

## Refinement of Steffensen's Inequality for Superquadratic functions

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### Abstract

In this paper, we discuss superquadratic functions and some results related to Steffensen's inequality. Particularly, we state and prove the superquadratic form of the refined Steffensen's inequality. Applications of the new results are established.

**Mathematics Subject Classification:** 26D10, 26D15

**Keywords:** Steffensen's inequality, superquadratic functions, Convex functions, refinement

## 1 Introduction

The concept of superquadratic functions was first introduced by S. Abramovich et al. in [1] and [2] and it has since been dealt with in numerous papers (see for example [3]), [4] and [5]. The definition of a superquadratic function is a simple modification of the geometrical notion of a convex function. In the case of a superquadratic function, it is required that  $\varphi$  lies above its tangent line plus a translation of  $\varphi$  itself. Our task in this paper is to present some refinements of the Steffensen's inequality

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b g(t)f(t)dt \leq \int_a^{a+\lambda} f(t)dt, \quad (1)$$

where  $\lambda = \int_a^b g(t)dt$ ,  $f$  and  $g$  are integrable functions defined on  $(a, b)$ ,  $f$  is decreasing and for each  $t \in (a, b)$ ,  $0 \leq g(t) \leq 1$  (see also [6], [7]), [8] and [9].

## 2 Preliminary Notes

We give some definitions here.

**Definition 2.1** (*Convex functions*)

Let  $I$  be an interval in  $\mathfrak{R}$ . Then  $\psi : I \rightarrow \mathfrak{R}$  is said to be convex if for all  $t_1, t_2 \in I$  and for all positive  $\lambda$  and  $\mu$  satisfying  $\lambda + \mu = 1$ , we have

$$\psi(\lambda t_1 + \mu t_2) \leq \lambda \psi(t_1) + \mu \psi(t_2). \quad (2)$$

Geometrically, a convex function is defined as

$$\psi(t_2) \geq \psi(t_1) + C_{t_1}(t_2 - t_1)$$

where  $C_{t_1}$  is a slope for each  $t_1 \in I$  and  $t_2 \in I$ . [Note: If  $\psi$  is differentiable at  $t_1$  then  $C_{t_1} = \psi'(t_1)$ .] A function  $\psi$  is said to be concave if  $-\psi$  is convex (i.e. if the inequality (2) is reversed). If it is strict for all  $t_1 \neq t_2$ ,  $\psi$  is said to be strictly concave. Some examples of convex functions are:  $|t|$ ,  $t^k$  for  $k > 1$  and  $-t^k$  for  $0 < k < 1$ ,  $e^t$ ,  $t(\log t)^k$  for  $k \geq 1$ ,  $-\log t$ , etc. Concave functions are  $t^k$  for  $0 < k < 1$ ,  $\log t$ ,  $\sqrt{t}$  for  $t \geq 0$  etc.

**Definition 2.2** A function  $\varphi : [0, \infty) \rightarrow \mathfrak{R}$  is superquadratic provided that for all  $t \geq 0$  there exists a constant  $C_{t_1} \in \mathfrak{R}$  such that

$$\varphi(t_2) - \varphi(t_1) - \underbrace{\varphi(|t_2 - t_1|)}_{\text{extra term}} \geq C_{t_1}(t_2 - t_1) \quad (3)$$

for all  $t_2 \geq 0$ .

The absolute values in the definition of superquadratic functions are employed instead of extending  $\varphi : [0, b) \rightarrow \mathfrak{R}$  to be an even function.

If  $\varphi(u) = u^2$ , we have the identity  $v^2 - u^2 - (v - u)^2 = 2u(v - u)$  where  $C_u = 2u$ . We observe that if  $\varphi(u)$  is superquadratic and  $a, b \geq 0$  then  $\varphi(u) - (au + b)$  is also superquadratic. Any function  $\varphi(u)$  satisfying  $-2 \leq \varphi(u) \leq -1$  for all  $u \geq 0$  is superquadratic. Some examples of superquadratic functions are  $u^p$  for  $p \geq 2$ ,  $-u^p$  for  $0 \leq p < 2$ ,  $\varphi(u) = u^2 \log u$  for  $u > 0$  and  $\varphi(0) = 0$  (See also [1],[2],[3]).

### 3 Results and Discussion

In [8], J.E. Pecaric gave a refinement of inequality (1) as

$$\left( \int_0^1 f(t)g(t)dt \right)^p \leq \int_0^\lambda f(t)^p dt \quad (4)$$

where  $\lambda = \left( \int_0^1 g(t)dt \right)^p$ ,  $f : [0, 1] \rightarrow \mathfrak{R}$  is a nonnegative and nonincreasing function,  $g : [0, 1] \rightarrow \mathfrak{R}$  is an integrable function with  $0 \leq g(t) \leq 1$  ( $\forall t \in [0, 1]$ ) and  $p \geq 1$ .

Putting  $p = 1$  and replace  $f(t)$  with  $-\psi'(t)$  in (4), we obtain the following theorem:

**Theorem 3.1** *Let the function  $g : [0, 1] \rightarrow \mathfrak{R}$  be continuous such that  $0 \leq g(t) \leq 1$ . If  $\psi : [0, 1] \rightarrow \mathfrak{R}$  is a convex, differentiable function with  $\psi(0) = 0$ . Then*

$$\psi \left( \int_0^1 g(t)dt \right) \leq \int_0^1 g(t)\psi'(t)dt \quad (5)$$

for all  $t \in [0, 1]$ .

**Proof** Let  $\psi'(t)$  denote the differential of  $\psi(t)$  which is increasing and therefore  $-\psi'(t)$  is nonincreasing for all  $t \in [0, 1]$ . Then by (4) we have

$$-\int_0^1 g(t)\psi'(t)dt \leq \int_0^\lambda -\psi'(t)dt$$

This simplifies to

$$\int_0^\lambda \psi'(t)dt \leq \int_0^1 g(t)\psi'(t)dt$$

Thus

$$\psi \left( \int_0^1 g(t)dt \right) \leq \int_0^1 g(t)\psi'(t)dt$$

Let us also establish a refinement of (5), in this case, by considering a superquadratic function. We first give a lemma.

**Lemma 3.2** Suppose that  $\varphi$  is a superquadratic function and that its differential exists. Then for all  $t_1 \geq 0$ , there exists  $\varphi'(t_2), \varphi'(t_1) \in \mathfrak{R}$  such that

$$\varphi'(t_2) \leq \varphi'(t_1) - \frac{2\varphi(|t_1 - t_2|)}{t_1 - t_2}$$

for all  $t_2 \geq 0, t_1 \neq t_2$ .

**Proof** By definition 2.2, if  $\varphi$  is superquadratic, then

$$\varphi(t_1) \geq \varphi(t_2) + \varphi'(t_2)(t_1 - t_2) + \varphi(|t_1 - t_2|). \quad (6)$$

for all  $t_1, t_2 \geq 0$ . Interchanging  $t_1$  and  $t_2$ , we have

$$\varphi(t_2) \geq \varphi(t_1) + \varphi'(t_1)(t_2 - t_1) + \varphi(|t_2 - t_1|) \quad (7)$$

Adding inequalities (6) and (7) we obtain

$$0 \geq \varphi'(t_2)(t_1 - t_2) + \varphi'(t_1)(t_2 - t_1) + 2\varphi(|t_1 - t_2|)$$

or

$$0 \geq [\varphi'(t_2) - \varphi'(t_1)](t_1 - t_2) + 2\varphi(|t_1 - t_2|)$$

Assume that  $t_1 > t_2$ , then

$$\varphi'(t_2) \leq \varphi'(t_1) - \frac{2\varphi(|t_1 - t_2|)}{t_1 - t_2}.$$

**Theorem 3.3** Let  $g : [0, 1] \rightarrow \mathfrak{R}$ , be an integrable function such that  $0 \leq g(t) \leq 1, (\forall t \in [0, 1])$ . If  $\varphi : [0, 1] \rightarrow \mathfrak{R}$  is superquadratic and differentiable with  $\varphi(0) = 0$ , then

$$\varphi\left(\int_0^1 g(t)dt\right) + 2 \int_0^1 g(t) \frac{\varphi(|t - \int_0^1 gdt|)}{t - \int_0^1 gdt} dt \leq \int_0^1 g(t)\varphi'(t)dt.$$

**Proof** Let  $G(t) = \int_0^t f(x)dx \leq t$ . The function  $g(t)$  is continuous and the differential of  $G(t)$  denoted  $G'(t) = g(t)$ . Let

$$F(t) = \varphi\{G(t)\} = \varphi\left(\int_0^t f(x)dx\right). \quad (8)$$

Now differentiating (8) and integrating the result, we obtain

$$F(1) = \int_0^1 F'(t)dt = \int_0^1 g(t)\varphi'(G(t))dt.$$

$\Rightarrow$

$$\varphi\left(\int_0^1 g(t)dt\right) = \int_0^1 g(t)\varphi'(G(t))dt. \quad (9)$$

Apply Lemma 3.2 by letting  $t_2 = G(t)$  and  $t_1 = t$ , then

$$\varphi'(G(t)) \leq \varphi'(t) - \frac{2\varphi(|t - G(t)|)}{t - G(t)}. \quad (10)$$

Substituting (10) into (9), we obtain

$$\varphi\left(\int_0^1 g(t)dt\right) \leq \int_0^1 g(t) \left(\varphi'(t) - \frac{2\varphi(|t - G(t)|)}{t - G(t)}\right) dt$$

Therefore

$$\varphi\left(\int_0^1 g(t)dt\right) + \int_0^1 g(t) \frac{2\varphi(|t - G(t)|)}{t - G(t)} dt \leq \int_0^1 g(t)\varphi'(t)dt$$

as required.

**Remark 3.4** By choosing  $\varphi(u) = u^p$ , for  $p \geq 2$ , Theorem 3.3 becomes

$$\left(\int_0^1 g(t)dt\right)^p + \underbrace{2 \int_0^1 g(t) \frac{(|t - G(t)|)^p}{t - G(t)} dt}_{\text{extra term}} \leq p \int_0^1 g(t)t^{p-1} dt. \quad (11)$$

**Remark 3.5** Consider  $p = 2$  in Remark 3.4 and write  $\{G^2(t)\}' = 2G(t)G'(t) = 2g(t)G(t)$  since  $G'(t) = g(t)$ . Then the extra term becomes

$$\begin{aligned} 2 \int_0^1 g(t)[t - G(t)]dt &= 2 \int_0^1 tg(t)dt - 2 \int_0^1 g(t)G(t)dt \\ &= 2 \int_0^1 tg(t)dt - \int_0^1 \{G^2(t)\}' dt \\ &= 2 \int_0^1 tg(t)dt - \left(\int_0^1 g(t)dt\right)^2. \end{aligned}$$

Thus inequality (11) becomes

$$\left(\int_0^1 g(t)dt\right)^2 + 2 \int_0^1 tg(t)dt - \left(\int_0^1 g(t)dt\right)^2 = 2 \int_0^1 tg(t)dt.$$

Therefore, equality is attained for  $p = 2$ .

**Applications.** For  $n \geq 1$ , let  $g(t) = \frac{1}{2}(1 + \varepsilon \sin nt)$  for  $\varepsilon = -1$  or  $1$ . Then

$$\varphi\left(\int_0^{2\pi} g(t)dt\right) = \varphi(\pi),$$

$$\int_0^{2\pi} g(t)\varphi'(t)dt = \frac{\varphi(2\pi)}{2} - \frac{n\varepsilon}{2} \int_0^{2\pi} \varphi(t) \cos(nt)dt$$

Also, let  $g(t) = \frac{1}{2}(1 + \varepsilon \cos nt)$  for  $\varepsilon = -1$  or  $1$ . Then

$$\int_0^{2\pi} g(t)\varphi'(t)dt = \frac{\varphi(2\pi)}{2}(1 + \varepsilon) + \frac{n\varepsilon}{2} \int_0^{2\pi} \varphi(t) \sin(nt)dt.$$

If  $\varphi$  is superquadratic, then

1.

$$\varphi(\pi) + \int_0^{2\pi} (1 + \varepsilon \sin nt) \frac{\varphi(|t - G(t)|)}{t - G(t)} dt \leq \frac{\varphi(2\pi)}{2} - \frac{n\varepsilon}{2} \int_0^{2\pi} \varphi(t) \cos(nt) dt \quad (12)$$

2.

$$\varphi(\pi) + \int_0^{2\pi} (1 + \varepsilon \cos nt) \frac{\varphi(|t - G(t)|)}{t - G(t)} dt \leq \frac{\phi(2\pi)}{2}(1 + \varepsilon) + \frac{n\varepsilon}{2} \int_0^{2\pi} \phi(t) \sin(nt) dt. \quad (13)$$

**Remark 3.6** Using Remark 3.5, If  $\varphi(t) = t^2$ , the extra term for (12) is

$$\begin{aligned} \int_0^{2\pi} (1 + \varepsilon \sin nt)(t - G(t))dt &= \int_0^{2\pi} t(1 + \varepsilon \sin nt)dt - \pi^2 \\ &= \pi^2 - \frac{2\pi\varepsilon}{n} \end{aligned}$$

and the extra term for (13) is

$$\begin{aligned} \int_0^{2\pi} (1 + \varepsilon \cos nt)(t - G(t))dt &= \int_0^{2\pi} t(1 + \varepsilon \cos nt)dt - \pi^2 \\ &= \pi^2 \end{aligned}$$

Thus, (12) attains equality

$$\pi^2 + \pi^2 - \frac{2\pi\varepsilon}{n} = 2\pi^2 - \frac{n\varepsilon}{2} \left( \frac{4\pi}{n^2} \right)$$

and (13) also attains equality

$$\pi^2 + 2\pi^2 - \pi^2 = 2\pi^2(1 + \varepsilon) + \frac{n\varepsilon}{2} \left( \frac{-4\pi^2}{n} \right)$$

for  $\varepsilon = -1$  or  $1$ .

## 4 Conclusion

A refinement of the Steffensen's inequality is thus presented. As well, superquadratic functions are discussed and established for the new Steffensen's inequality (5). This led to the applications of the results in superquadratic form.

**Acknowledgements.** The first author wish to thank Frederic SYMESAK and the French Government for the attention and financial assistance during his research period at the University of Angers, France.

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**Received: February 17, 2014**