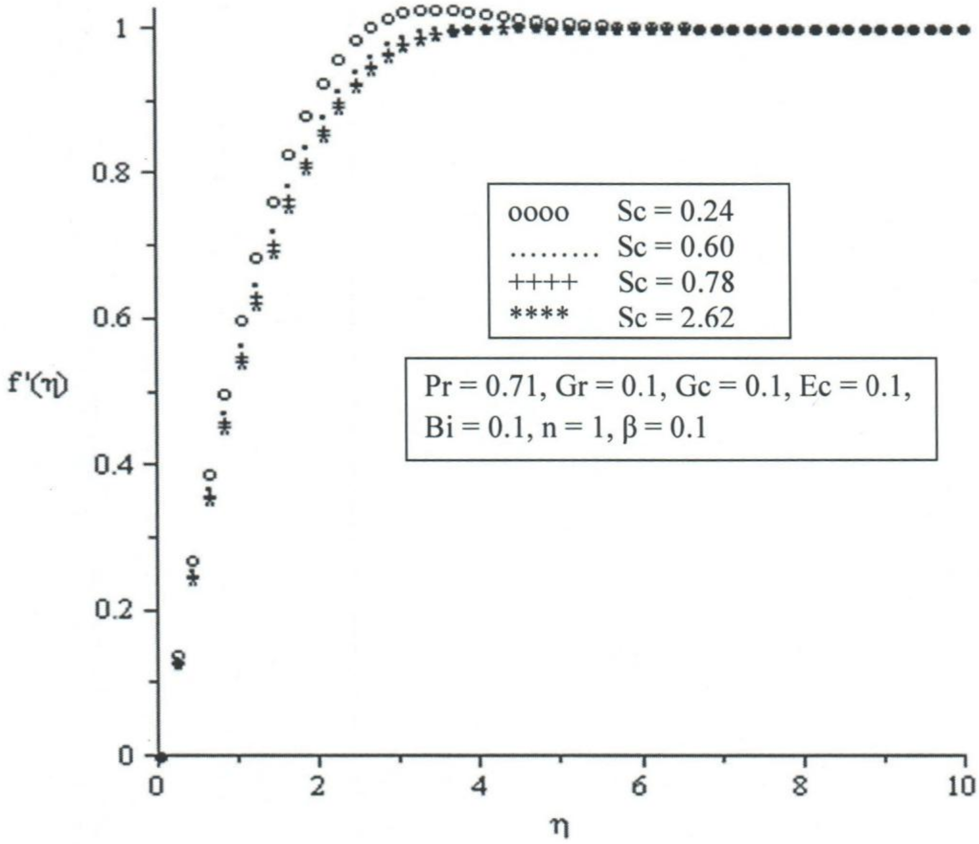

Figure 4. 1 Velocity profiles for varying values of Schmidt number (Sc)

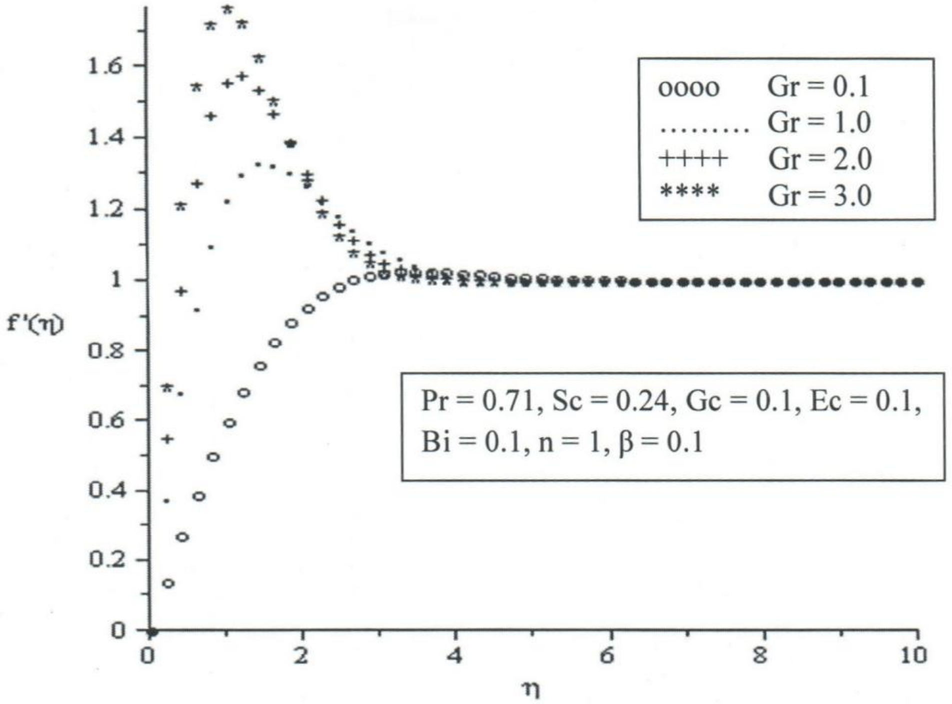

Figure 4. 2 Velocity profiles for varying values of thermal Grashof number (Gr)

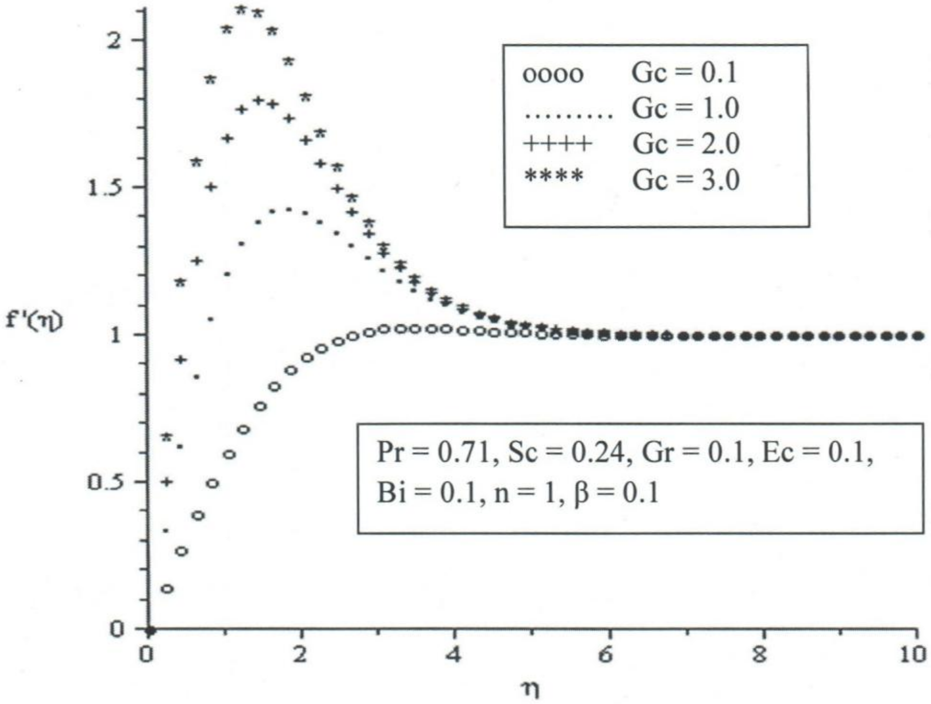

Figure 4. 3 Velocity profiles for varying values of solutal Grashof number (Gr)

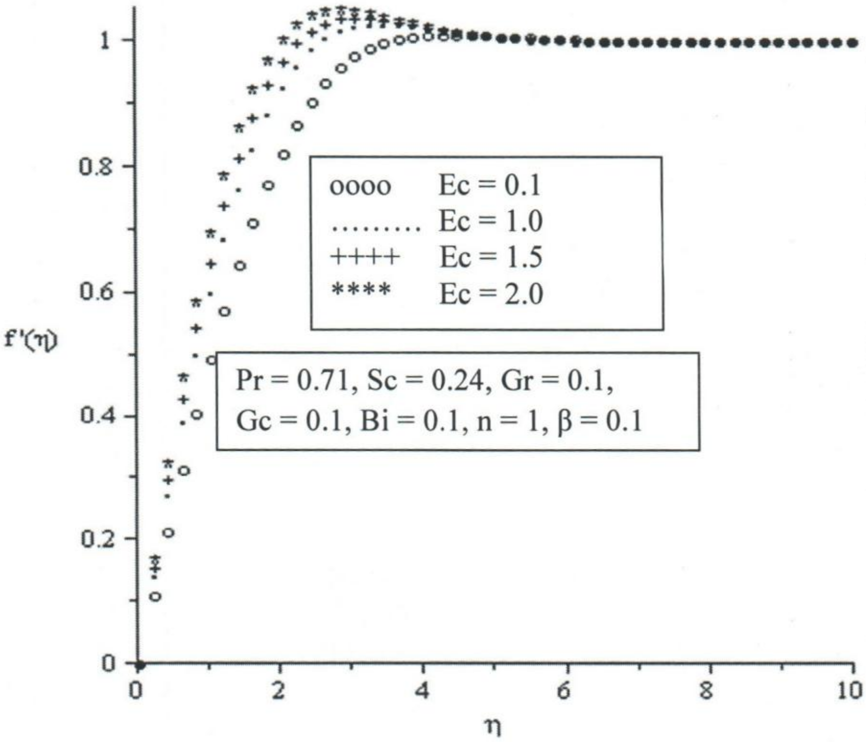

Figure 4. 4 Velocity profiles for varying values of Eckert number (*Ec*)

4.2.2 Effects of Parameter Variation on Temperature Profiles

The effects of parameter variation on temperature profiles are shown in Figures 4.5 to 4.9. In Figure 4.5, increasing values of the Prandtl number causes a decrease in the fluid temperature leading to a decaying thermal boundary layer. The reason is that smaller values of Prandtl number are equivalent to increasing thermal conductivity and therefore heat is able to diffuse away from the heated surface more rapidly than for higher values of Prandtl number. In Figure 4.8 and 4.9, it is also observed that increasing both the Eckert number and the convective heat transfer parameter increases the thickness of the temperature boundary layer. Eckert number is the ratio of the kinetic energy of the flow to the boundary layer enthalpy difference. The effect of viscous dissipation on the flow field is to increase the energy, yielding a greater fluid temperature and as a consequence greater buoyancy force. It is observed in Figure 4.6 and 4.7 that with the increase in buoyancy parameters, the surface temperature increases but slightly decreases towards the free stream the increase is due to the increase in the dissipation parameter and hence it enhances the temperature.

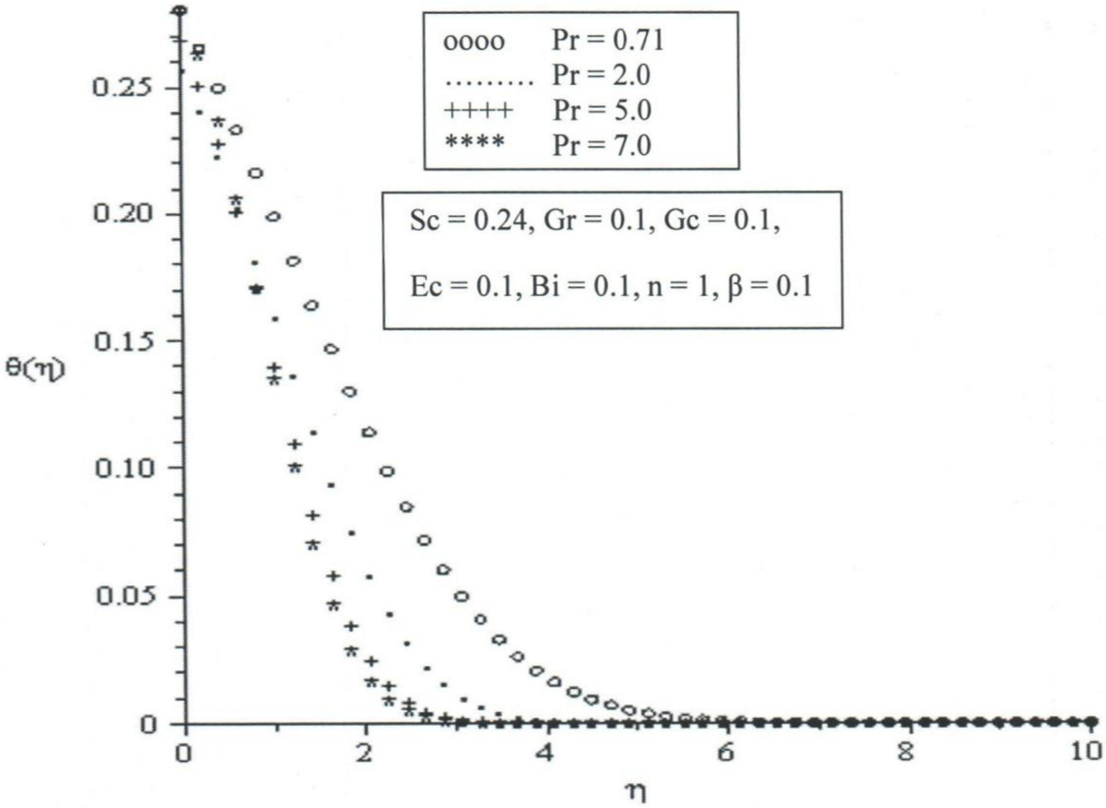

Figure 4. 5 Temperature profiles for varying values of Prandtl number ($\Pr$)

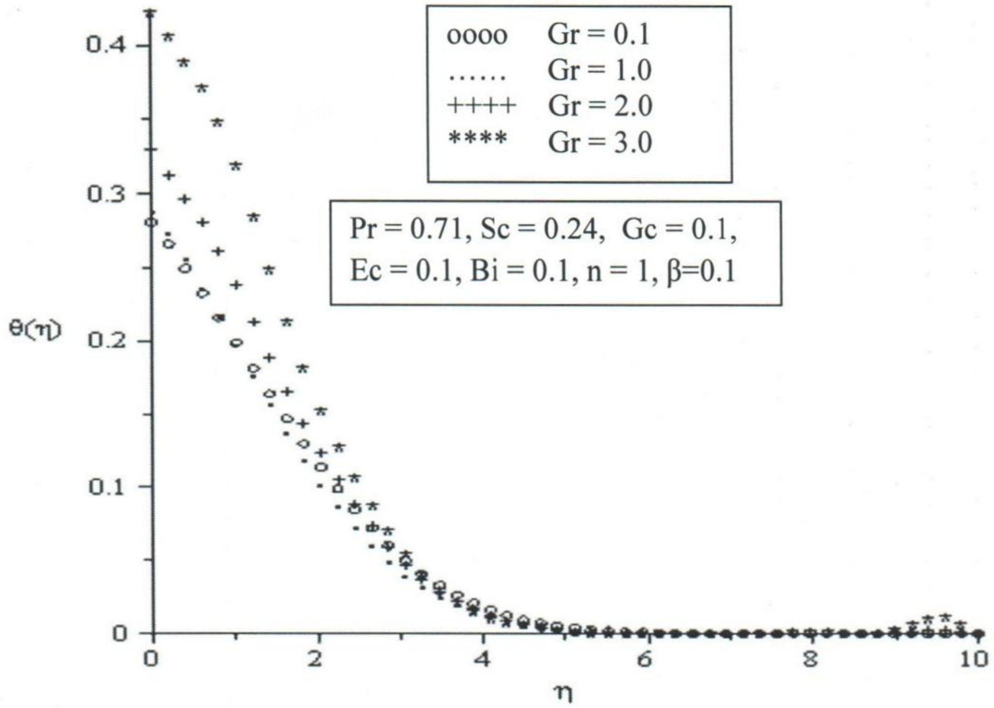

Figure 4. 6 Temperature profiles for varying values of thermal Grashof number *Gr*)

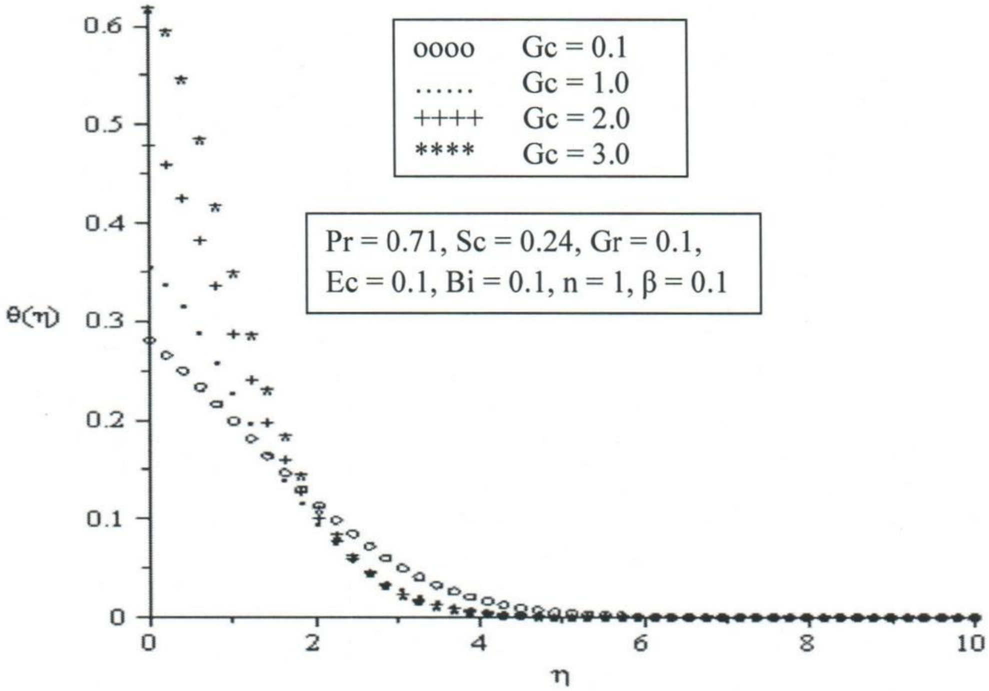

Figure 4. 7 Temperature profiles for varying values of solutal Grashof number (*Gc*)

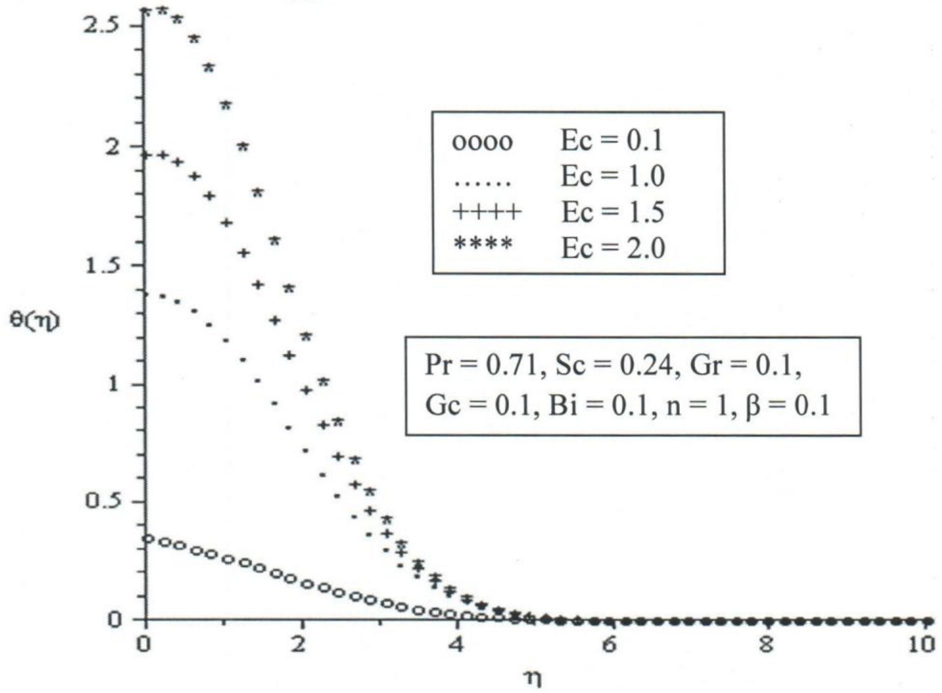

Figure 4. 8 Temperature profiles for varying values of Eckert number (*Ec*)

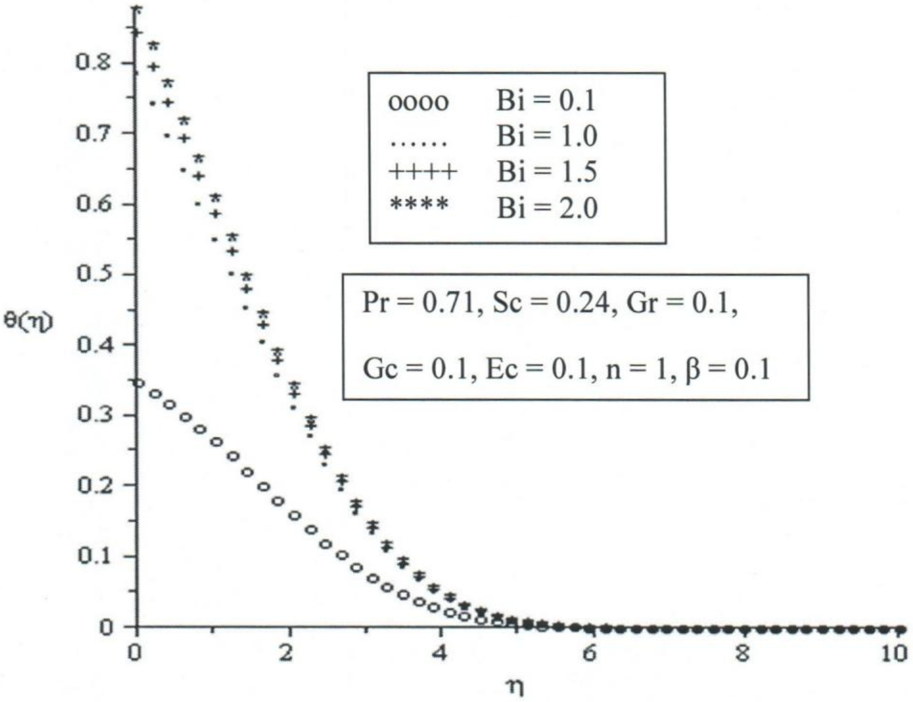

Figure 4. 9 Temperature profiles for varying values of convective heat transfer parameter (*Bi*)

4.2.3 Effects of Parameter Variation on Concentration Profiles

Figures 4.10 to 4.15 depict the effects of varying parameters on the thickness of the concentration boundary layer. It is observed in Figure 4.10 to 4.15 that, increasing the buoyancy forces, the convective heat transfer parameter and the reaction rate parameter have adverse effect of decaying the concentration boundary layer thickness. Also, a small Schmidt number implies that the fluid has high diffusion coefficient for the species and hence in steady state the concentration of the species is higher in the fluid which thereby reduces the concentration gradient at the surface, hence the concentration decreases with increasing Schmidt number. Moreover, the boundary layer thickness increases slightly when the order of reaction increases. Its effect on fluid velocity and temperature is negligible and hence those figures have not been included. Therefore it is concluded that effect of parameter *n* on species consumption is not significant.

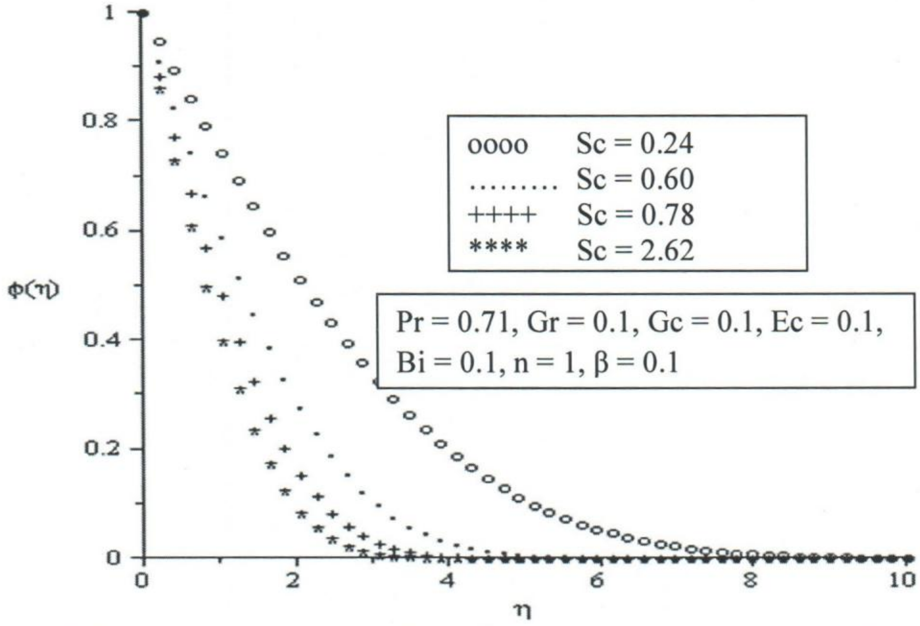

Figure 4. 10 Concentration profiles for varying values of Schmidt number (Sc)

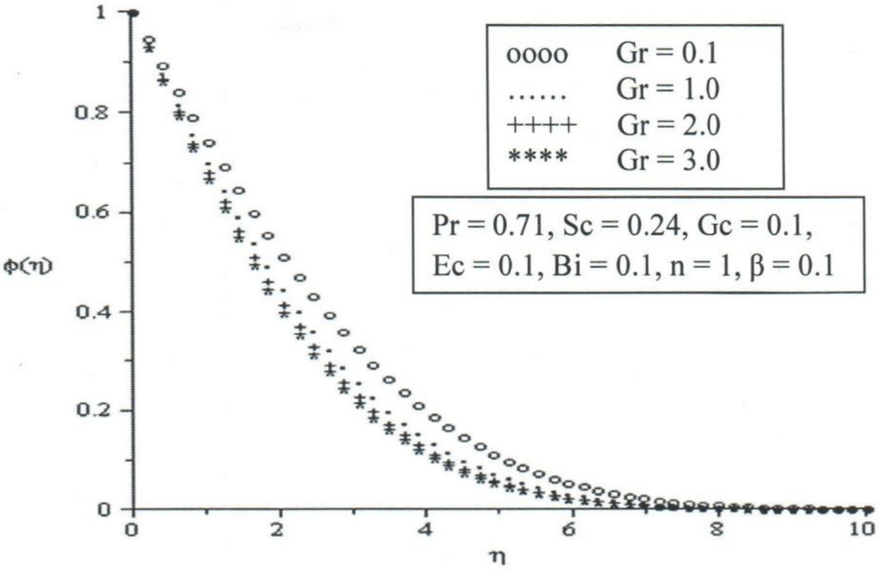

Figure 4. 11 Concentration profiles for varying values of thermal Grashof number (Gr)

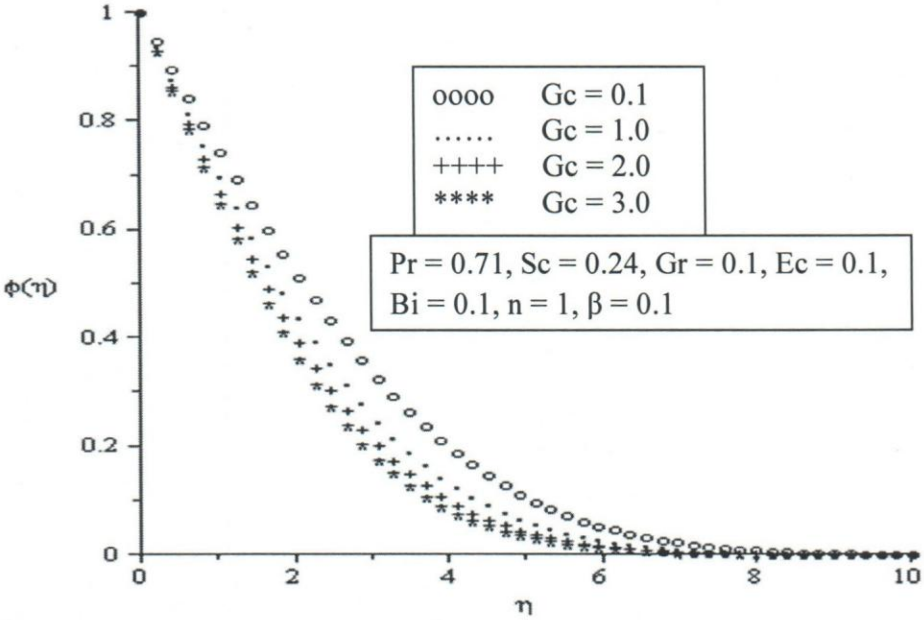

Figure 4. 12 Concentration profiles for varying values of Solutal Grashof number (Gc)

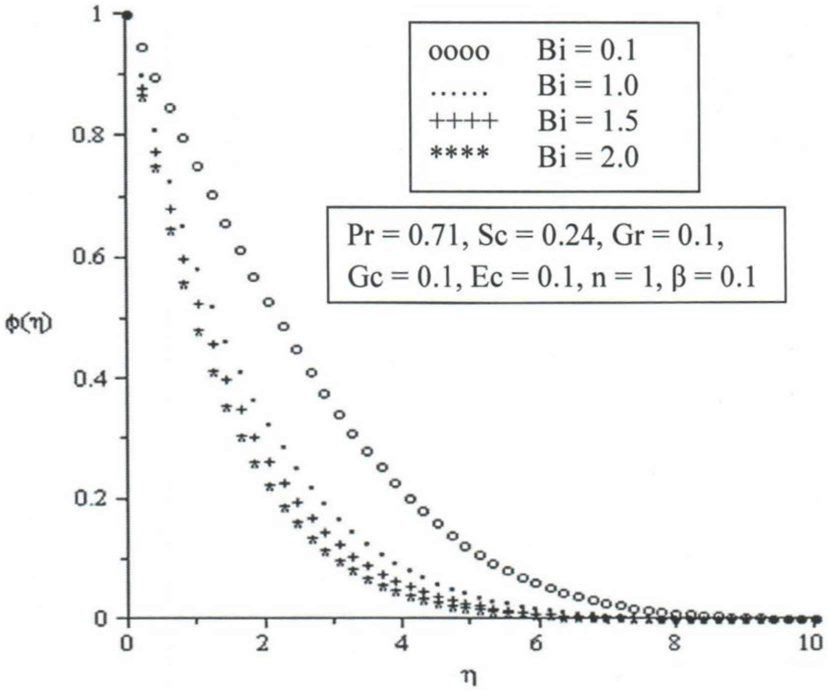

Figure 4. 13 Concentration profiles for varying values of convective heat transfer parameter (*Bi*)

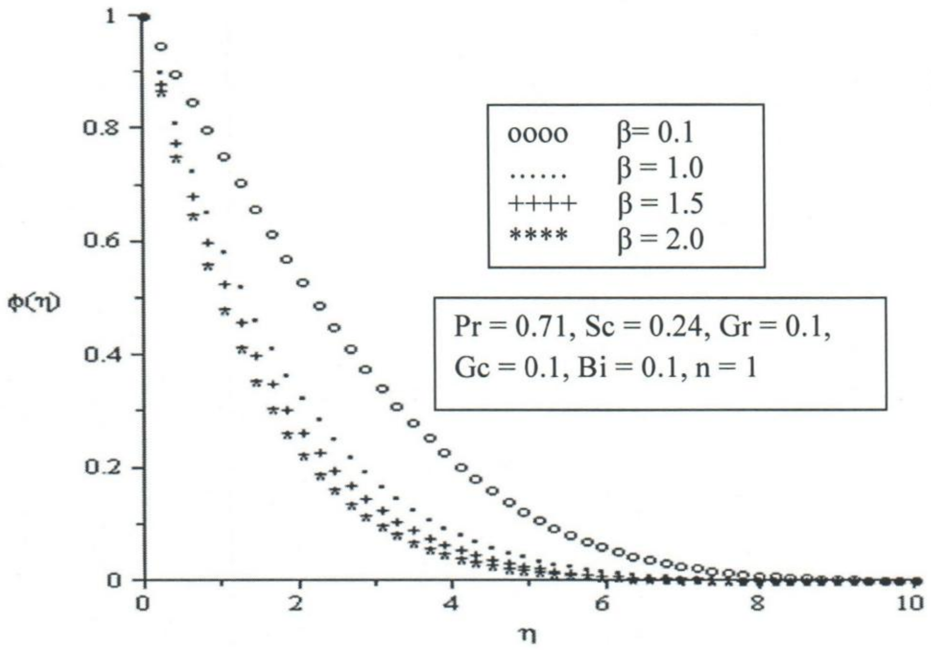

Figure 4. 14 Concentration profiles for varying values of reaction rate parameter (*β*)

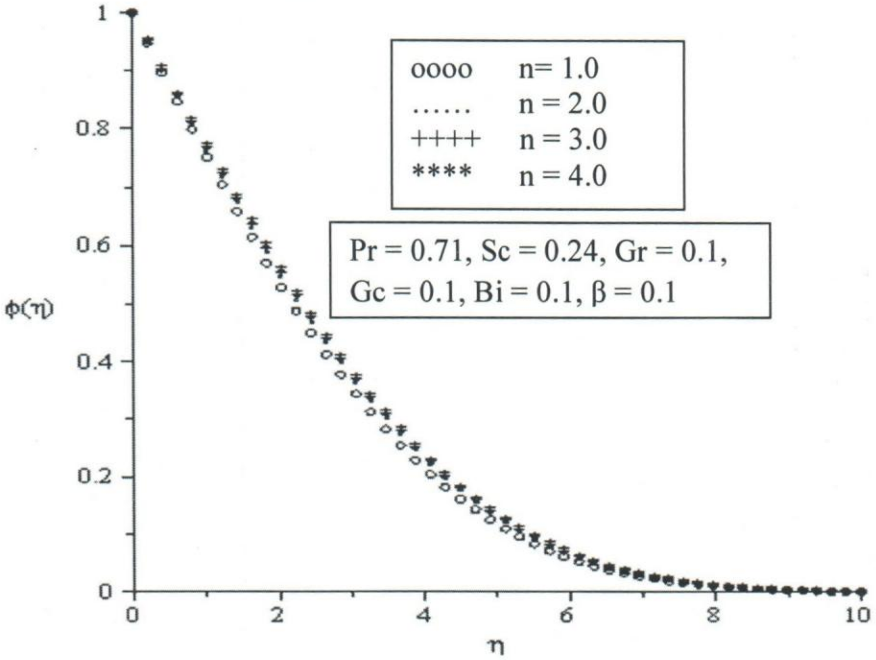

Figure 4. 15 Concentration profiles for varying values of order of chemical reaction (*n*)

4.3 Boundary conditions

It is observed from Figures 4.1 to .4.4 that the velocity starts from a zero value at the plate surface and increases to a peak value and then decreases slightly towards the free stream value far away from the plate surface satisfying the far field velocity boundary condition for all parameter values. Generally, the fluid temperature attains its maximum value at the plate surface and decreases exponentially to the free stream zero value away from the plate satisfying the thermal boundary condition and this is observed in Figures 4.5 to 4.9. Figures 4.10 to 5.15 depict concentration profiles of species against spanwise coordinate (η) for varying values of physical parameters in the boundary layer. The concentration of the species is highest at the plate surface and decreases to zero far away from the plate satisfying the concentration boundary condition.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

Heat and mass transfer over a vertical surface with convective boundary conditions in the presence of viscous dissipation and n^{th} order chemical reaction has been studied. Numerical results have been compared to earlier results published in literature and a perfect agreement was achieved. Among others, the results reveal that:

i. The velocity of the fluid increases with the increase in Eckert, thermal and concentration Grashof numbers. It also decreases with an increase in Schmidt number. This implies proper control of the Eckert, thermal and concentration Grashop numbers will enchance the efficient flow of fluids whilst high Schmidt numbers will impede easy flow.

ii. The temperature of the vertical surface reduces with increasing Prandlt number and increases with increasing Eckert, convective heat transfer parameter, thermal and concentration Grashof numbers. So a major improvement in efficiency of lasers for the treatment of cancer will be obtained if high Prandtl numbers are used.

iii. The concentration boundary layer of the fluid decreases with increase in reaction rate parameter, Schmidt, convective heat transfer parameter, thermal and solutal Grashof numbers; and increases slightly with increasing order of chemical reaction. These facts should be taken into account for the practical application in mass transfer processes.

iv. The Skin friction at the surface increases with the increase in the convective heat transfer parameter, order of chemical reaction, Eckert, thermal and solutal Grashof numbers; and decreases for increasing Prandtl, reaction rate parameter and Schmidt numbers. High skin friction has the potential to impair easy movement of fluid thereby increasing the temperature of surfaces hence should be controlled.

v. The Nusselt number increases with an increase in the convective heat transfer parameter and decreases with increasing: reaction rate parameter, order of chemical reaction, Schmidt, Eckert, Prandtl, thermal and solutal Grashof numbers. An increase in Nusselt number means heat will be lost easily from the system hence an increase in convective heat transfer will be an advantage in efficient cooling.

vi. The Sherwood number increases with an increase in reaction rate parameter, Schmidt, convective heat transfer parameter, Eckert, thermal and solutal Grashof numbers; and decreases with increasing Prandtl and order of chemical reaction. High Sherwood number implies high rate of mass transfer and this can adversely affect the quality of the final product if not properly controlled.

5.2 Recommendations

In conclusion, the following recommendations are made:

i. The chemical reaction parameters of the fluid such as the Schmidt number should be well controlled to achieve desired product characteristics.

ii. In the transportation of oil products through ducts, the effect of viscous dissipation is enormous so it is recommended that the viscous dissipation parameter (that is the Eckert number) should be well controlled to enhance the transportation process to avoid overheating,

iii. This study was conducted on a vertical plate; it is therefore recommended that it should be extended to different geometries such as vertical cylinder or vertical infinite channel.

iv. The research can serve as a reference material to future researchers.

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APPENDIX I

LIST OF PUBLICATIONS FROM THE RESEACH WORK

Imoro R., Arthur E.M. and Seini Y.I. (2014), Heat and Mass Transfer over a Vertical Surface with Convective Boundary Conditions in the Presence of Viscous Dissipation and nth Order Chemical Reaction, International Journal of Computational and Applied Mathematics. ISSN 1819-4966 9(2) , pp. 101-118, http://www.ripublication.com.

APPENDIX II

MAPLE CODE FOR NUMERICAL RESULTS

$$Pr := 0.71 : Sc := 0.24 : Gr := 0.1 : Gc := 0.1 : Ec := 0.1 : Bi := 0.10 : \beta := 0.1 : n := 1 :$$

$$fcns := \{F(y), \theta(y), \phi(y)\} :$$

$$sys1 := diff(F(y), y\$3) + \frac{1}{2}\cdot F(y)\cdot diff(F(y), y\$2) + Gr\cdot\theta(y) + Gc\cdot\phi(y) = 0, diff(\theta(y), y\$2) + \frac{1}{2}\cdot Pr\cdot F(y)\cdot diff(\theta(y), y) + Pr\cdot Ec\cdot diff(F(y), y\$2)^2 = 0, diff(\phi(y), y\$2) + \frac{1}{2}\cdot Sc\cdot F(y)\cdot diff(\phi(y), y) - Sc\cdot\beta\cdot(\phi(y))^n = 0, \mathrm{D}(F)(0) = 0, F(0) = 0, \phi(0) = 1, \mathrm{D}(\theta)(0) = -Bi\cdot(1 - \theta(0)), \mathrm{D}(F)(10) = 1, \phi(10) = 0, \theta(10) = 0 :$$

$$p := dsolve(\{sys1, \mathrm{D}(F)(0) = 0, F(0) = 0, \phi(0) = 1, \mathrm{D}(\theta)(0) = -Bi\cdot(1 - \theta(0)), \mathrm{D}(F)(10) = 1, \phi(10) = 0, \theta(10) = 0\}, fcns, type = numeric, method = bvp, abserr = 1\mathrm{e}-6)$$

proc(*x_bvp*) ... **end proc**

$$dsol1 := dsolve(\{sys1\}, numeric, output = operator)$$

[*y* = **proc**(*y*) ... **end proc**, *F* = **proc**(*y*) ... **end proc**, D(*F*) = **proc**(*y*)

...

end proc, $\mathrm{D}^{(2)}(F)$ = **proc**(*y*) ... **end proc**, ϕ = **proc**(*y*)

...

end proc, D(ϕ) = **proc**(*y*) ... **end proc**, θ = **proc**(*y*) ... **end proc**, D(θ) = **proc**(*y*) ... **end proc**]

$$dsol1(0);$$

$$[y = 0, F(0) = 0., \mathrm{D}(F)(0) = 0., \mathrm{D}^{(2)}(F)(0) = 0.539342718923407792, \phi(0) = 0.999999999999999966, \mathrm{D}(\phi)(0) = -0.255722468099067402, \theta(0) = 0.280711231942019190, \mathrm{D}(\theta)(0) = -0.0719288768057980422]$$

APPENDIX III

MAPLE CODE FOR GRAPHICAL RESULTS

>

$$\begin{array}{l}
\mathit{with}(\mathit{plots}):\\
\mathit{Pr} := 0.71 : \mathit{Sc} := 0.24 : \mathit{Gr} := 0.1 : \mathit{Gc} := 0.10 : \mathit{Ec} := 0.1 : \mathit{Bi}\\
\quad := 0.10 : \beta := 0.1 : n := 1 :\\
\mathit{fcns} := \{F(y), \theta(y), \phi(y)\} :\\
\mathit{sys} := \mathit{diff}(F(y), y\$3) + \frac{1}{2} \cdot F(y) \cdot \mathit{diff}(F(y), y\$2) + \mathit{Gr} \cdot \theta(y)\\
\quad + \mathit{Gc} \cdot \phi(y) = 0, \mathit{diff}(\theta(y), y\$2) + \frac{1}{2} \cdot \mathit{Pr} \cdot F(y) \cdot \mathit{diff}(\theta(y), y)\\
\quad + \mathit{Pr} \cdot \mathit{Ec} \cdot (\mathit{diff}(F(y), y\$2))^2 = 0, \mathit{diff}(\phi(y), y\$2) + \frac{1}{2} \cdot \mathit{Sc}\\
\quad \cdot F(y) \cdot \mathit{diff}(\phi(y), y) - \mathit{Sc} \cdot \beta \cdot (\phi(y))^n = 0 :\\
\mathit{p1} := \mathit{dsolve}(\{\mathit{sys}, \mathrm{D}(F)(0) = 0, F(0) = 0, \phi(0) = 1, \mathrm{D}(\theta)(0) = -\mathit{Bi}\\
\quad \cdot (1 - \theta(0)), \mathrm{D}(F)(10) = 1, \phi(10) = 0, \theta(10) = 0\}, \mathit{fcns}, \mathit{type}\\
\quad = \mathit{numeric}, \mathit{method} = \mathit{bvp}, \mathit{abserr} = 1\text{e-}10) : \mathit{p1t} := \mathit{odeplot}(\mathit{p1},\\
\quad [y, \theta(y)], 0..10, \mathit{numpoints} = 50, \mathit{labels} = [\text{"}\eta\text{"}, \text{"}\theta(\eta)\text{"}], \mathit{style}\\
\quad = \mathit{point}, \mathit{symbol} = \mathit{circle}, \mathit{color} = \mathit{black}) :\\
\mathit{p1f} := \mathit{odeplot}(\mathit{p1}, [y, F'(y)], 0..10, \mathit{numpoints} = 50, \mathit{labels} = [\text{"}\eta\text{"},\\
\quad \text{"f'}(\eta)\text{"}], \mathit{style} = \mathit{point}, \mathit{symbol} = \mathit{circle}, \mathit{color} = \mathit{black}) : \mathit{p1c}\\
\quad := \mathit{odeplot}(\mathit{p1}, [y, \phi(y)], 0..10, \mathit{numpoints} = 50, \mathit{labels} = [\text{"}\eta\text{"},\\
\quad \text{"}\phi(\eta)\text{"}], \mathit{style} = \mathit{point}, \mathit{symbol} = \mathit{circle}, \mathit{color} = \mathit{black}) :
\end{array}$$

$$\begin{array}{l}
\mathit{with}(\mathit{plots}):\\
\mathit{Pr} := 0.71 : \mathit{Sc} := 0.24 : \mathit{Gr} := 0.1 : \mathit{Gc} := 0.1 : \mathit{Ec} := 0.1 : \mathit{Bi}\\
\quad := 0.1 : \beta := 0.1 : n := 2 :\\
\mathit{fcns} := \{F(y), \theta(y), \phi(y)\} :\\
\mathit{sys} := \mathit{diff}(F(y), y\$3) + \frac{1}{2} \cdot F(y) \cdot \mathit{diff}(F(y), y\$2) + \mathit{Gr} \cdot \theta(y)\\
\quad + \mathit{Gc} \cdot \phi(y) = 0, \mathit{diff}(\theta(y), y\$2) + \frac{1}{2} \cdot \mathit{Pr} \cdot F(y) \cdot \mathit{diff}(\theta(y), y)\\
\quad + \mathit{Pr} \cdot \mathit{Ec} \cdot (\mathit{diff}(F(y), y\$2))^2 = 0, \mathit{diff}(\phi(y), y\$2) + \frac{1}{2} \cdot \mathit{Sc}\\
\quad \cdot F(y) \cdot \mathit{diff}(\phi(y), y) - \mathit{Sc} \cdot \beta \cdot (\phi(y))^n = 0 :\\
\mathit{p2} := \mathit{dsolve}(\{\mathit{sys}, \mathrm{D}(F)(0) = 0, F(0) = 0, \phi(0) = 1, \mathrm{D}(\theta)(0) = -B\\
\quad \cdot (1 - \theta(0)), \mathrm{D}(F)(10) = 1, \phi(10) = 0, \theta(10) = 0\}, \mathit{fcns}, \mathit{type}\\
\quad = \mathit{numeric}, \mathit{method} = \mathit{bvp}, \mathit{abserr} = 1\text{e-}10) : \mathit{p2t} := \mathit{odeplot}(\mathit{p2},\\
\quad [y, \theta(y)], 0..10, \mathit{numpoints} = 50, \mathit{labels} = [\text{"}\eta\text{"}, \text{"}\theta(\eta)\text{"}], \mathit{style}\\
\quad = \mathit{point}, \mathit{symbol} = \mathit{point}, \mathit{color} = \mathit{black}) :\\
\mathit{p2f} := \mathit{odeplot}(\mathit{p2}, [y, F'(y)], 0..10, \mathit{numpoints} = 50, \mathit{labels}\\
\quad = [\text{"}\eta\text{"}, \text{"f'}(\eta)\text{"}], \mathit{style} = \mathit{point}, \mathit{symbol} = \mathit{point}, \mathit{color} = \mathit{black}) :
\end{array}$$

$$\begin{array}{l}
\mathit{p2c} := \mathit{odeplot}(\mathit{p2}, [y, \phi(y)], 0..10, \mathit{numpoints} = 50, \mathit{labels} = [\text{"}\eta\text{"},\\
\quad \text{"}\phi(\eta)\text{"}], \mathit{style} = \mathit{point}, \mathit{symbol} = \mathit{point}, \mathit{color} = \mathit{black}) :
\end{array}$$

```
with(plots) :
Pr := 0.71 : Sc := 0.24 : Gr := 0.1 : Gc := 0.1 : Ec := 0.1 : Bi
    := 0.1 : β := 0.1 : n := 3 :
fcns := {F(y), θ(y), ϕ(y)} :
sys := diff(F(y), y$3) + 1/2 · F(y)·diff(F(y), y$2) + Gr·θ(y)
    + Gc·ϕ(y) = 0, diff(θ(y), y$2) + 1/2 · Pr·F(y)·diff(θ(y), y)
    + Pr·Ec·(diff(F(y), y$2))^2 = 0, diff(ϕ(y), y$2) + 1/2 · Sc
    ·F(y)·diff(ϕ(y), y) − Sc·β·(ϕ(y))^n = 0 :
p3 := dsolve({sys, D(F)(0) = 0, F(0) = 0, ϕ(0) = 1, D(θ)(0) = −B
    ·(1 − θ(0)), D(F)(10) = 1, ϕ(10) = 0, θ(10) = 0}, fcns, type
    = numeric, method = bvp, abserr = 1e-10) : p3t := odeplot(p3,
    [y, θ(y)], 0 ..10, numpoints = 50, labels = ["η", "θ(η)"], style
    = point, symbol = cross, color = black) :
p3f := odeplot(p3, [y, F'(y)], 0 ..10, numpoints = 50, labels = ["y",
    "f'(η)"], style = point, symbol = cross, color = black) :


p3c := odeplot(p3, [y, ϕ(y)], 0 ..10, numpoints = 50, labels = ["y",
    "ϕ(η)"], style = point, symbol = cross, color = black) :

with(plots) :
Pr := 0.71 : Sc := 0.24 : Gr := 0.1 : Gc := 0.1 : Ec := 0.1 : Bi
    := 0.1 : β := 0.1 : n := 4 :
fcns := {F(y), θ(y), ϕ(y)} :
sys := diff(F(y), y$3) + 1/2 · F(y)·diff(F(y), y$2) + Gr·θ(y)
    + Gc·ϕ(y) = 0, diff(θ(y), y$2) + 1/2 · Pr·F(y)·diff(θ(y), y)
    + Pr·Ec·(diff(F(y), y$2))^2 = 0, diff(ϕ(y), y$2) + 1/2 · Sc
    ·F(y)·diff(ϕ(y), y) − Sc·β·(ϕ(y))^n = 0 :
p4 := dsolve({sys, D(F)(0) = 0, F(0) = 0, ϕ(0) = 1, D(θ)(0) = −B
    ·(1 − θ(0)), D(F)(10) = 1, ϕ(10) = 0, θ(10) = 0}, fcns, type
    = numeric, method = bvp, abserr = 1e-10) : p4t := odeplot(p4,
    [y, θ(y)], 0 ..10, numpoints = 50, labels = ["η", "θ(η)"], style
    = point, symbol = asterisk, color = black) :
p4f := odeplot(p4, [y, F'(y)], 0 ..10, numpoints = 50, labels
    = ["η", "f'(η)"], style = point, symbol = asterisk, color = black) :


p4c := odeplot(p4, [y, ϕ(y)], 0 ..10, numpoints = 50, labels = ["η",
    "ϕ(η)"], style = point, symbol = asterisk, color = black) :

> plots[display]({p1c, p2c, p3c, p4c});
```