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Perturbation Analysis of Hilbert Linear Systems Arising in Applications

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Abstract – **Perturbation analysis of Hilbert linear systems arising in applications is reported. This paper shows the impact of error bound in relation to the perturbation of the linear system. The goal is to bound the relative error of Ax = b when both the right-hand side (RHS) vector b and the coefficient matrix A are perturbed slightly and to determine the relevance of small residual vector. We considered a Hilbert system for the conditioning due to its unique sensitivity to perturbation. From our numerical results ran on MATLAB version 7.01, the condition number of the Hilbert matrix gets larger as size (n) of the matrix increases. We also showed that small residual does not necessarily mean that the approximate solution is 'close' to the exact solution and again small residual does not imply a small error vector of the linear system.**

Keywords – **Condition Number, Perturbation Analysis, Dense Linear Systems, Norm, Error Analysis, Small Residual.**

I. Introduction

It is widely known that the solutions of systems of linear equations are sensitive [4] to round-off error. For some linear systems a small change in one of the values of the coefficient matrix or the right-hand side vector causes a large change in the solution vector. When the solution is highly sensitive to the values of the coefficient matrix A or the right-hand side constant vector b, the equations are said to be ill-conditioned. Ill-conditioned linear systems frequently arise in many real world applications [8, 5].Some ill-conditioned linear systems of equations in the form Ax = b come from discretization of Fredholm integral equations of the first kind [16, 18], where A, x and b are discretizations of continuous functions. Solving such systems has been of great interest for many years and various approaches have been developed to do so. Therefore, we cannot easily rely on the solutions coming out of an ill-conditioned system. These systems pose particular problems when the coefficients are estimated from experimental results [6].The standard method to solve ill-conditioned systems known as Regularization has been studied [17]. Regularization methods use known information about the solution for solving ill-conditioned systems. The mission of regularization is to point to the most desirable solution by incorporating all the prior information of$x$.

The conditioning of a problem is a measure of how sensitive the problem is to small perturbations in the data. This attribute of the problem is important since most problem are approximate to target ones and one would like to feel that good approximation to a target one will leave a solution close to that of the target. If however the problem is highly sensitive to small perturbations in the sense that small perturbations in the data cause large changes in the solution, then the problem is ill-conditioned, otherwise it is well-conditioned. The matrix A is called well-conditioned, if the condition number defined by κ(A) = $\|A\|_2\|A^{-1}\|_2$ is relatively small and ill-conditioned if κ(A) is large. Since I = $AA^{-1}$and $\|AA^{-1}\| \leq \|A\|\|A^{-1}\|$ it implies κ(A) ≥ 1. When using the 2-norm, then the condition number of a square non-singular matrix [17] can be expressed in terms of its non-singular values as κ(A) =$\|A\|_2\|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_2}$ where σ_1 and $\sigma_2$are the largest and the smallest non-singular values of A, respectively.

The directed rounding method is used to compute and verify error bounds for the solution of a linear system Ax = b with an extremely ill-conditioned matrix [10].Therefore, the need to estimate the condition number or sensitivity of numerical problems is one of the most fundamental issues in numerical analysis. Together with knowledge of the underlying stability of the algorithm employed to solve a problem, a good condition estimate can be used to provide a means to comment quantitatively about the accuracy of a computed solution [11].

II. Motivation

Consider the linear system A=$\begin{bmatrix}0.913 & 0.659\\ 0.457 & 0.330\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0.254\\ 0.127\end{bmatrix}$= b. If matrix A is perturbed with the element $A_{11} = 1$, then A=$\begin{bmatrix}1 & 0.659\\ 0.457 & 0.330\end{bmatrix}$ and the approximate solutions is $\hat{x}_1 = \begin{bmatrix}0.00440\\ 0.37870\end{bmatrix}$. The exact solution is $x = (1, -1)^T$ and the residual is $(0.00038, 0.00000)^T$. The issue is: does the small residual imply:

I) The approximate solution is 'close' to the exact solution?

II) Small error?

These are the issues this study seeks to examine.

III. Overview of Condition Number

Condition number is basically a measure of stability or sensitivity of a matrix to numerical operations [2]. The condition number of a problem measures the sensitivity of the solution to small changes in the input [1]. When a problem has a low condition number, it is called well-conditioned and problems with a high condition number are said to be ill conditioned. Norm- wise condition

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estimation consolidates all sensitivity information into a single number. Thus, important information may be lost if individual solution components have widely disparate sensitivity [9]. Algorithm for the computation of the condition of the average eigenvalue and eigenvectors has been studied [3]. The condition of structure-specific linear equations [7] is very significant in the solution of systems of linear equations. MATLAB codes to produce more accurate statistical estimates for the sensitivity of certain structures has been studied [9]. It reflects the maximum possible relative change in the exact solution of a linear system induced by a change in the data. Condition number plays a vital role in numerical solution of linear systems since it measures the sensitivity of the linear system A$x = b$ to the perturbation of A or b. We say a matrix is nearly singular if its condition number is very large. Therefore, ill-conditioning (near singularity) has a much bigger impact on solving a linear system than matrix-vector multiplication. It provides an approximate upper bound on the error in a computed solution and can also be used to predict the convergence of iterative methods. .The value of condition number is dependent on the choice of a matrix norm, and indirectly on the choice of a vector norm. There are several ways of estimating the condition number of a linear system. Statistical condition estimation (SCE) has been introduced in [7, 14]. SCE provides a systematic way of estimating structured condition numbers Properties of Condition Number are:

1. For any matrix **A**, cond ($\boldsymbol{A}$) ≥ 1
2. For identity matrix, cond (I) = 1
3. For any matrix A and scalar γ, cond (γ A) = cond (A)

IV. Vector and Matrix Norms

A norm is a function [20] that assigns a positive length to all the vectors in a vector space. Matrix and vector norms have the same symbol ‖.‖. However vector-norm and matrix-norm are computed very differently. Thus before computing a norm, we need to examine carefully whether it is applied to a vector or a matrix, [19]. A vector norm $\|x\|$ measures the size of a vector $x \in R^n$ by a nonnegative quantity and has the following properties:

1. $\|x\| \geq 0$ and $\|x\| = 0$ implies $x = 0$ (positive definiteness)
2. $\|\alpha x\| = \|\alpha\|\|x\|$ (Positive homogeneity)
3. $\|x + y\| \leq \|x\| + \|x\|$(Triangular Inequality)

Matrix norms corresponding to the vector norms above are defined by the general relationship:

$$\|A\|_p = \max_{\|x\|_{p=1}} \|[A]\{x\}\|_p$$

1. Largest column sum: $\| A \|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$
2. Largest row sum: $\| x \|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$
3. Spectral norm: $\| A \|_2 = (\mu_{max})^{1/2}$

where μ_{max} is the largest eigenvalue of $[A]^T[A]$.

If [A] is symmetric, $(\mu_{max})^{1/2} = \lambda_{max}$,the largest eigenvalue of [A].

1. Norms permit us to express the accuracy of the solution {x} in terms of ‖x‖
2. Norms allow us to bound the magnitude of the product [A] {x} and the associated errors.

V. Perturbation Analysis

Recently, structured perturbations for linear systems have attracted much attention [13, 12, and 15]. When solving linear systems of equations, it is important to analyze [17] how small perturbations of the matrix and right-hand side affect the solution. Perturbation theory has developed a number of diagnostic measures. One of the most important of these is the condition number. The condition number of a matrix can be regarded as a worst-case indicator of the sensitivity of the result of a matrix inversion to a perturbation of the coefficients of the matrix itself. The eigenvalues of some matrices are sensitive to perturbations. Small changes in the matrix elements can lead to large changes in the eigenvalues. Round-off errors introduced during the computation of eigenvalues with floating-point arithmetic have the same effect as perturbations in the original matrix. Consequently, these round-off errors are magnified in the computed values of sensitive eigenvalues. The sensitivity of the eigenvalues is estimated by the condition number of the matrix of eigenvectors.

I. Perturbation in the right hand side

The right-hand side (RHS) vector may be slightly perturbed to observe the nature of the error bound and the conditioning of the linear system. The condition number is a key factor in determining the relative error of the linear system. Thus:

$$\vec{b} \to \vec{b} + \delta\vec{b} \Rightarrow \vec{x} \to \vec{x} + \delta\vec{x} \quad \Leftrightarrow \quad A\,\vec{x} = \vec{b} \;\&\; A(\vec{x} + \delta\vec{x}) = \vec{b} + \delta\vec{b}$$

$$A\,\delta\vec{x} = \delta\vec{b} \Rightarrow \frac{\|\delta\vec{x}\|}{\|\vec{x}\|} = \frac{\|A^{-1}\,\delta\vec{b}\|}{\|\vec{x}\|} \leq \|A^{-1}\|\|A\|\frac{\|\delta\vec{b}\|}{\|A\|\|\vec{x}\|} \leq \|A^{-1}\|\|A\|\frac{\|\delta\vec{b}\|}{\|\vec{b}\|}$$

II. Perturbation on Matrix A

The coefficient matrix A is perturbed in the system Ax=b which shows a variation in the approximate solution. In this case:

$$\text{k(A)} := \text{cond(A)} = \|A^{-1}\|\|A\| = \sqrt{\lambda_{\max} / \lambda_{\min}} \geq 1$$

$$A \to A + \delta A \Rightarrow \vec{x} \to \vec{x} + \delta\vec{x}$$

$$A\,\vec{x} = \vec{b} \;\&\; (A + \delta A)(\vec{x} + \delta\vec{x}) = \vec{b}$$

$$\delta\vec{x} = -A^{-1}(\delta A\,\vec{x} + \delta A\,\delta\vec{x}) \Rightarrow \frac{\|\delta\vec{x}\|}{\|\vec{x}\|} \leq \frac{k(A)\dfrac{\|\delta A\|}{\|A\|}}{1 - k(A)\dfrac{\|\delta A\|}{\|A\|}}$$

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V. Error Analysis

Suppose we want to solve Ax = b. Let $\hat{x}$ denote an approximated solution and r = A$\hat{x}$-b is called residual. Let e=$\hat{x}$-x denote the actual error in the solution, then $\frac{\|e\|}{\|x\|}$ is called relative error. Now Ae = A$(\hat{x}-x)$, Ae = A$\hat{x}$ – Ax , Ae = A$\hat{x}$ – b , Ae = r

Hence e=A^{-1}r

$\|e\|$ = $\|A^{-1}r\|$ ≤ $\|A^{-1}\|\|r\|$, b=A x which implies $\|b\|$=$\|Ax\|$ ≤ $\|A\|\|x\|$, hence

$$\frac{\|e\|}{\|x\|} \le \frac{\|A^{-1}\|\|r\|}{\frac{\|b\|}{\|A\|}} = \|A^{-1}\|\|A\|.\frac{\|r\|}{\|b\|} = \text{Cond(A)}\frac{\|r\|}{\|b\|}$$

$$\frac{\|e\|}{\|x\|} \le \text{Cond(A)}\frac{\|r\|}{\|b\|}.$$

The bound, $\frac{\|e\|}{\|x\|} \le$ Cond(A) $\frac{\|r\|}{\|b\|}$ implies that the linear system Ax=b is well-conditioned if the condition number is small. In particular, if the condition number is small and the relative residual norm, $\frac{\|r\|}{\|b\|}$ is also small, then the approximate solution has a small error (in normwise relation sense). However, if the condition number is large, then the linear system is ill-conditioned.

VII. Numerical Experiments

In this section, we present some numerical results to show the performance of error bound, condition number, relative error and residuals of ill-conditioned and well-conditioned linear systems. We ran our algorithm using MATLAB software version 7.0.1 on Intel(R) Pentium (R) CPU P600 @ 1.87GHz 1.87 and Installed Memory (RAM): 4.00GB.

Example 1

We considered the Hilbert system of linear equations using the MATLAB command for the coefficient matrix A and RHS vector b where the exact solution is x = ones (n,1) for the discussion to test for ill-conditioning of the system as shown in Table 1.The Hilbert matrix H$\epsilon R^{n\times n}$ with entries $h_{ij} = \int_0^1 x^{i+j-2}\,dx = \frac{1}{i+j-1}$.

Table 1: Hilbert System and Conditioning.

n	Condition Number	Relative Error	Relative Residual	Error Bound	EMF
2	19.2815	7.7716e-016	0.0000	0.0000	∞
3	524.0568	1.4433e-014	0.0000	0.0000	∞
4	1.5514e+004	6.3771e-013	0.0000	0.0000	∞
5	4.7661e+005	6.0325e-012	0.0000	0.0000	∞
6	1.4951e+007	3.9584e-010	0.0000	0.0000	∞
7	4.7537e+008	1.7710e-008	0.0000	0.0000	∞
8	1.5258e+010	8.3549e-007	0.0000	0.0000	∞
9	4.9315e+011	2.1290e-005	0.0000	0.0000	∞
10	1.6025e+013	5.2781e-004	0.0000	0.0000	∞
11	5.2237e+014	0.0117	0.0000	0.0000	∞
12	1.6335e+016	0.0812	0.0000	0.0000	∞
13	1.3442e+018	5.8393	0.0000	0.0000	∞
14	2.6741e+017	18.3823	6.8289e-017	18.2612	2.6919e+017
15	3.9824e+017	11.5222	3.3458e-017	13.3245	3.4438e+017
16	2.3717e+017	8.4047	3.2840e-017	7.7887	2.5593e+017
17	9.4039e+017	12.5040	6.4556e-017	60.7081	1.9369e+017
18	6.9787e+017	45.4945	9.5295e-017	66.5038	4.7741e+017
19	4.4423e+018	13.2892	6.2588e-017	278.0313	2.1233e+017
20	2.9373e+018	31.9708	6.1718e-017	181.2837	5.1802e+017
50	2.6060e+019	353.2111	3.4546e-016	9.0027e+003	1.0224e+018
100	4.2276e+019	1.7247e+003	1.1985e-015	5.0670e+004	1.4390e+018
150	6.6433e+019	887.8749	1.4297e-015	9.4977e+004	6.2103e+017
200	1.5712e+020	3.5558e+003	1.5866e-015	2.4927e+005	2.2412e+018
250	1.5532e+020	3.0947e+003	3.0573e-015	4.7487e+005	1.0122e+018
300	2.2004e+020	4.9432e+003	4.6652e-015	1.0265e+006	1.0596e+018
350	6.4285e+020	4.2322e+004	3.8637e-014	2.4838e+007	1.0954e+018
400	1.4107e+021	7.3601e+004	4.9073e-014	6.9229e+007	1.4998e+018
450	1.8079e+021	2.0016e+004	7.8358e-015	1.4166e+007	2.5544e+018
500	5.1045e+020	1.8573e+004	1.2421e-014	6.3406e+006	1.4952e+018

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Table 1 shows the behavior of the condition number of the Hilbert matrix which increases when n gets larger. The error bound for between n = 2 and n = 13 is zero because the relative residual, $\frac{\|r\|}{\|b\|}$ is zero, which implies that the error magnification factor (EMF) that is the ratio of relative error and relative residual should go to infinity due to division by zero. This also means that the approximate and exact solutions for that interval are equal. Again, from n=14 to n=500 there were variations between the approximate and the exact solutions and since the condition numbers got larger and larger, the error bound must automatically increase.

Example 2

Consider the linear system A x =b, where A= $\begin{pmatrix} 2 & 6 \\ 2 & 6.00001 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, b=$\begin{pmatrix} 8 \\ 8.00001 \end{pmatrix}$

The exact solution is $x = (1,1)^T$ and the condition number of coefficient matrix A is 4000000 and let x^* be the approximate solution. Table 2 illustrates the perturbation of the right-hand side (RHS) vector. The system was solved using MATLAB command: x = inv (A) $* \, b$

Table 2: RHS Perturbation

RHS	**Approximate Solution (x^*)**	**Exact Solution (x)**
8.00002	$(-2\ ,2)^T$	$(1\ ,1\ )^T$
8.00003	$(-5, 3)^T$	$(1\ ,1\ )^T$
8.00004	$(-8, 4)^T$	$(1\ ,1\ )^T$
8.00005	$(-11, 5)^T$	$(1\ ,1\ )^T$

From Table 2, as the RHS vector is perturbed slightly, the approximate solutions increased as compared to the exact solutions. This are an indication of an ill-conditioned linear system.

Table 3: Perturbation of Matrix A

Matrix A	**Approximate solution (x^*)**	**Exact solution (x)**
6.00002	$(3.4, 0.2)^T$	$(1\ ,1\ )^T$
6.00003	$(3.000, 0.333)^T$	$(1\ ,1\ )^T$
6.00004	$(3.25, 0.25)^T$	$(1\ ,1\ )^T$
6.00005	$(3.40, 0.20)^T$	$(1\ ,1\ )^T$

Table 3 shows that, as the matrix A is perturbed slightly from 6.00001 to 6.00002 , 6.00003 , 6.00004 , and 6.00005 respectively , the approximate solutions increased as compared to the exact solutions. This is also an indication of an ill-conditioned linear system

Table 4: Perturbation of both RHS and Matrix A

RHS Perturbation	Perturbation on Matrix A	Approximate solution (x^*)	Exact solution (x)
8.00006	6.00004	$(2.00,0.67\ )^T$	$(1\ ,1\ )^T$
8.00002	6.00003	$(-0.50,1.50\ )^T$	$(1\ ,1\ )^T$
8.00005	6.00004	$(1.60,0.80\ )^T$	$(1\ ,1\ )^T$
8.00005	6.00003	$(2.20,0.60\ )^T$	$(1\ ,1\ )^T$

Table 4 shows that as both the RHS vectors and the coefficient matrix are perturbed, the approximate solutions increased as compared to the exact solutions.

Example 3

Consider the linear system A x = b, where A =$\begin{bmatrix} 1 & 4 \\ 1.0001 & 4 \end{bmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $b = \begin{bmatrix} 5 \\ 5.0001 \end{bmatrix}$

The exact solution is $x = (1,1)^T$. If $b = \begin{bmatrix} 5 \\ 5.0001 \end{bmatrix}$ is perturbed as

$b1 = \begin{bmatrix} 5 \\ 5.0002 \end{bmatrix}$ the approximate solution, x^* = $(2.00,\ \ 0.75)^T$.

Table 5: Small Residual 1

Residual	Relative Error	Relative Residual	Condition Number	Error Bound	EMF
$(0, -0.0001)^T$.	1.0000	1.9999e-005	8.5001e+004	1.6999	5.0002e+004

Table 5 shows that the residual vector, $(0, -0.0001)^T$ of the system stated in example 3 is small and the condition number, 8.5001e+004 is very large. As well as EMF. The error bound must always be greater than or equal to the relative error as:

Cond(A) $\frac{\|r\|}{\|b\|} \geq \frac{\|e\|}{\|x\|}$ which is shown in Table 5. The value of the condition number, 8.5001e+004 indicates that the linear system is ill-conditioned.

Example 4

Consider the linear system A x = b, where A =$\begin{bmatrix} 0.4 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $b = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$

The exact solution is $x = (0,1)^T$. The element A_{12} is perturbed as $b1 = \begin{bmatrix} 0.31 \\ 0.7 \end{bmatrix}$ and the approximate solution is x^*= $(-7.00,\ \ 10.00)^T$.

Table 6: Small Residual 2

Residual	Relative Error	Relative Residual	Condition Number	Error Bound	EMF
$(0.10, -0.00)^T$.	9.0000	0.1429	154.9935	22.1419	63

Table 6 shows that small residual vector, $(0.10, -0.00)^T$ does not mean that the approximate solution, x^* = $(-7.00,\ 10.00)^T$ is 'close' to the exact solution, $x = (0,1)^T$. The error vector is $(7.000, -9.000)^T$. Therefore, a small residual did not necessarily result in a small error. The error bound is related to the relative error and the condition number of 154.9935 confirmed that the above linear system is ill-conditioned.

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Example 5

Consider the linear system Ax = b,

where A $=\begin{bmatrix} 1 & 1 \\ 0 & 0.999 \end{bmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $b = \begin{bmatrix} 2 \\ 0.999 \end{bmatrix}$

The exact and approximate solutions are $x = (1,1)^T$ and $x = (1.0256,1.000)^T$ respectively

The perturbed matrix is A$=\begin{bmatrix} 0.95 & 1 \\ 0 & 0.999 \end{bmatrix}$

Table 7: Small Residual 3

Residual	Relative Err.	Relative Res.	Cond.Num	Error Bound	EMF
$(-0.0526, 0)^T$	0.0526	0.0263	2.6192	0.0689	2.0000

From Table 7, the condition number, 2.6192 is small which implies that the above linear system is well-conditioned. The EMF is also small which has a direct link with the relationship between error bound and relative error. That is Error Bound ≥ Relative Error.

Again, RHS vector perturbed as b $=\begin{bmatrix} 1.98 \\ 0.999 \end{bmatrix}$ results in Table 8 below.

Table 8: Small Residual 4

Residual	Relative Error	Relative Res.	Cond.Num	Error Bound	EMF
$(0.0200,0.0000)^T$	0.0200	0.0100	2.6192	0.0262	2.0000

The residual vector, $(0.0200,0.0000)^T$ is small compared to the EMF (2.0000), the errorbound (0.0262) and relative error (0.0200)confirmed that the system in example 5 is well-conditioned. That is to say that, for well-conditioned linear system, EMF and condition number sufficiently smaller as compared to example 3

VIII. Conclusion

In this paper, we studied error bounds in relation to relative error of an ill-conditioned and well-conditioned linear system as a quantitative measure of sensitivity. From the numerical results, the condition number plays a major role in the problem of solving linear systems and relates relative error and error bounds. From results obtained, we have observed that large relative residual implies large backward error in the matrix and the algorithm used to compute the solution is unstable. Another way of saying this is that a stable algorithm will invariably produce a solution with small relative residual, irrespective of the conditioning of the problem, and hence a small residual by itself, sheds little light on the quality of the approximate solution and how close the approximate solution is to the exact solution.

Future Work

In this study only Hilbert matrix was experimented with due to it sensitivity to slight perturbation and large condition number. The error bound as well as the relative residual of this matrix need further analysis more especially the first 13 matrices of the Hilbert system. The QR factorization, singular Value Decomposition and Regularization method could as well be used to determine the relative and residual errors. Furthermore, the analysis of Vander monde and Pascal matrices in terms of conditioning of linear systems would also be areas of further research.

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