

MHD Thermal Boundary Layer Flow over a Flat Plate with Internal Heat Generation, Viscous Dissipation and Convective Surface Boundary Conditions

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Abstract—This paper investigates the effects of exponentially decaying internal heat generation and viscous dissipation on magnetohydrodynamic thermal boundary layer flow over a flat plate with convective surface boundary condition. The governing partial differential equations were transformed into coupled nonlinear differential equations which were solved numerically using the fourth order Runge-Kutta algorithm with a shooting method. Numerical results for the skin friction coefficient, the rate of heat transfer represented by the local Nusselt number and the plate surface temperature were presented whilst the velocity and temperature profiles illustrated graphically and analyzed. The effects of the Biot number, magnetic field parameter, Prandtl number, internal heat generation parameter and Brinkmann number on the flow field were discussed.

Keywords—Magnetohydrodynamic, Magnetic Field, Brinkmann Number, Viscous dissipation, Boundary Layer Flow.

I. INTRODUCTION

Magnetohydrodynamic (MHD) thermal boundary layer flow with internal heat generation and viscous dissipation continues to receive considerable attention due to its numerous applications in both engineering and geophysical fields. Some of such applications include: polymer extrusion, cooling of electronic devices and nuclear reactors, enhanced oil recovery and the design of heat exchangers. These among other applications led researchers to investigate various aspect of the problem. Pioneering work in this area can be traced to Blasius [1] who presented a theoretical result for the boundary layer flow over a flat plate in a uniform stream and on a circular cylinder. Thereafter, many researchers have extended his work by introducing parameters of interest to the industry. Aziz [2] presented similarity solution for laminar thermal boundary layer over a flat plate with a convective surface boundary condition. Ajadi *et al.*, [3] studied slip boundary layer flow of non-Newtonian fluid over a flat plate with convective thermal boundary condition whilst Makinde and Olanrewaju [4] investigated the effects of buoyancy forces on thermal boundary layer over a flat plate with a convective boundary condition.

Their results revealed that the buoyancy effects tend to reduce the thermal boundary layer thickness. Makinde [5] presented similarity solution for natural convection from a moving vertical plate with internal heat generation and a convective boundary condition whilst Olanrewaju *et al.*, [6] investigated the effects of internal heat generation on thermal boundary layer with a convective surface boundary condition. Their results revealed that an increase in the internal heat generation prevents the rapid flow of heat from the lower surface to the upper surface of the plate.

Arthur and Seini [7] studied magnetohydrodynamic thermal stagnation point flow towards a stretching porous surface whilst Seini and Makinde [8] investigated the boundary layer flow near stagnation-points on a vertical surface with slip in the presence of transverse magnetic field. Abel and Mareppa [9] looked at magnetohydrodynamic flow and heat transfer of the mixed hydrodynamic and thermal slip over a linear vertically stretching sheet whilst Seini and Makinde [10] investigated the hydromagnetic flow with dufour and solet effects past a vertical plate embedded in porous media. Morteza *et al.*, [11] provided an analytic solution of hydrodynamic and thermal boundary layers over a flat plate in a uniform stream of fluid with convective surface boundary condition. Jat and Chand [12] studied MHD flow and heat transfer over an exponentially stretching sheet with viscous dissipation and radiation effects. Etwire and Seini [13] investigated the effects of thermal radiation on magneto hydrodynamic flow over a vertical plate with convective boundary conditions. Christian *et al.*, [14] analyzed the magnetohydrodynamic boundary layer stagnation point flow with radiation and chemical reaction towards a heated shrinking porous surface. Emmanuel *et al.*, [15] also investigated the chemically reacting hydromagnetic flow over a flat surface in the presence of radiation with viscous dissipation and convective boundary conditions.

This paper investigates the effects of exponentially decaying internal heat generation and viscous dissipation on magneto hydrodynamic thermal boundary layer flow over a flat plate with convective surface boundary condition. Section 2 presents the mathematical model of the problem. The numerical procedure is outlined in section 3 whilst results and discussions are presented in section 4. Section 5 presents some useful conclusions.

II. PROBLEM FORMULATION

We consider a two-dimensional steady incompressible and electrically conducting fluid flow with heat transfer by convection over a flat plate in the presence of a constant magnetic field of strength B_0 applied in the positive y direction as shown in Figure 1. A stream of cold fluid at temperature T_∞ moving over the upper surface of the plate with a uniform velocity U_∞ while the lower surface of the plate is heated by convection from a hot fluid at temperature T_f , which provides a heat transfer coefficient h_f (see Fig. 1). The cold fluid in contact with the upper surface of the plate generates heat internally at the volumetric rate \dot{q} .

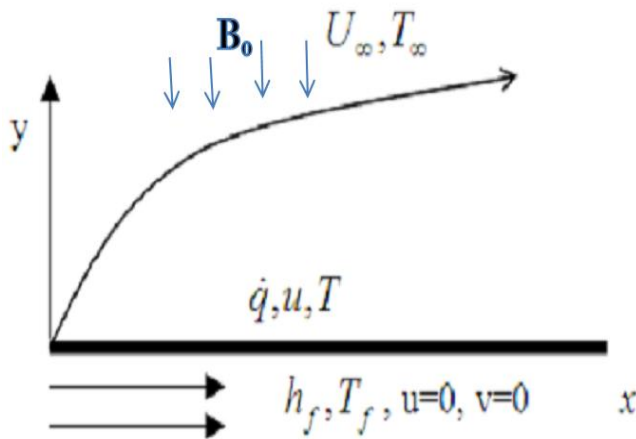


Figure 1. Flow configuration and coordinate system

The induced magnetic field due to the motion of the electrically conducting fluid and the pressure gradient are neglected. The density variation in this fluid is taken into account using the Boussinesq approximation. The continuity, momentum, and energy equations describing the flow can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho} (U_\infty - u) \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\sigma B_0^2}{\rho c_p} (U_\infty - u)^2 + \dot{q} \quad (3)$$

where u and v are the x (along the plate) and the y (normal to the plate) components of velocities respectively, T is the temperature, ρ is the fluid density, c_p is the specific heat at constant pressure, ν is the kinematics viscosity of the fluid, α is the thermal diffusivity of the fluid and β is the thermal expansion coefficient. The velocity boundary conditions can be expressed as:

$$u(x,0) = v(x,0) = 0, u(x, \infty) = U_\infty \quad (4)$$

The thermal boundary conditions at the plate lower surface and far into the cold fluid at the plate upper surface can be written as

$$-k \frac{\partial T}{\partial y}(x,0) = h_f [T_f - T(x,0)], T(x, \infty) = T_\infty \quad (5)$$

Introducing a similarity variable η , a dimensionless stream function $f(\eta)$ and dimensionless temperature $\theta(\eta)$ as:

$$\eta = y \sqrt{\frac{U_\infty}{\nu x}} = \frac{y}{x} \sqrt{\text{Re}_x}, \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}, u = U_\infty f',$$

$$v = \frac{1}{2} \sqrt{\frac{U_\infty \nu}{x}} (\eta f' - f), \lambda_x = \frac{\dot{q} x^2 e^{-\eta}}{k \text{Re}_x (T_f - T_\infty)} \quad (6)$$

After substituting equation (6) into equations (1) to (5), we obtain the following similar equations:

$$f''' + \frac{1}{2} f f'' - M(f' - 1) = 0, \quad (7)$$

$$\theta'' + \frac{1}{2} \text{Pr} f \theta' + \text{Br} f''^2 + \text{Br} M(1 - f'^2) + \lambda_x e^{-\eta} = 0 \quad (8)$$

The associated boundary conditions then become:

$$f'(0) = f(0) = 0, \theta'(0) = -Bi_x(1 - \theta(0)), f'(\infty) = 1, \theta(\infty) = 0 \quad (9)$$

Where

$$Pr = \frac{\nu}{\alpha}, \quad M = \frac{\sigma B_o^2 x}{\rho U_\infty}, \quad Br = \frac{U_\infty^2 \mu}{k(T_f - T_\infty)},$$

$$Bi_x = \frac{h_f}{k} \sqrt{\frac{\nu x}{U_\infty}} \quad (10)$$

In the above equations, the prime symbol denotes differentiation with respect to η . The local internal heat generation parameter λ_x is defined so that the internal heat generation \dot{q} decays exponentially with the similarity variable η . In order to obtain a true similarity solution, the parameters Bi_x and λ_x must be constants and independent of x . This can be achieved if the heat transfer coefficient h_f is proportional to $x^{\frac{1}{2}}$ and the internal heat generation \dot{q} is proportional to x^{-1} . Thus, we assume that

$$h_f = cx^{\frac{1}{2}}, \dot{q} = lx^{-1} \quad (11)$$

Where c and l are constants. Substituting Equation (11) into Equations (6) and (10), we obtain

$$Bi = \frac{c}{k} \sqrt{\frac{\nu}{U_\infty}}, \quad \lambda = \frac{l\nu e^\eta}{kU_\infty(T_f - T_\infty)} \quad (12)$$

The Biot number (Bi) lumps together the effects of convection resistance of the hot fluid and the resistance of the flat plate. The parameter λ measures the strength of the internal heat generation.

III. NUMERICAL PROCEDURE

The coupled nonlinear equations (7) and (8) are observed to be of higher order and can be reduced to a system of first order differential equations by letting:

$$f = x_1, f' = x_2, f'' = x_3, \theta = x_4, \theta' = x_5, \quad (13)$$

Using equation (13), equations (7) and (8) are reduced to first order differential equations by letting:

$$f' = x_1' = x_2,$$

$$f'' = x_2' = x_3,$$

$$f''' = x_3' = -\frac{1}{2}x_1x_3 + M(x_2 - 1),$$

$$\theta' = x_4' = x_5,$$

$$\theta'' = x_5' = -\left(\frac{1}{2}Prx_1x_5 + Brx_3^2 + BrM(1 - x_2)^2 + \lambda e^{-\eta}\right), \quad (14)$$

Subject to the boundary conditions

$$x_1(0) = 0, x_2(0) = 0, x_3(0) = s_1, x_4(0) = s_2, x_5(0) = -Bi_x(1 - s_2), \quad (15)$$

In the shooting method, the unspecified initial conditions; s_1 and s_2 in equation (15) were assumed and equation (14) integrated numerically as an initial valued problem to a given terminal point. The accuracy of the assumed missing initial conditions is checked by comparing the calculated value of the dependent variable at the terminal point with its given value there. If differences exist, improved values of the missing initial conditions are obtained and the process repeated. The computations were done by a written programme which uses a symbolic and computational computer language (MAPLE). A step size of $\Delta\eta = 0.001$ was selected to be satisfactory for a convergence criterion of 10^{-7} in nearly all cases. The maximum value of η_∞ to each group of parameters are determined when the values of unknown boundary conditions at $\eta_\infty = 0$ not change to successful loop with error less than 10^{-7} . From the process of numerical computations, the local skin-friction coefficient $f''(0)$, the rate of heat transfer $\theta'(0)$ represented by the local Nusselt number and the plate surface temperature $\theta(0)$ were worked out and their numerical values presented in tables.

IV. RESULTS AND DISCUSSIONS

Numerical results for different values of thermo-physical parameters controlling the dynamics in the flow regime have been computed.

The thermo-physical parameters considered in this study include: Prandtl number (Pr), Biot number (Bi_x), Brinkman number (Br) and Magnetic field parameter (M).

A. Numerical Results

The plate surface temperature ($\theta(0)$) and the rate of heat transfer ($-\theta'(0)$) represented by the local Nusselt number for different values of the Biot number (Bi_x) are

presented in Table I for $\lambda_x = M = Br = 0$ and $Pr = 0.72$ where a comparison of the results of the present work with the works of Aziz [2] and Olanrewaju *et al.*, [6] exhibit a perfect agreement. This degree of closeness vouches for the high accuracy of the present computational scheme.

Table I.
Computations showing comparison with Aziz [2] and Olanrewaju *et al.*, [6] for $\lambda_x = M = Br = 0$ and $Pr = 0.72$

| Bi_x | Aziz [2] | | Olanrewaju <i>et al.</i> , [6] | | Present Work | |
|--------------|-------------|---------------|--------------------------------|---------------|--------------|---------------|
| | $\theta(0)$ | $-\theta'(0)$ | $\theta(0)$ | $-\theta'(0)$ | $\theta(0)$ | $-\theta'(0)$ |
| 0.05 | 0.1447 | 0.0428 | 0.1447 | 0.0428 | 0.1447 | 0.0428 |
| 0.10 | 0.2528 | 0.0747 | 0.2528 | 0.0747 | 0.2528 | 0.0747 |
| 0.20 | 0.4035 | 0.1193 | 0.4035 | 0.1193 | 0.4035 | 0.1193 |
| 0.40 | 0.5750 | 0.1700 | 0.5750 | 0.1700 | 0.5750 | 0.1700 |
| 0.60 | 0.6699 | 0.1981 | 0.6699 | 0.1981 | 0.6699 | 0.1981 |
| 0.80 | 0.7302 | 0.2159 | 0.7302 | 0.2159 | 0.7302 | 0.2159 |
| 1.00 | 0.7718 | 0.2282 | 0.7718 | 0.2282 | 0.7718 | 0.2282 |
| 5.00 | 0.9442 | 0.2791 | 0.9442 | 0.2791 | 0.9442 | 0.2791 |
| 10.00 | 0.9713 | 0.2871 | 0.9713 | 0.2871 | 0.9713 | 0.2871 |
| 20.00 | 0.9854 | 0.2913 | 0.9854 | 0.2913 | 0.9854 | 0.2913 |

Table II.
Computation showing $f''(0)$, $\theta'(0)$ and $\theta(0)$ for different parameter values.

| Bi_x | Pr | λ_x | M | Br | $f''(0)$ | $\theta'(0)$ | $\theta(0)$ |
|-------------|------|-------------|-----|-----|----------|--------------|-------------|
| 0.1 | 0.72 | 0.1 | 1 | 0.1 | 1.044009 | 0.118981 | 2.189810 |
| 1.0 | 0.72 | 0.1 | 1 | 0.1 | 1.044009 | 0.403491 | 1.403491 |
| 10.0 | 0.72 | 0.1 | 1 | 0.1 | 1.044009 | 0.530297 | 1.053030 |
| 0.1 | 1.00 | 0.1 | 1 | 0.1 | 1.044009 | 0.081152 | 1.811521 |
| 0.1 | 3.00 | 0.1 | 1 | 0.1 | 1.044009 | -0.005621 | 0.943795 |
| 0.1 | 7.10 | 0.1 | 1 | 0.1 | 1.044009 | -0.043582 | 0.564180 |
| 0.1 | 0.72 | 0.2 | 1 | 0.1 | 1.044009 | 0.297766 | 3.977655 |
| 0.1 | 0.72 | 0.3 | 1 | 0.1 | 1.044009 | 0.476550 | 5.765501 |
| 0.1 | 0.72 | 0.5 | 1 | 0.1 | 1.044009 | 0.834119 | 9.341192 |
| 0.1 | 0.72 | 0.1 | 1.5 | 0.1 | 1.260190 | 0.118147 | 2.181468 |
| 0.1 | 0.72 | 0.1 | 2 | 0.1 | 1.444652 | 0.118345 | 2.183450 |
| 0.1 | 0.72 | 0.1 | 3 | 0.1 | 1.756666 | 0.119938 | 2.199383 |
| 0.1 | 0.72 | 0.1 | 1 | 1.0 | 1.044009 | 0.285870 | 3.858700 |
| 0.1 | 0.72 | 0.1 | 1 | 2.0 | 1.044009 | 0.471302 | 5.713023 |
| 0.1 | 0.72 | 0.1 | 1 | 3.0 | 1.044009 | 0.656735 | 7.567346 |

The effects of various embedded thermo-physical parameters on the skin friction coefficient ($f''(\theta)$), the rate of heat transfer ($\theta'(0)$) represented by the local Nusselt number and the plate surface temperature ($\theta(0)$) are presented in Table II. It is evident in the results that with the exception of the magnetic field parameter, the various values of the other embedded thermo-physical parameters have the same skin friction coefficient, $f''(\theta) = 1.044009$.

It is also observed that increasing the Biot number from 0.1 to 10 increases the rate of heat transfer due to an increase in convective heating of the surface whilst the plate upper surface temperature reduces sharply. Furthermore, increasing the Prandtl number from 0.72 (Air) to 7.1 (Water) decreases the rate at which heat is transferred at the surface of the plate thereby preventing the back heat flow into the plate. Moreover, increasing the internal heat generation from 0.1 to 0.5 increases the plate surface temperature due to an increase in convective heat exchange at the surface of the plate. Finally, increasing the magnetic field parameter from 1 to 3 decreases the rate of heat transfer at the surface of the plate due to distractive force known as Lorentz force whilst increasing the Brinkman number from 0.1 to 3.0, increases the plate surface temperature.

B. Graphical Results

Effects of Parameter Variation on the Velocity Profiles

Figure 1 presents the velocity profile for increasing the magnetic field strength. Generally, the velocity starts from a zero value at the plate surface and increases to the free stream value far away from the plate surface satisfying the far field boundary conditions for all parameter values. It is observed in Figure 1 that increasing the intensity of the magnetic field parameter decreases the velocity longitudinally with all profiles tending asymptotically to the free stream value away from the plate.

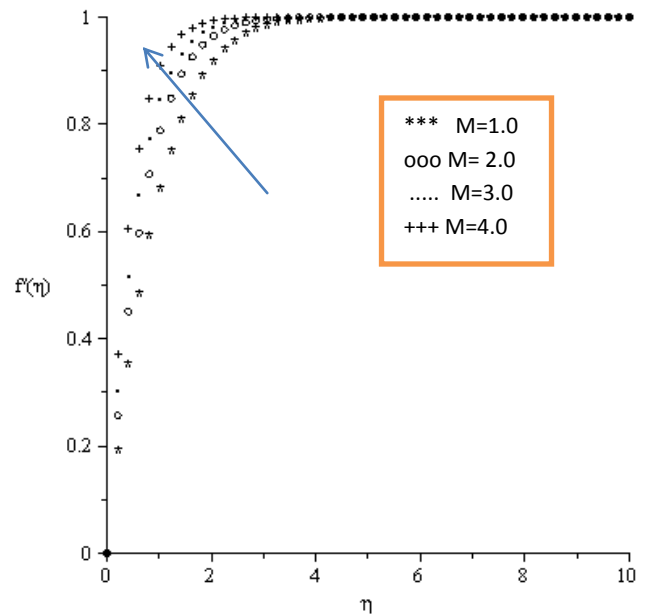


Figure 1. Velocity Profile for increasing Magnetic field parameter for $Pr = 0.72$, $Bi = 0.1$, $\lambda = 0.1$ and $Br = 0.1$.

Effects of Parameter Variation on Temperature Profiles

Figures 2 – 6 present the temperature profiles for various parameter variations. Generally, the temperature of the fluid reaches its maximum at the surface of the plate and decreases exponentially to the free stream zero value away from the plate where it attained its minimum, satisfying the boundary condition. In figure 2, the thermal boundary layer thickness increases as the internal heat generation is increased. Also, it is observed in figure 3 that increasing the Brinkman number increases the fluid temperature thereby increasing the thermal boundary layer thickness due to Ohmic heating of the fluid. Finally, it is noted in figures 4, 5 and 6 that increasing the Magnetic field parameter, Prandtl and Biot numbers decrease the thermal boundary layer thickness far from the boundary.

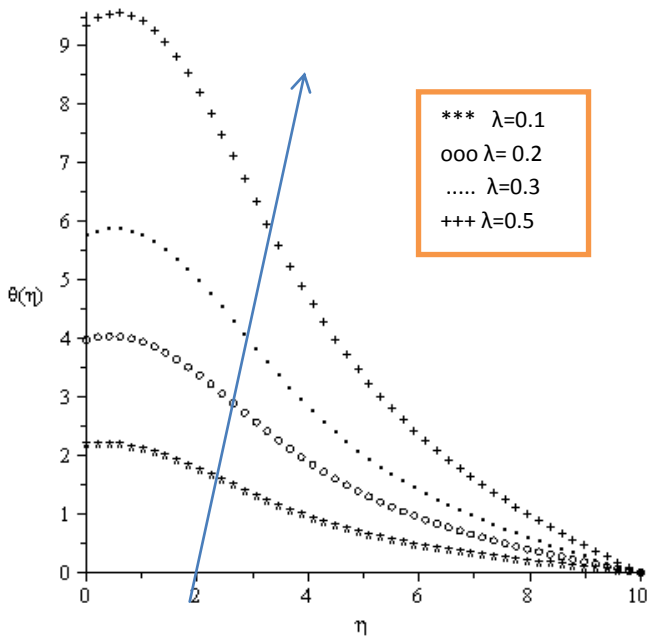


Figure 2. Temperature Profile for varying internal heat generation for $Pr = 0.72$, $Bi = 0.1$, $M = 1$ and $Br = 0.1$.

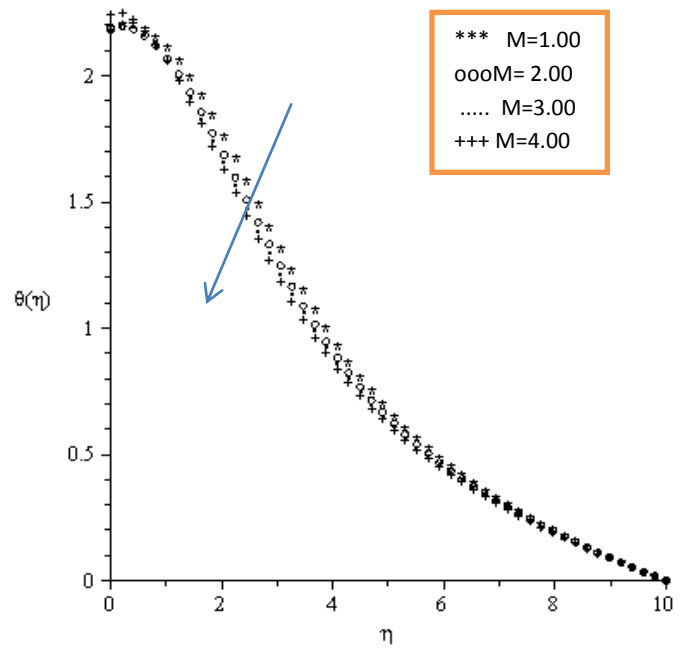


Figure 4. Temperature Profile for varying magnetic field parameter for $Pr = 0.72$, $Bi = 0.1$, $\lambda = 0.1$ and $Br = 0.1$

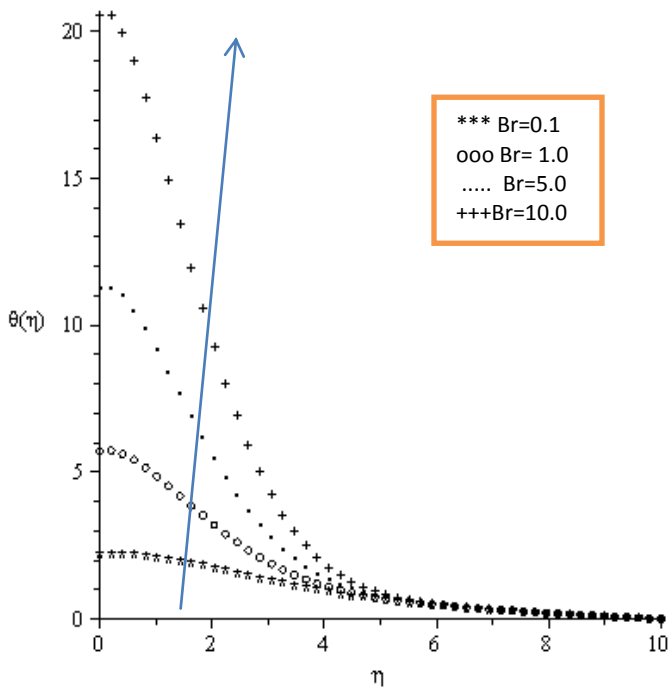


Figure 3. Temperature Profile for varying Brinkman number for $Pr = 0.72$, $Bi = 0.1$, $\lambda = 0.1$ and $M = 1$.

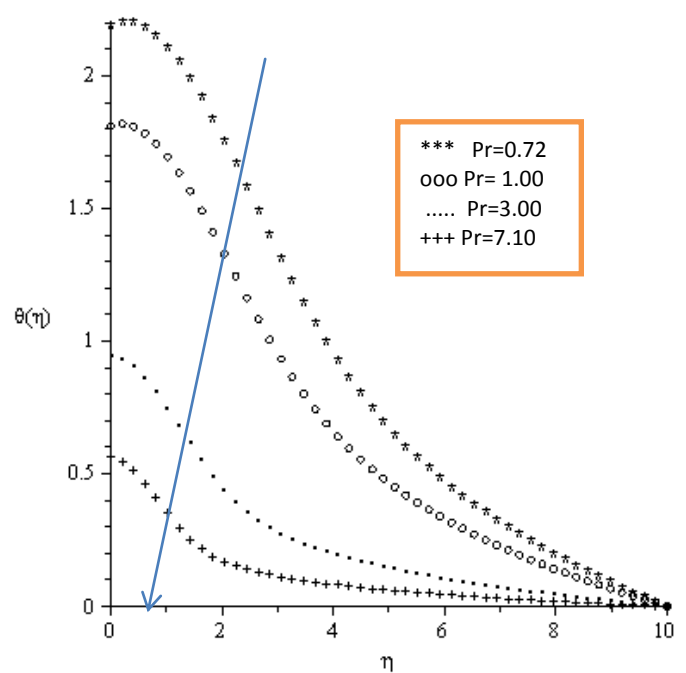


Figure 5. Temperature Profile for varying Prandtl number for $M = 1$, $Bi = 0.1$, $\lambda = 0.1$ and $Br = 0.1$.

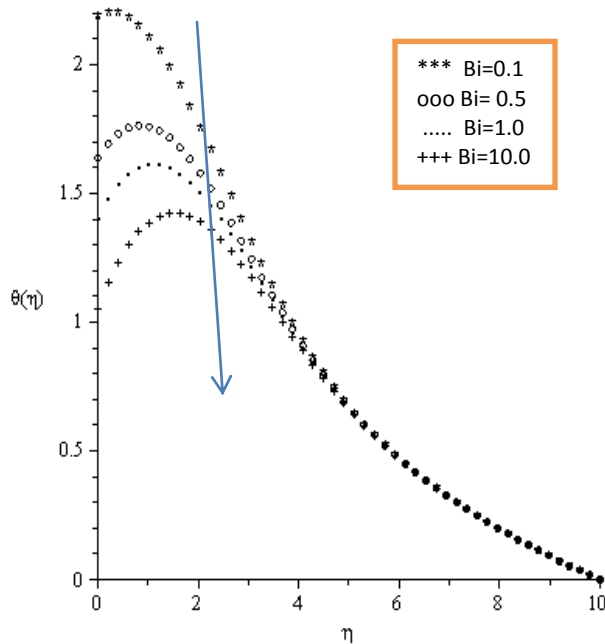


Figure 6. Temperature Profile for varying Biot number for $Pr = 0.72$, $Bi = 0.1$, $\lambda = 0.1$ and $M = 1$.

V. CONCLUSIONS

In this study, the effect of exponentially decaying internal heat generation and viscous dissipation on magnetohydrodynamic thermal boundary layer flow over a flat plate with convective surface boundary conditions has been investigated. The nonlinear and coupled governing differential equations were solved numerically using the fourth order Runge-Kutta algorithm with a shooting method. Numerical results were presented whilst the velocity and temperature profiles illustrated graphically and analyzed. The following conclusions can be made.

- Increasing the Biot number, internal heat generation parameter, Prandtl number and Brinkman number have no effect on the skin friction coefficient whilst increasing the magnetic field parameter decreases the skin friction coefficient.
- Increasing the internal heat generation parameter, magnetic field parameter and Brinkman number increases the rate of heat transfer and plate surface temperature whilst increasing the Prandtl number decreases both the rate of heat transfer and plate surface temperature
- Increasing the Magnetic field parameter decreases the velocity boundary layer thickness.

- Increasing the Brinkman number and internal heat generation parameter increase the thermal boundary layer thickness whilst increasing the Magnetic field parameter, Biot and Prandtl numbers decrease the thermal boundary layer thickness of the fluid flow.

Nomenclature

- (x, y) = Cartesian coordinates
- (u, v) = Velocity components
- T_∞ = Freestream temperature
- T_f = Hot fluid temperature
- T = Fluid temperature
- Pr = Prandtl number
- U_∞ = Freestream velocity
- Bi_x = Biot number
- Br = Brinkmann number
- M = Magnetic field parameter
- k = Thermal conductivity
- C_p = Specific heat at constant pressure
- Re_x = Reynolds number

Greek Symbols

- α = Thermal diffusivity of the fluid
- β = Thermal expansion coefficient
- ν = Kinematic viscosity
- σ^* = The mean absorption coefficient
- λ_x = The internal heat generation parameter

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