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GENERALIZED ERLANG-TRUNCATED EXPONENTIAL DISTRIBUTION

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Abstract

In this study, a new continuous distribution called the Kumaraswamy Erlang-truncated exponential distribution is introduced and studied. The mathematical properties of the new model such as the quantile function, moments and moment generating function and order statistics are derived. The estimation of the parameters of the model is approached by the method of maximum likelihood. The importance of the model is illustrated by means of application to real data set.

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1. Introduction

The knowledge of the statistical distribution any phenomenon follows, greatly improves the sensitivity, efficiency and the power of the test associated with it. Because of this, considerable efforts over the years have been made in the development of large classes of standard probability distributions along with relevant methodologies.

In recent years, new classes of distributions have been proposed by modifying existing distributions using the Kumaraswamy family of generalized distributions proposed by Cordeiro and de Castro [1] to cope with bathtub failure rates. Among these are: Kumaraswamy linear exponential distribution [8], Kumaraswamy exponentiated Pareto distribution [2] and Kumaraswamy generalized gamma distribution [9]. The Kumaraswamy family has similar properties as the beta-G distribution (see [4]) but has some advantages in terms of tractability, since it does not involve any special function such as the beta function.

In this paper, we combine the works of Kumaraswamy [6] and Cordeiro and de Castro [1] to derive the mathematical properties of a new model, called the *Kumaraswamy Erlang-truncated exponential* (*Kw-ETE*) *distribution*. The Erlang-truncated exponential (ETE) distribution was developed by El-Alosey [3]. A non-negative random variable X is said to have the *ETE distribution* with shape parameter $\beta > 0$ and scale parameter $\lambda > 0$ if its probability density function (PDF) is given by

$$g(x; \beta, \lambda) = \beta(1 - e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}, \quad x > 0. \tag{1}$$

The corresponding cumulative distribution function (CDF) is given by

$$G(x; \beta, \lambda) = 1 - e^{-\beta(1-e^{-\lambda})x}, \quad x > 0. \tag{2}$$

2. Kumaraswamy Erlang-truncated Exponential (Kw-ETE) Distribution

A non-negative random variable X has a Kw-ETE distribution with parameters $\alpha > 0$, $\theta > 0$, $\beta > 0$ and $\lambda > 0$ $(Kw\text{-}ETE(\alpha, \theta, \beta, \lambda))$ if its CDF is given by

$$F_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda) = 1 - [1 - (1 - e^{-\beta(1-e^{-\lambda})x})^{\theta}]^{\alpha}, \quad x > 0. \tag{3}$$

The parameters α, θ and β are shape parameters and the parameter λ is a scale parameter. If the parameter $\alpha = 1$, then we obtain exponentiated Erlang-truncated exponential (EETE) distribution and if $\alpha = 1$, $\theta = 1$, then the ETE distribution is obtained. The corresponding PDF is given by

$$f_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda) = \alpha\theta\beta(1 - e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}[1 - e^{-\beta(1-e^{-\lambda})x}]^{\theta-1}$$

$$\times [1 - (1 - e^{-\beta(1-e^{-\lambda})x})^{\theta}]^{\alpha-1}, \quad x > 0. \tag{4}$$

The survival and the hazard rate functions of the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ are

$$S_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda) = [1 - (1 - e^{-\beta(1-e^{-\lambda})x})^{\theta}]^{\alpha}, \quad x > 0 \tag{5}$$

and

$$h_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda)$$

$$= \frac{\alpha\theta\beta(1 - e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}[1 - e^{-\beta(1-e^{-\lambda})x}]^{\theta-1}}{1 - [1 - e^{-\beta(1-e^{-\lambda})x}]^{\theta}}, \quad x > 0, \tag{6}$$

respectively. The reverse hazard has been shown to play a useful role in reliability analysis (see [5]). The reverse hazard function of the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution is

$$\tau_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda)$$

$$= \frac{\alpha\theta\beta(1 - e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}[1 - e^{-\beta(1-e^{-\lambda})x}]^{\theta-1}[1 - (1 - e^{-\beta(1-e^{-\lambda})x})^{\theta}]^{\alpha-1}}{1 - [1 - (1 - e^{-\beta(1-e^{-\lambda})x})^{\theta}]^{\alpha}},$$

$$x > 0. \tag{7}$$

Figure 1 and Figure 2 display the PDF and hazard function of the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution, for different parameter values, respectively. From the figures, it is obvious that the PDF can be decreasing or unimodal and the hazard can exhibit decreasing, increasing or constant failure rates.

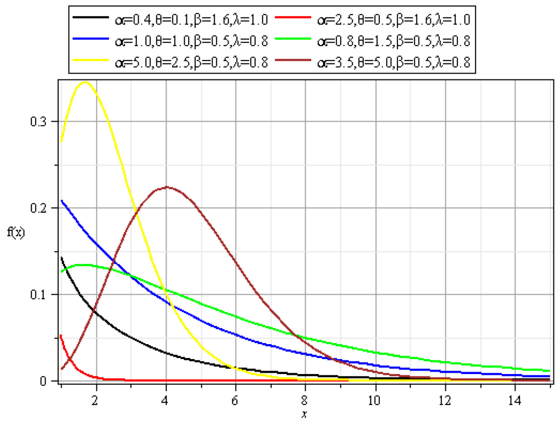

Figure 1. PDF of Kw-ETE distribution.

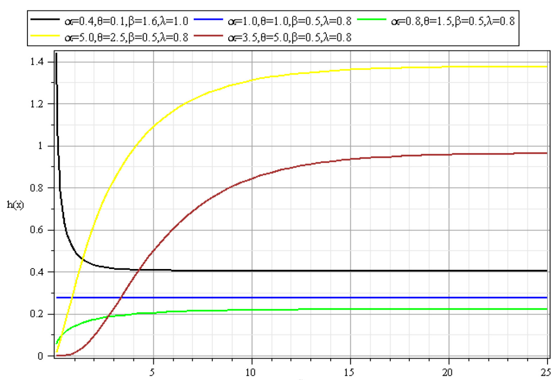

Figure 2. Hazard function plot of Kw-ETE distribution.

The PDF of the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution can be written as a linear combination of the PDFs of the ETE distribution. This result is important in providing the mathematical properties of the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ model directly from those properties of the ETE distribution. For $d > 0,$ a series expansion for $(1 - z)^{d-1},$ for $|z| < 1$ is

$$(1-z)^{d-1} = \sum_{k=0}^{\infty}(-1)^k \binom{d-1}{k} z^k = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(d)}{k!\,\Gamma(d-k)} z^k, \tag{8}$$

where $\Gamma(\cdot)$ is the gamma function. Since $0 < e^{-\beta(1-e^{-\lambda})x} < 1$, for $x > 0$, using the series expansion (8) in (4) yields

$$f_{Kw\text{-}ETE}(x;\, \alpha,\, \theta,\, \beta,\, \lambda)$$

$$= \theta\beta(1-e^{-\lambda}) \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\Gamma(\alpha+1)\Gamma(\theta(k+1))}{k!\, j!\, \Gamma(\alpha-k)\Gamma(\theta(k+1)-j)} e^{-\beta(j+1)(1-e^{-\lambda})x},\ x > 0$$

$$= \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))}{k!\,(j+1)!\,\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)} f_{ETE}(x;\, \beta_{j+1},\, \lambda), \tag{9}$$

where $f_{ETE}(x;\, \beta_{j+1},\, \lambda)$ is the PDF of the ETE distribution with shape parameter $\beta_{j+1} = \beta(j+1)$ and scale parameter λ. When $\alpha > 0$ is an integer, the index k stops at $\alpha - 1$ and when $\theta(k+1) > 0$ is an integer, the j stops at $\theta(k+1) - 1$.

3. Statistical Properties

In this section, the statistical properties of the newly developed distribution were derived.

3.1. Quantile, median and mode

The characteristics of a distribution such as the median, skewness and kurtosis can be studied through the quantile function of the distribution. The quantile function of a distribution can also be used to generate random numbers from the distribution. The $Kw\text{-}ETE(\alpha,\, \theta,\, \beta,\, \lambda)$ quantile function, say $Q(p) = F^{-1}(p)$, is straightforward and to be computed by inverting (3). The pth quantile is given by

$$x_p = \frac{1}{\beta(1 - e^{-\lambda})} \ln\left[\frac{1}{1 - (1 - (1 - p)^{\frac{1}{\alpha}})^{\frac{1}{\theta}}}\right] \tag{10}$$

which is used for data generation from the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution. The random variable *p* is uniformly distributed on the $(0, 1)$ interval. Using (10), the median of the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution can be obtained as

$$x_{0.5} = \frac{1}{\beta(1 - e^{-\lambda})} \ln\left[\frac{1}{1 - (1 - (0.5)^{\frac{1}{\alpha}})^{\frac{1}{\theta}}}\right]. \tag{11}$$

The mode, which is defined as the maximum value of the PDF, denoted by x_0 can be obtained by numerically solving the following non-linear equation (12) since it is not possible to obtain the explicit solution in the general case: For different special cases, the explicit form may be obtained:

$$\frac{(\theta - 1)\beta(1 - e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}}{1 - e^{-\beta(1-e^{-\lambda})x}}$$

$$- \frac{(\alpha - 1)\theta\beta(1 - e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}[1 - e^{-\beta(1-e^{-\lambda})x}]^{\theta-1}}{1 - (1 - e^{-\beta(1-e^{-\lambda})x})^{\theta}} = \beta(1 - e^{-\lambda}). \tag{12}$$

3.2. Moments

It is customary to derive the moments when a new distribution is proposed. Moments play an important role in any statistical analysis, especially in applications. They are used for finding measures of central tendency, dispersion, skewness and kurtosis among others. In this subsection, the *r*th non-central moment for the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution was derived.

Proposition 1. *If X has a* $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ *distribution, then the rth non-central moment of X is given by the following*:

$$\mu'_r = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))\Gamma(r+1)}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)[\beta_{j+1}(1-e^{-\lambda})]^r}, \quad r = 1, 2, \ldots. \tag{13}$$

Proof. Let X be a random variable having density function (4). The rth non-central moment of $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution is given by

$$\mu'_r = E(X^r) = \int_0^{\infty} x^r f_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda)dx.$$

Using (9),

$$\mu'_r = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)}$$

$$\times \int_0^{\infty} x^r f_{ETE}(x; \beta_{j+1}, \lambda)dx. \tag{14}$$

Then

$$\mu'_r = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)}$$

$$\times \int_0^{\infty} x^r \beta_{j+1}(1-e^{-\lambda})e^{-\beta_{j+1}(1-e^{-\lambda})x}dx.$$

Now, define the following substitution:

$$y = \beta_{j+1}(1-e^{-\lambda})x \Rightarrow dy = \beta_{j+1}(1-e^{-\lambda})dx.$$

Clearly,

$$x = \frac{y}{\beta_{j+1}(1-e^{-\lambda})}.$$

Thus,

$$\mu'_r = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)} \int_0^{\infty} \left[\frac{y}{\beta_{j+1}(1-e^{-\lambda})}\right]^r e^{-y} dy$$

$$= \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))\Gamma(r+1)}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)[\beta_{j+1}(1-e^{-\lambda})]^r}, \; r = 1, 2, \ldots.$$

This completes the proof.

The mean of the random variable X is obtained by putting $r = 1$ in (13). Hence, the mean is

$$\mu = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)\beta_{j+1}(1-e^{-\lambda})}. \tag{15}$$

The second non-central moment of the random variable X is obtained by putting $r = 2$ in (13). Hence, the second non-central moment is

$$\mu'_2 = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))\Gamma(3)}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)[\beta_{j+1}(1-e^{-\lambda})]^2}. \tag{16}$$

The variance of the random variable X is given by

$$\sigma^2 = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))\Gamma(3)}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)[\beta_{j+1}(1-e^{-\lambda})]^2} - \mu^2. \tag{17}$$

Based on the first four non-central moments of the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution, the coefficient of skewness and kurtosis can be obtained as

$$Skewness(X) = \frac{\mu'_3 - 3\mu'_2\mu + 2\mu^3}{(\mu'_2 - \mu^2)^{\frac{3}{2}}}$$

and

$$Kurtosis(X) = \frac{\mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4}{(\mu'_2 - \mu^2)^2},$$

respectively.

3.3. Moment generating function

In this subsection, the moment generating function of a random variable X having a $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution was derived.

Proposition 2. *If X has* $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ *distribution, then the moment generating function* $M_X(t)$ *has the form*

$$M_X(t) = \sum_{j,k,r=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))\Gamma(r+1)t^r}{r!k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)[\beta_{j+1}(1-e^{-\lambda})]^r}. \quad (18)$$

Proof. The moment generating function is obtained using the definition

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda)dx. \quad (19)$$

Using the Taylor series expansion of e^{tx}, (19) can be written as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$$

$$= \sum_{j,k,r=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))\Gamma(r+1)t^r}{r!k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)[\beta_{j+1}(1-e^{-\lambda})]^r}.$$

This completes the proof.

3.4. Incomplete moment

In this subsection, the incomplete moment for a random variable X having a $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution was derived. The incomplete moment is useful in calculating the mean and median deviations, and measures of inequalities such as the Lorenz and Bonferroni curves.

Proposition 3. *If* X *has* $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ *distribution, then the incomplete moment* $M_r(z)$ *has the form*

$$M_r(z) = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))\gamma(r+1, \beta_{j+1}(1-e^{-\lambda})z)}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)[\beta_{j+1}(1-e^{-\lambda})]^r},$$

$$r = 1, 2, \ldots, \tag{20}$$

where $\gamma(\vartheta, z) = \int_0^z x^{\vartheta-1}e^{-x}dx$ *is the lower incomplete gamma function.*

Proof. Let X be a random variable having density function (4). The incomplete moment of the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution is given by

$$M_r(z) = \int_0^z x^r f_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda)dx.$$

Using (9),

$$M_r(z) = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)}$$

$$\times \int_0^z x^r f_{ETE}(x; \beta_{j+1}, \lambda)dx$$

$$= \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)}$$

$$\times \int_0^z x^r \beta_{j+1}(1-e^{-\lambda})e^{-\beta_{j+1}(1-e^{-\lambda})x}dx. \tag{21}$$

Using similar concept for proving the moments

$$M_r(z) = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)}$$

$$\times \int_0^{\beta_{j+1}(1-e^{-\lambda})z} \left[\frac{y}{\beta_{j+1}(1-e^{-\lambda})}\right]^r e^{-y}dy.$$

Thus,

$$M_r(z) = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))\gamma(r+1, \beta_{j+1}(1-e^{-\lambda})z)}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)[\beta_{j+1}(1-e^{-\lambda})]^r}, \quad r = 1, 2, \ldots.$$

This completes the proof.

3.5. Mean and median deviations

Let $X \sim Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$. The amount of scatter in X is evidently measured by the totality of deviations from the mean and median. They are known as the mean deviation and median deviation defined by

$$\delta_1(X) = \int_0^\infty |x - \mu| f_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda)dx$$

and

$$\delta_2(X) = \int_0^\infty |x - \varphi| f_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda)dx,$$

respectively, $\mu = E(X)$ and φ is the median of X. The measures $\delta_1(X)$ and $\delta_2(X)$ can be determined by $\delta_1(X) = 2\mu F_{EETE}(\mu) - 2M_1(\mu)$ and $\delta_2(X) = \mu - 2M_1(\varphi)$. It is easy to compute $M_1(\mu)$ and $M_1(\varphi)$ from (20).

3.6. Inequality measures

The Lorenz and Bonferroni curves have many applications not only in economics to study income and poverty but also in other fields like reliability, medicine and insurance. The Lorenz curve, $L_F(x)$ can be defined as the proportion of total income volume accumulated by those units with income lower than or equal to the volume, and the Bonferroni curve, $B_F(x)$ is the scaled conditional mean curve, that is the ratio of group mean income of the population.

Proposition 4. *If a random variable X has a* $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ *distribution, then the Lorenz curve* $L_F(x)$ *is given by*

$$L_F(x) = \frac{1}{\mu}\sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))\gamma(2, \beta_{j+1}(1-e^{-\lambda})z)}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)\beta_{j+1}(1-e^{-\lambda})}. \quad (22)$$

Proof. By definition, the Lorenz curve can be obtained using the relationship

$$L_F(x) = \frac{\int_0^x tf(t)dt}{\mu}.$$

The integral $\int_0^x tf(t)dt$ is the first incomplete moment which can be obtained from (20). Thus, this proof is complete.

Proposition 5. *If a random variable X has a* $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ *distribution, then the Bonferroni curve* $B_F(x)$ *is given by*

$$B_F(x) = \frac{1}{\mu F_{Kw\text{-}ETE}(x; \alpha, \theta, \beta, \lambda)}$$

$$\times \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\theta\Gamma(\alpha+1)\Gamma(\theta(k+1))\gamma(2, \beta_{j+1}(1-e^{-\lambda})z)}{k!(j+1)!\Gamma(\alpha-k)\Gamma(\theta(k+1)-j)\beta_{j+1}(1-e^{-\lambda})}. \quad (23)$$

Proof. The proof can easily be obtained from the relationship

$$B_F(x) = \frac{L_F(x)}{F(x)},$$

where $F(x)$ is the CDF. This completes the proof.

3.7. Entropy

Statistical entropy is a probabilistic measure of uncertainty or ignorance about the outcome of a random experiment and is a measure of reduction in that uncertainty. Various entropy and information indices exist, among them the Rényi entropy has been developed and used in many disciplines and context. For a random variable X having a PDF $f(x)$, the Rényi entropy is defined by

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_{\Re} f^{\delta}(x)dx\right] \quad (24)$$

for $\delta > 0$ and $\delta \neq 1$.

Proposition 6. *If a random variable X has a Kw-ETE*$(\alpha, \theta, \beta, \lambda)$ *distribution, then the Rényi entropy is given by*

$$I_R(\delta) = \frac{\delta}{1-\delta}\log(\alpha\theta) - \log[\beta(1-e^{-\lambda})]$$

$$+\frac{1}{1+\delta}\log\left[\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}\Gamma(\delta(\alpha-1)+1)\Gamma(\theta(\delta+k)-\delta+1)}{j!k!(\delta+j)\Gamma(\delta(\alpha-1)-k+1)\Gamma(\theta(\delta+k)-\delta-j+1)}\right]. \tag{25}$$

Proof. Using (24), the Rényi entropy is given by

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_0^\infty \{\alpha\theta\beta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}[1-e^{-\beta(1-e^{-\lambda})x}]^{\theta-1}\right.$$

$$\left.\times[1-(1-e^{-\beta(1-e^{-\lambda})x})^{\theta}]^{\alpha-1}\}^{\delta}dx\right]. \tag{26}$$

Using the binomial expansion, (26) can be written as

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[(\alpha\theta)^{\delta}[\beta(1-e^{-\lambda})]^{\delta}\right.$$

$$\times\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}\Gamma(\delta(\alpha-1)+1)\Gamma(\theta(\delta+k)-\delta+1)}{j!k!\Gamma(\delta(\alpha-1)-k+1)\Gamma(\theta(\delta+k)-\delta-j+1)}$$

$$\left.\times\int_0^\infty e^{-\beta(\delta+k)(1-e^{-\lambda})x}dx\right]$$

$$= \frac{1}{1-\delta}\log\left[(\alpha\theta)^{\delta}[\beta(-e^{-\lambda})]^{\delta}\right.$$

$$\left.\times\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}\Gamma(\delta(\alpha-1)+1)\Gamma(\theta(\delta+k)-\delta+1)}{j!k!\beta(1-e^{-\lambda})(\delta+j)\Gamma(\delta(\alpha-1)-k+1)\Gamma(\theta(\delta+k)-\delta-j+1)}\right].$$

Thus,

$$I_R(\delta) = \frac{\delta}{1-\delta}\log(\alpha\theta) - \log[\beta(1-e^{-\lambda})]$$

$$+\frac{1}{1-\delta}\log\left[\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}\Gamma(\delta(\alpha-1)+1)\Gamma(\theta(\delta+k)-\delta+1)}{j!k!(\delta+j)\Gamma(\delta(\alpha-1)-k+1)\Gamma(\theta(\delta+k)-\delta-j+1)}\right].$$

This completes the proof.

3.8. Reliability

The estimation of reliability is important in stress-strength models. If X_1 is the strength of a component and X_2 is the stress, then the component fails when $X_2 > X_1$. Then the estimation of the reliability of the component R is $P(X_2 < X_1)$. When X_1 and X_2 are distributed independently as

$$X_1 \sim Kw\text{-}ETE(\alpha_1, \theta_1, \beta_1, \lambda_1) \quad \text{and} \quad X_2 \sim Kw\text{-}ETE(\alpha_2, \theta_2, \beta_2, \lambda_2),$$

then the reliability is given by

$$R = \int_0^{\infty} f_1(x)F_2(x)dx = 1 - \int_0^{\infty} f_1(x)S_2(x)dx. \tag{27}$$

Proposition 7. *If X_1 and X_2 are the strength and stress of a component, respectively, and are distributed independently as*

$$X_1 \sim Kw\text{-}ETE(\alpha_1, \theta_1, \beta_1, \lambda_1) \quad \textit{and} \quad X_2 \sim Kw\text{-}ETE(\alpha_2, \theta_2, \beta_2, \lambda_2),$$

then the reliability of the component is given by

$$R = 1 - \theta_1\beta_1(1-e^{-\lambda_1})\sum_{i,j,k,\ell=0}^{\infty}\omega_{i,j,k,\ell}\frac{1}{[\beta_1(j+1)(1-e^{-\lambda_1})+\beta_2\ell(1-e^{-\lambda_2})]}, \tag{28}$$

where

$$\omega_{i,j,k,\ell} = \frac{(-1)^{i+j+k+\ell}\Gamma(\alpha_1+1)\Gamma(\theta_1(k+1))\Gamma(\alpha_2+1)\Gamma(\theta_2 i+1)}{i!\,j!\,k!\,\ell!\,\Gamma(\alpha_1-k)\Gamma(\theta_1(k+1)-j)\Gamma(\alpha_2-i+1)\Gamma(\theta_2 i-\ell+1)}.$$

Proof. Using (27),

$$R = 1 - \int_0^\infty f_1(x) S_2(x) dx,$$

$$f_1(x) = \alpha_1\theta_1\beta_1(1 - e^{-\lambda_1}) e^{-\beta_1(1-e^{-\lambda_1})x} (1 - e^{-\beta_1(1-e^{-\lambda_1})x})^{\theta_1 - 1}$$

$$\times [1 - (1 - e^{-\beta_1(1-e^{-\lambda_1})x})^{\theta_1}]^{\alpha_1 - 1} \quad (29)$$

and

$$S_2(x) = [1 - (1 - e^{-\beta_2(1-e^{-\lambda_2})x})^{\theta_2}]^{\alpha_2}. \quad (30)$$

Using the binomial series expansion, (27) can be written as

$$R = 1 - \theta_1\beta_1(1 - e^{-\lambda_1}) \sum_{i, j, k, \ell=0}^{\infty} \omega_{i, j, k, \ell} \int_0^\infty e^{-[\beta_1(j+1)(1-e^{-\lambda_1}) + \beta_2\ell(1-e^{-\lambda_2})]x} dx, \quad (31)$$

where

$$\omega_{i, j, k, \ell} = \frac{(-1)^{i+j+k+\ell}\,\Gamma(\alpha_1 + 1)\Gamma(\theta_1(k+1))\Gamma(\alpha_2 + 1)\Gamma(\theta_2 i + 1)}{i!\, j!\, k!\, \ell!\, \Gamma(\alpha_1 - k)\Gamma(\theta_1(k+1) - j)\Gamma(\alpha_2 - i + 1)\Gamma(\theta_2 i - \ell + 1)}.$$

Thus,

$$R = 1 - \theta_1\beta_1(1 - e^{-\lambda_1}) \sum_{i, j, k, \ell=0}^{\infty} \omega_{i, j, k, \ell} \frac{1}{[\beta_1(j+1)(1 - e^{-\lambda_1}) + \beta_2\ell(1 - e^{-\lambda_2})]}.$$

This completes the proof.

3.9. Probability weighted moments

In this subsection, the probability weighted moments (PWMs) of the $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution were derived. The PWMs method can be used generally to estimate parameters of a distribution whose inverse form cannot be expressed explicitly. For a random variable X with PDF $f(x)$ and CDF $F(x)$, the (r, s)th PWM of X $(\text{for } r \geq 1,\ s \geq 0)$ is formally defined as

$$\rho_{r,s} = E[X^r, F^s(x)] = \int_0^\infty x^r F^s(x) f(x) dx. \tag{32}$$

Proposition 8. *If a random variable X has a Kw-ETE*$(\alpha, \theta, \beta, \lambda)$ *distribution, then the* (r, s)th *PWM of X* $($*for* $r \geq 1,\ s \geq 0)$ *is given by*

$$\rho_{r,s} = \alpha\theta \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(s+1)\Gamma(\alpha(k+1))\Gamma(\theta(j+1))\Gamma(r+1)}{(i+1)!\, j!\, k!\, \Gamma(s-k+1)\Gamma(\alpha(k+1)-j)\Gamma(\theta(j+1)-i)[\beta(i+1)(1-e^{-\lambda})]^r}. \tag{33}$$

Proof. Using (32),

$$\rho_{r,s} = E[X^r, F^s(x)] = \int_0^\infty x^r F^s(x) f(x) dx,$$

$$F^s(x) = [1-(1-(1-e^{-\beta(1-e^{-\lambda})x})^{\theta})^{\alpha}]^s.$$

Using the binomial series expansion, (32) can be written as

$$\rho_{r,s} = \alpha\theta\beta(1-e^{-\lambda})$$

$$\times \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(s+1)\Gamma(\alpha(k+1))\Gamma(\theta(j+1))}{i!\, j!\, k!\, \Gamma(s-k+1)\Gamma(\alpha(k+1)-j)\Gamma(\theta(j+1)-i)}$$

$$\times \int_0^\infty x^r e^{-\beta(i+1)(1-e^{-\lambda})x} dx. \tag{34}$$

Now, define the following substitution:

$$y = \beta(i+1)(1-e^{-\lambda})x \Rightarrow \beta(1-e^{-\lambda})dx = \frac{dy}{(i+1)}.$$

Clearly,

$$x = \frac{y}{\beta(i+1)(1-e^{-\lambda})},$$

$$\rho_{r,s} = \alpha\theta \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(s+1)\Gamma(\alpha(k+1))\Gamma(\theta(j+1))}{i!\,j!\,k!\,\Gamma(s-k+1)\Gamma(\alpha(k+1)-j)\Gamma(\theta(j+1)-i)}$$

$$\times \int_0^{\infty} \left(\frac{y}{\beta(i+1)(1-e^{-\lambda})}\right)^r e^{-y} \frac{dy}{(i+1)}$$

$$= \alpha\theta \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(s+1)\Gamma(\alpha(k+1))\Gamma(\theta(j+1))\Gamma(r+1)}{(i+1)!\,j!\,k!\,\Gamma(s-k+1)\Gamma(\alpha(k+1)-j)\Gamma(\theta(j+1)-i)[\beta(i+1)(1-e^{-\lambda})]^r}.$$

This completes the proof.

4. Distribution of Order Statistics

In this subsection, the PDF of the *i*th order statistic was derived. Let $X_1, X_2, \ldots, X_n$ be a random sample from an EETE distribution and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the corresponding order statistics obtained from the sample. Then the PDF, $f_{i:n}(x)$, of the *i*th order statistic $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{1}{B(i,\, n-i+1)}[F(x)]^{i-1}[1-F(x)]^{n-i} f(x),$$

where $F(x)$ and $f(x)$ are the CDF and PDF given by (3) and (4), respectively, and $B(\cdot,\, \cdot)$ is the beta function. Since $0 < F(x) < 1$ for $x > 0$, by using the binomial series expansion of $[1-F(x)]^{n-i}$, given by

$$[1-F(x)]^{n-i} = \sum_{k=0}^{n-i}(-1)^k \binom{n-i}{k}[F(x)]^k,$$

we have

$$f_{i:n}(x) = \frac{1}{B(i,\, n-i+1)} \sum_{k=0}^{n-i}(-1)^k \binom{n-i}{k}[F(x)]^{i+k-1} f(x), \tag{35}$$

therefore substituting (3) and (4) into (35), one gets

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \sum_{j=0}^{i+k-1} \frac{(-1)^{j+k} \Gamma(n+1)\Gamma(i+k)}{k!\,(j+1)!\,(i-1)!\,(n-i-k)!} \times f_{Kw\text{-}ETE}(x;\, \alpha_{j+1},\, \theta,\, \beta,\, \lambda), \tag{36}$$

where $f_{Kw\text{-}ETE}(x;\, \alpha_{j+1},\, \theta,\, \beta,\, \lambda)$ is the PDF of the Kw-ETE distribution with parameters $\alpha_{j+1} = \alpha(j+1),\, \theta,\, \beta$ and λ. Relation (36) revealed that $f_{i:n}(x)$ is the weighted average of the Kw-ETE distribution with different shape parameters.

Proposition 9. *The rth non-central moment of the ith order statistic* $X_{i:n}$ *is given by*

$$\mu_r'^{(i:n)} = \theta \sum_{m,\ell=0}^{\infty} \sum_{k=0}^{n-i} \sum_{j=0}^{i+k-1} \varpi_{m,\ell,j,k} \frac{\Gamma(r+1)}{[\beta(m+1)(1-e^{-\lambda})]^r},\ r = 1, 2, \ldots, \tag{37}$$

where

$$\varpi_{m,\ell,j,k} = \frac{(-1)^{j+\ell+k+m} \Gamma(n+1)\Gamma(i+k)\Gamma(\alpha_{j+1}+1)\Gamma(\theta(\ell+1))}{k!\,\ell!\,m!\,(j+1)!\,(i-1)!\,(n-i-k)!\,\Gamma(\alpha_{j+1}-\ell)\Gamma(\theta(\ell+1)-m)}.$$

Remark. The proof for Proposition 9 can be derived by using the concept for the *rth* non-central moment of the Kw-ETE distribution.

5. Estimation and Inference

In this section, the method of maximum likelihood estimation was established for estimating the parameters of the Kw-ETE distribution developed. Let $X_1, X_2, \ldots, X_n$ be a random sample with observed values $x_1, x_2, \ldots, x_n$ from Kw-ETE distribution with parameters α, θ, β and λ. Let $\Theta = (\alpha, \theta, \beta, \lambda)^T$ be the parameter vector. The log-likelihood function is given by

$$\ell = n\ln\alpha + n\ln\theta + n\ln\beta + n\ln(1-e^{-\lambda})$$

$$-\beta(1-e^{-\lambda})\sum_{i=1}^{n} x_i + (\theta-1)\sum_{i=1}^{n}\ln[1-e^{-\beta(1-e^{-\lambda})x_i}]$$

$$+(\alpha-1)\sum_{i=1}^{n}\ln[1-(1-e^{-\beta(1-e^{-\lambda})x_i})^{\theta}]. \quad (38)$$

The associated score functions are given by

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln[1-(1-e^{-\beta(1-e^{-\lambda})x_i})^{\theta}], \quad (39)$$

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln[1-e^{-\beta(1-e^{-\lambda})x_i}]$$

$$-(\alpha-1)\sum_{i=1}^{n}\frac{(1-e^{-\beta(1-e^{-\lambda})x_i})^{\theta}\ln[1-e^{-\beta(1-e^{-\lambda})x_i}]}{1-(1-e^{-\beta(1-e^{-\lambda})x_i})^{\theta}}, \quad (40)$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - (1-e^{-\lambda})\sum_{i=1}^{n}x_i + (\theta-1)\sum_{i=1}^{n}\frac{x_i(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x_i}}{1-e^{-\beta(1-e^{-\lambda})x_i}}$$

$$-(\alpha-1)\sum_{i=1}^{n}\frac{\theta x_i(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x_i}(1-e^{-\beta(1-e^{-\lambda})x_i})^{\theta-1}}{1-(1-e^{-\beta(1-e^{-\lambda})x_i})^{\theta}}, \quad (41)$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{ne^{-\lambda}}{1-e^{-\lambda}} - \beta e^{-\lambda}\sum_{i=1}^{n}x_i + (\theta-1)\sum_{i=1}^{n}\frac{\beta x_i e^{-\lambda-\beta(1-e^{-\lambda})x_i}}{1-e^{-\beta(1-e^{-\lambda})x_i}}$$

$$-(\alpha-1)\sum_{i=1}^{n}\frac{\theta\beta x_i e^{-\lambda-\beta(1-e^{-\lambda})x_i}(1-e^{-\beta(1-e^{-\lambda})x_i})^{\theta-1}}{1-(1-e^{-\beta(1-e^{-\lambda})x_i})^{\theta}}. \quad (42)$$

The maximum likelihood estimate of $\boldsymbol{\Theta}$, say $\hat{\boldsymbol{\Theta}}$, is obtained by equating (39), (40), (41) and (42) to zero and solving the non-linear system of equations numerically. For interval estimation and hypothesis tests on the model parameters, the information matrix is required. The information matrix is given by

$$I_n(\boldsymbol{\Theta}) = -\begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\theta} & I_{\alpha\beta} & I_{\alpha\lambda} \\ I_{\theta\alpha} & I_{\theta\theta} & I_{\theta\beta} & I_{\theta\lambda} \\ I_{\beta\alpha} & I_{\beta\theta} & I_{\beta\beta} & I_{\beta\lambda} \\ I_{\lambda\alpha} & I_{\lambda\theta} & I_{\lambda\beta} & I_{\lambda\lambda} \end{pmatrix}.$$

Let $G(x_i) = 1 - e^{-\beta(1-e^{-\lambda})x_i}$. Then

$$I_{\alpha\alpha} = \frac{\partial^2 \ell}{\partial^2 \alpha} = -\frac{n}{\alpha^2}, \tag{43}$$

$$I_{\alpha\theta} = \frac{\partial^2 \ell}{\partial\alpha\partial\theta} = -\sum_{i=1}^{n} \frac{[G(x_i)]^{\theta} \ln[G(x_i)]}{1-[G(x_i)]^{\theta}}, \tag{44}$$

$$I_{\alpha\beta} = \frac{\partial^2 \ell}{\partial\alpha\partial\beta} = -\sum_{i=1}^{n} \frac{x_i\theta(1-e^{-\lambda})[GG(x_i)]^{\theta-1} e^{-\beta(1-e^{-\lambda})x_i}}{1-[G(x_i)]^{\theta}}, \tag{45}$$

$$I_{\alpha\lambda} = \frac{\partial^2 \ell}{\partial\alpha\partial\lambda} = -\sum_{i=1}^{n} \frac{x_i\theta\beta(1-e^{-\lambda})[G(x_i)]^{\theta-1} e^{-\lambda-\beta(1-e^{-\lambda})x_i}}{1-[G(x_i)]^{\theta}}, \tag{46}$$

$$I_{\theta\theta} = \frac{\partial^2 \ell}{\partial^2 \theta} = -\frac{n}{\theta^2} + (\alpha-1)\sum_{i=1}^{n}\left[-\frac{[G(x_i)]^{2\theta}\ln[G(x_i)]^2}{(1-[G(x)]^{\theta})^2} - \frac{[G(x)]^{\theta}\ln[G(x_i)]^2}{1-[G(x)]^{\theta}}\right], \tag{47}$$

$$I_{\theta\beta} = \frac{\partial^2 \ell}{\partial\theta\partial\beta} = \sum_{i=1}^{n} \frac{x_i(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x_i}}{G(x_i)}$$

$$+ (\alpha - 1)\sum_{i=1}^{n}\left[-\frac{x_i(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x_i}[G(x_i)]^{\theta-1}}{1-[G(x)]^{\theta}}\right.$$

$$-\frac{x_i\theta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x_i}[G(x_i)]^{2\theta-1}\ln[G(x_i)]}{(1-[G(x_i)]^{\theta})^2}$$

$$\left.-\frac{x_i\theta(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x_i}[G(x_i)]^{\theta-1}\ln[G(x_i)]}{1-[G(x_i)]^{\theta}}\right], \tag{48}$$

$$I_{\theta\lambda} = \frac{\partial^2 \ell}{\partial\theta\partial\lambda} = \sum_{i=1}^{n} \frac{x_i\beta e^{-\lambda-\beta(1-e^{-\lambda})x_i}}{G(x_i)}$$

$$+ (\alpha - 1)\sum_{i=1}^{n}\left[-\frac{x_i\beta[G(x_i)]^{\theta-1}e^{-\lambda-\beta(1-e^{-\lambda})x_i}}{1-[G(x_i)]^{\theta}}\right.$$

$$-\frac{x_i\theta\beta[G(x_i)]^{2\theta-1}e^{-\lambda-\beta(1-e^{-\lambda})x_i}\ln[G(x_i)]}{(1-[G(x_i)]^{\theta})^2}$$

$$\left.-\frac{x_i\theta\beta[G(x_i)]^{\theta-1}e^{-\lambda-\beta(1-e^{-\lambda})x_i}\ln[G(x_i)]}{1-[G(x_i)]^{\theta}}\right], \tag{49}$$

$$I_{\beta\beta} = \frac{\partial^2 \ell}{\partial^2\beta} = -\frac{n}{\beta^2} + (\theta - 1)$$

$$\times \sum_{i=1}^{n}\left[-\frac{x_i^2(1-e^{-\lambda})^2 e^{-2\beta(1-e^{-\lambda})x_i}}{[G(x_i)]^2} - \frac{x_i^2(1-e^{-\lambda})^2 e^{-\beta(1-e^{-\lambda})x_i}}{G(x_i)}\right]$$

$$+ (\alpha - 1)\sum_{i=1}^{n}\left[\frac{x_i^2\theta(1-e^{-\lambda})^2[G(x_i)]^{\theta-1}e^{-\beta(1-e^{-\lambda})x_i}}{1-[G(x_i)]^{\theta}}\right.$$

$$-\frac{x_i^2\theta(\theta-1)(1-e^{-\lambda})^2[G(x_i)]^{\theta-2}e^{-2\beta(1-e^{-\lambda})x_i}}{1-[G(x_i)]^{\theta}}$$

$$\left.-\frac{x_i^2\theta^2(1-e^{-\lambda})^2[G(x_i)]^{2\theta-2}e^{-2\beta(1-e^{-\lambda})x_i}}{(1-[G(x_i)]^{\theta})^2}\right], \tag{50}$$

$$I_{\beta\lambda} = \frac{\partial^2\ell}{\partial\alpha\partial\lambda} = e^{-\lambda}\sum_{i=1}^{n}x_i + (\theta-1)$$

$$\times\sum_{i=1}^{n}\left[\frac{x_i e^{-\lambda-\beta(1-e^{-\lambda})x_i}}{G(x_i)} - \frac{x_i^2\beta(1-e^{-\lambda})e^{-\lambda-2\beta(1-e^{-\lambda})x_i}}{[G(x_i)]^2}\right.$$

$$\left.-\frac{x_i^2\beta(1-e^{-\lambda})e^{-\lambda-\beta(1-e^{-\lambda})x_i}}{G(x_i)}\right]$$

$$+ (\alpha - 1)\sum_{i=1}^{n}\left[-\frac{x_i\theta[G(x_i)]^{\theta-1}e^{-\lambda-\beta(1-e^{-\lambda})x_i}}{1-[G(x_i)]^{\theta}}\right.$$

$$+\frac{x_i^2\theta\beta(1-e^{-\lambda})[G(x_i)]^{\theta-1}e^{-\lambda-\beta(1-e^{-\lambda})x_i}}{1-[G(x_i)]^{\theta}}$$

$$-\frac{x_i^2\theta(\theta-1)\beta(1-e^{-\lambda})[G(x_i)]^{\theta-2}e^{-\lambda-2\beta(1-e^{-\lambda})x_i}}{1-[G(x_i)]^{\theta}}$$

$$\left.-\frac{x_i^2\theta^2\beta(1-e^{-\lambda})[G(x_i)]^{2\theta-2}e^{-\lambda-2\beta(1-e^{-\lambda})x_i}}{(1-[G(x_i)]^{\theta})^2}\right], \tag{51}$$

$$I_{\lambda\lambda} = \frac{\partial^2 \ell}{\partial^2 \lambda}$$

$$= -\frac{ne^{-2\lambda}}{(1-e^{-\lambda})^2} - \frac{ne^{-\lambda}}{1-e^{-\lambda}} + \beta e^{-\lambda} \sum_{i=1}^{n} x_i + (\theta - 1)$$

$$\times \sum_{i=1}^{n} \left[-\frac{x_i^2 \beta^2 e^{-2\lambda - 2\beta(1-e^{-\lambda})x_i}}{[G(x_i)]^2} - \frac{x_i \beta (1 + x_i \beta e^{-\lambda}) e^{-\lambda - \beta(1-e^{-\lambda})x_i}}{G(x_i)} \right]$$

$$+ (\alpha - 1) \sum_{i=1}^{n} \left[-\frac{x_i^2 \theta(\theta-1)\beta^2 (1-e^{-\lambda})[G(x_i)]^{\theta-2} e^{-2\lambda - 2\beta(1-e^{-\lambda})x_i}}{1 - [G(x_i)]^{\theta}} \right.$$

$$- \frac{x_i^2 \theta^2 \beta^2 [G(x_i)]^{2\theta-2} e^{-2\lambda - 2\beta(1-e^{-\lambda})x_i}}{(1-[G(x_i)]^{\theta})^2}$$

$$\left. + \frac{x_i \theta \beta (1 + x_i \beta e^{-\lambda})[G(x_i)]^{\theta-1} e^{-\lambda - \beta(1-e^{-\lambda})x_i}}{(1-[G(x_i)]^{\theta})^2} \right]. \tag{52}$$

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\mathbf{\Theta}} - \mathbf{\Theta})$ is $N_4(\mathbf{0}, J_n(\mathbf{\Theta})^{-1})$, where $J_n(\mathbf{\Theta})$ is the expected information matrix. This asymptotic behavior is valid if $J_n(\mathbf{\Theta})$ is replaced by $I_n(\hat{\mathbf{\Theta}})$, that is, the observed information matrix is evaluated at $\hat{\mathbf{\Theta}}$. The asymptotic multivariate normal $N_4(\mathbf{0}, I_n(\hat{\mathbf{\Theta}})^{-1})$ distribution can be used to construct approximate confidence intervals and confidence regions for the individual parameters. A $100(1-\gamma)\%$ asymptotic confidence interval for each parameter Θ_i is given by

$$ACI_i = (\hat{\Theta}_i - Z_{\frac{\gamma}{2}} \sqrt{\hat{I}_{ii}},\, \hat{\Theta}_i - Z_{\frac{\gamma}{2}} \sqrt{\hat{I}_{ii}}),$$

where $\hat{I}_{ii}$ is the (i, i) diagonal element of $I_n(\hat{\mathbf{\Theta}})^{-1}$ for $i = 1, 2, 3,$ and $Z_{\frac{\gamma}{2}}$ is the quantile $1 - \frac{\gamma}{2}$ of the standard normal distribution. The likelihood ratio (LR) test can be used to compare the fit of the Kw-ETE distribution with its sub-model for a given data set. For example, to test $\alpha = 1,$ the LR statistic is

$$\omega = 2[\ell(\hat{\alpha}, \hat{\theta}, \hat{\beta}, \hat{\lambda}) - \ell(1, \tilde{\theta}, \tilde{\beta}, \tilde{\lambda})],$$

where $\hat{\alpha}, \tilde{\theta}, \tilde{\beta}, \tilde{\lambda}$ are the unrestricted estimates, and $\tilde{\theta}, \tilde{\beta}, \tilde{\lambda}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi_d^2,$ where χ_d^2 denotes the upper 100*d*% point of the χ^2 distribution with one degree of freedom.

6. Application

In this section, real data set was analyzed to illustrate the desirable performance of the Kw-ETE model in practice. The data set was cited from Lawless [7]. The data set consists of failure times for 36 appliances subjected to an automatic life test and are given as: 11, 35, 49, 170, 329, 381, 708, 958, 1062, 1167, 1594, 1925, 1990, 2223, 2327, 2400, 2451, 2471, 2551, 2565, 2568, 2694, 2702, 2761, 2831, 3034, 3059, 3112, 3214, 3478, 3504, 4329, 6367, 6976, 7846, 13403. Here to illustrate that the Kw-ETE distribution can be a reasonable model, we compared it with modified Weibull distribution (MWD), exponentiated Weibull distribution (EWD), new generalized linear exponential distribution (NGLED), Erlang-truncated exponential distribution (ETE) and exponentiated Erlang-truncated exponential distribution (EETE). Table 1 displays the maximum likelihood estimates (MLEs) of the parameters of the fitted models, their standard errors and their log-likelihood (ℓ).

Table 1. Maximum likelihood estimates of the parameters of the fitted models

Model	Parameter estimates	Standard error	ℓ
NGLED	$\hat{\alpha} = 12.093 \times 10^{-1}$	0.196	
	$\hat{\lambda} = 3.480 \times 10^{-4}$	6.233×10^{-5}	–358.99
	$\hat{\beta} = 12.209 \times 10^{-1}$	4.668×10^{-2}	
	$\hat{\gamma} = 6.263 \times 10^{-3}$	7.466×10^{-3}	
EWD	$\hat{\alpha} = 4.587$	1.003	
	$\hat{\beta} = 1.306$	0.234	–367.73
	$\hat{\gamma} = 0.079$	1.437×10^{-2}	
MWD	$\hat{\lambda} = 3.452 \times 10^{-4}$	6.208×10^{-5}	
	$\hat{\beta} = 1.700$	1.714×10^{-5}	–381.95
	$\hat{\gamma} = 4.540 \times 10^{-3}$	5.390×10^{-3}	
ETE	$\hat{\beta} = 3.629 \times 10^{-4}$	6.049×10^{-5}	–321.19
	$\hat{\lambda} = 9.342$	1.922×10^{-12}	
EETE	$\hat{\theta} = 0.960$	0.205	
	$\hat{\beta} = 3.535 \times 10^{-5}$	7.699×10^{-5}	–321.17
	$\hat{\lambda} = 10.706$	2.258×10^{-7}	
Kw-ETE	$\hat{\alpha} = 3.474$	2.631×10^{-3}	
	$\hat{\theta} = 0.321$	6.165×10^{-2}	–303.56
	$\hat{\beta} = 1.055$	9.193×10^{-3}	
	$\hat{\lambda} = 3.718 \times 10^{-5}$	1.256×10^{-5}	

The variance-covariance matrix of the MLEs under the Kw-ETE distribution is computed as

$$I(\hat{\mathbf{\Theta}})^{-1} = \begin{pmatrix} 6.920 \times 10^{-6} & 1.622 \times 10^{-4} & -2.418 \times 10^{-5} & 1.709 \times 10^{-8} \\ 1.622 \times 10^{-4} & 3.800 \times 10^{-3} & -5.667 \times 10^{-4} & 4.005 \times 10^{-7} \\ -2.418 \times 10^{-5} & -5.667 \times 10^{-4} & 8.451 \times 10^{-5} & -5.973 \times 10^{-8} \\ 1.709 \times 10^{-8} & 4.005 \times 10^{-7} & -5.973 \times 10^{-8} & 1.578 \times 10^{-10} \end{pmatrix}.$$

Therefore, 95% confidence intervals for α, θ, β and λ are $[3.469, 3.479]$, $[0.200, 0.442]$, $[1.037, 1.073]$ and $[1.256 \times 10^{-5}, 6.180 \times 10^{5}]$, respectively. In order to compare the fitted distributions, criteria like the Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICc), Bayesian Information Criterion (BIC) and -2ℓ were used. The better distribution corresponds to smaller AIC, AICc, BIC and -2ℓ values. The values in Table 2 indicate that the Kw-ETE distribution leads to a better fit than the other models.

Table 2. Criteria for comparison

Model	-2ℓ	AIC	AICc	BIC
NGLED	717.986	725.986	727.276	732.320
EWD	735.457	741.457	742.225	746.208
MWD	763.898	769.898	770.648	774.649
ETE	642.372	646.372	646.736	649.539
EETE	642.335	648.335	649.085	653.086
Kw-ETE	607.114	615.114	616.404	621.448

Further, the likelihood ratio test was performed to compare the Kw-ETE distribution with its sub-models (EETE and ETE). The results from Table 3, clearly revealed that the Kw-ETE distribution provides a better fit than its sub-models.

Table 3. Likelihood ratio test

Model	Test statistics	*P*-value
ETE versus Kw-ETE	35.260	0.000
EETE versus Kw-ETE	35.220	0.000

7. Simulation

Simulation studies were performed in this section to investigate the performance of the accuracy of point estimates of $Kw\text{-}ETE(\alpha, \theta, \beta, \lambda)$ distribution. The simulation studies were performed with sample sizes $n = 100, 200, 300, 400$ and 500. For each of the true parameter value

$\alpha = 1.0$, $\theta = 0.5$, $\beta = 1.5$ and $\lambda = 0.8$, we simulate 1000 samples. Table 4 displays the Average Estimate (AE) and the Root Mean Square Error (RMSE). From the results, it was clear that as the sample size increases, the AE of the parameters approaches the true parameter values and the RMSE decay towards zero.

Table 4. Simulation results (AE and RMSE)

n	Parameters	AE	RMSE
100	α	1.380	0.399
	θ	0.420	0.796
	β	0.662	0.838
	λ	0.541	0.259
200	α	1.313	0.314
	θ	0.439	0.608
	β	0.963	0.807
	λ	0.603	0.224
300	α	1.297	0.295
	θ	0.536	0.438
	β	1.205	0.713
	λ	0.672	0.135
400	α	1.256	0.214
	θ	0.524	0.311
	β	1.443	0.543
	λ	0.775	0.119
500	α	1.013	0.187
	θ	0.511	0.251
	β	1.467	0.411
	λ	0.813	0.103

Also, we simulated random numbers of size $n = 30$ and employed the techniques of maximum likelihood estimate to compare the performance of Kw-ETE distribution with NGLED, EWD, MWD, ETE and EETE. Table 5 displays the parameter estimates of the various models with their standard errors in bracket. It was obvious that the Kw-ETE distribution performs better than the other candidate models since it has the highest log-likelihood value and smallest AIC value.

Table 5. Estimates of model parameters for simulated data

Model	$\hat{\alpha}$	$\hat{\lambda}$	α	$\hat{\gamma}$	$\hat{\theta}$	ℓ	AIC
NGLED	0.025 (0.017)	0.058 (0.023)	5.991 (0.008)	0.012 (0.003)	-	–329.18	666.36
EWD	0.013 (0.0026)	-	4.204 (0.0071)	0.244 (0.104)	-	–330.88	667.76
MWD	-	1.133 (0.00163)	0.086 (0.036)	2.477 (0.102)	-	–327.16	660.32
ETE		0.042 (0.0013)	0.356 (0.113)	-	-	–321.51	647.02
EETE	-	0.0004 (0.000006)	1.735 (0.339)	-	0.175 (0.043)	–320.80	647.60
Kw-ETE	0.914 (0.0051)	6.345 (0.0001)	0.245 (0.0005)	-	0.595 (0.128)	–314.80	637.60

8. Conclusion

We introduced and studied a new lifetime model called the *Kumaraswamy Erlang-truncated exponential distribution*. The structural properties of this new model, including the expressions for the moments, moment generating functions and order statistics were derived. The method of maximum likelihood was employed for estimating the model parameters. We demonstrated the application of the new model using real data set. The new model provided a better fit than its sub-models and other competing models. It is our hope that the new model will attract wider application in different areas such as engineering and economics.

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