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INEQUALITIES FOR THE (q, k)-DEFORMED GAMMA FUNCTION EMANATING FROM CERTAIN PROBLEMS OF TRAFFIC FLOW

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Abstract. In this paper, the authors establish some double inequalities concerning the (q, k)-deformed Gamma function. These inequalities emanate from certain problems of traffic flow. The procedure makes use of the integral representation of the (q, k)-deformed Gamma function.

1. Introduction

The celebrated classical Euler's Gamma function, $\Gamma(x)$ is usually defined for $x > 0$ by

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1} e^{-t}\, dt \\ &= \lim_{n\to\infty} \left[\frac{n! n^x}{x(x+1)\dots(x+n)} \right].\end{aligned}$$

The k-deformed Gamma Function, $\Gamma_k(x)$ (also known as the k-analogue of the Gamma function or simply the k-Gamma function) is defined by (see [3])

$$\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{k}} t^{x-1}\, dt, \quad k > 0, \quad x > 0.$$

It satisfies the following properties (see [3]).

$$\begin{aligned}\Gamma_k(x+k) &= x\Gamma_k(x), \\ \Gamma_k(k) &= 1.\end{aligned}$$

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The Jackson's q-integral from 0 to a and from 0 to ∞ are defined as follows

$$\int_0^a f(t)\,d_qt = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n,$$

$$\int_0^{\infty} f(t)\,d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n$$

provided that the sums converge absolutely.

In a generic interval $[a,b]$, the Jackson's q-integral takes the following form:

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt.$$

For more information on this special integral, reference is made to [7].

The q-deformed Gamma function is also defined for $q \in (0,1)$ and $x > 0$ by

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1}E_q^{-qt}\,d_qt$$
$$= \int_0^{[\infty]_q} t^{x-1}E_q^{-qt}\,d_qt$$

where $[x]_q = \frac{1-q^x}{1-q}$, and $E_q^t = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\frac{t^n}{[n]_q!} = (-(1-q)t;q)_\infty$ is a q-analogue of the classical exponential function. See also [1], [2], [5], [6] and the references therein. For $a \in C$, the set of complex numbers, we have the following notations.
$(a;q)_0 = 1$, $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$, $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$ and $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$.

Just as the k-deformed Gamma function, the q-deformed Gamma function also satisfies the following properties:

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x),$$
$$\Gamma_q(1) = 1.$$

Similarly, the (q,k)-deformed Gamma function, $\Gamma_{q,k}(t)$ was defined by Díaz and Teruel [4] for $x > 0$, $q \in (0,1)$ and $k > 0$ as (See also [9])

$$\Gamma_{q,k}(x) = \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{x-1}E_{q,k}^{-\frac{q^kt^k}{[k]_q}}\,d_qt.$$